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Normal Fans of Polyhedral Convex Sets

Structures and Connections

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Dedicated to Boris Mordukhovich in honour of his 60th birthday

Abstract The normal fan of a polyhedral convex set in $\mathbb{R}^n$ is the collection of its normal cones. The structure of the normal fan reflects the geometry of that set. This paper reviews and studies properties about the normal fan. In particular, it investigates situations in which the normal fan of a polyhedral convex set refines, or is a subfan of, that of another set. It then applies these techniques in several examples. One of these concerns the face structure and normal manifold of the critical cone of a polyhedral convex set associated with a point in $\mathbb{R}^n$. Another concerns how perturbation of the right hand side of the linear constraints defining such a set affects the normal fan and the face structure.

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1 Introduction

This paper studies the structure of the *normal fans* of polyhedral convex sets in $\mathbb{R}^n$ and the connections between those of two polyhedral convex sets. The normal fan of a polyhedral convex set is the collection of normal cones to this set; we will explain more about this in Section 2. This introductory section describes the notation we will use, reviews prior related work, and explains what is new here.

By definition, a face of a convex set C in $\mathbb{R}^n$ is a convex subset F of C such that if x_1 and x_2 belong to C and $\lambda x_1 + (1-\lambda)x_2 \in F$ for some $\lambda \in (0,1)$, then x_1 and x_2 actually belong to F. We denote the collection of faces of C by $\mathscr{F}(C)$. Both the empty set $\emptyset$ and C itself belong to $\mathscr{F}(C)$. It is easy to show that for each $v \in \mathbb{R}^n$ the set

$$F = \operatorname*{argmax}_{C} \langle v, \cdot \rangle \tag{1}$$

is a (possibly empty) face of C. Here $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^n$.

Except where we explicitly state otherwise, let A be a fixed $m \times n$ matrix with its rows denoted by $a_1, \cdots, a_m$. Let I and J be two disjoint subsets of $\{1, \cdots, m\}$, and b be a vector in $\mathbb{R}^m$. For each such triple (b,I,J), define a possibly empty polyhedral convex set in $\mathbb{R}^n$,

$$P(b,I,J) = \{x \in \mathbb{R}^n \mid A_I x \le b_I, A_J x = b_J\}, \tag{2}$$

where A_I and A_J are submatrices of A consisting of those rows with indices in I and J respectively, and b_I and b_J are subvectors of b with the same convention.

Suppose that for a given triple (b,I,J) the set $P(b,I,J)$ defined in (2) is nonempty. For notational simplicity write $P = P(b,I,J)$. For any subset I' of I, the set $P(b, I \setminus I', J \cup I')$ is the collection of points x in P satisfying $\langle a_i, x \rangle = b_i$ for each $i \in I'$. It follows from the definition of face that $P(b, I \setminus I', J \cup I')$ is a (possibly empty) face of P. On the other hand, for any nonempty face F of P there is a maximal subset I' of I such that $\langle a_i, x \rangle = b_i$ for every point x of F and for each $i \in I'$. We call this I' the *active index set* for the face F of P. Note that I' depends not only on the set P and the face F, but also on the representation of P: there is generally more than one way of writing P in the form of (2), and those may give rise to different I' for a given F. When we speak of active index sets for a face of a polyhedral convex set in the rest of this paper, these will refer to the representation of that polyhedral convex set in the context.

If we let $\mathfrak{I}(b,I,J)$ be the collection of subsets I' of I for which there exists $\bar{x} \in \mathbb{R}^n$ with

$$\langle a_i, \bar{x} \rangle < b_i,\ i \in I \setminus I', \quad \langle a_j, \bar{x} \rangle = b_j, j \in J \cup I', \tag{3}$$

then there exists a well known one-to-one correspondence between $\mathfrak{I}(b,I,J)$ and $\mathscr{F}(P) \setminus \{\emptyset\}$:

(a) Each $I' \in \mathfrak{I}(b,I,J)$ defines a nonempty face $P(b, I \setminus I', J \cup I')$ of P whose active set is just I';

(b) Conversely, for each nonempty face F of P, its active index set I' belongs to $\mathfrak{I}(b,I,J)$ and satisfies $P(b, I \setminus I', J \cup I') = F$.

To see why (a) holds, recall that for each subset I' of I the set $P(b,I\setminus I',J\cup I')$ is a face of P. If I' belongs to $\mathfrak{I}(b,I,J)$, then there exists $\bar{x}$ that satisfies (3) and therefore belongs to $P(b,I\setminus I',J\cup I')$, so $P(b,I\setminus I',J\cup I')$ is nonempty. The existence of $\bar{x}$ also implies that I' is the active index set of $P(b,I\setminus I',J\cup I')$.

For (b), the definition of active index set implies that (i) $\langle a_i,x\rangle = b_i$ for each $i\in I'$ and each $x\in F$, so that $F\subset P(b,I\setminus I',J\cup I')$; (ii) for each $j\in I\setminus I'$ there exists some point $x_j\in F$ such that $\langle a_j,x_j\rangle < b_j$. Let $\bar{x}$ be the average of such x_j for $j\in J\setminus I'$ (let $\bar{x}$ be an arbitrary point of F if $I\setminus I'$ is empty); then $\bar{x}$ belongs to F and satisfies (3). This shows that $I'\in\mathfrak{I}(b,I,J)$. Now let f be any point of $P(b,I\setminus I',J\cup I')$. For sufficiently small positive μ the point $f_\mu := \bar{x}-\mu(f-\bar{x})$ belongs to P, so that $\bar{x}$ is a convex combination of f and f_μ with positive coefficients. But $\bar{x}$ belongs to F, which is a face of P, so $f\in F$ and therefore $P(b,I\setminus I',J\cup I')\subset F$.

Now let F be a nonempty face of P with I' being its active index set, and let a be the average of a_i for $i\in I'$ (if $I'=\emptyset$, then $F=P$, in which case we just let a be the origin). The fact that $F=P(b,I\setminus I',J\cup I')$ implies that $F=\operatorname{argmax}_P\langle a,\cdot\rangle$. This shows that each nonempty face of a polyhedral convex set can be written in the form (1). In addition, as is shown in Scholtes [14, Proposition 2.1.2 - 2.1.3], the affine hull of F is given by

$$\operatorname{aff} F = \{x\in\mathbb{R}^n \mid \langle a_i,x\rangle = b_i,\ i\in J\cup I'\}, \tag{4}$$

the relative interior of F is given by

$$\operatorname{ri} F = \{x\in\mathbb{R}^n \mid \langle a_i,x\rangle < b_i,\ i\in I\setminus I',\ \ \langle a_i,x\rangle = b_i,\ i\in J\cup I'\}, \tag{5}$$

and on $\operatorname{ri} F$ the normal cone to P takes the constant value,

$$N_P(F) = \operatorname{pos}\{a_i,\ i\in I'\} + \operatorname{span}\{a_j, j\in J\}, \tag{6}$$

where for a finite set $\{a_1,\dots,a_k\}$,

$$\operatorname{pos}\{a_1,\dots,a_k\} = \{0\}\cup\{\sum_{i=1}^k \tau_i a_i \mid \tau_i\in\mathbb{R}_+\},$$

and

$$\operatorname{span}\{a_1,\dots,a_k\} = \{0\}\cup\{\sum_{i=1}^k \tau_i a_i \mid \tau_i\in\mathbb{R}\}.$$

These definitions ensure that $\operatorname{pos}\emptyset = \operatorname{span}\emptyset = \{0\}$.

By the definition of normal cone, for each vector y in $\mathbb{R}^n$,

$$(N_P)^{-1}(y) \ = \ \{x\in P : y\in N_P(x)\} \ = \ \operatorname*{argmax}_P\langle y,\cdot\rangle. \tag{7}$$

Hence, for each vector y in $N_P(F)$, we have $\operatorname{ri} F\subset \operatorname{argmax}_P\langle y,\cdot\rangle$; the latter set is closed, so $F=\operatorname{cl}\operatorname{ri} F\subset\operatorname{argmax}_P\langle y,\cdot\rangle$. This implies that

$$N_P(F)\subset N_P(x) \text{ for each } x\in F. \tag{8}$$

We will briefly describe the *normal manifolds* of polyhedral convex sets in $\mathbb{R}^n$. The normal manifold of P is the collection of polyhedral convex sets

$$C_P(F) = F + N_P(F), \tag{9}$$

for nonempty faces F of P. Each such $C_P(F)$ is of dimension n, and we call it an n-cell of the normal manifold of P. For more on normal manifolds of polyhedral convex sets, see Robinson [9, Section 2] or Scholtes [14, Section 2.4.2].

The concept of fans arose in the field of algebraic geometry, especially in the theory of toric varieties (see Ewald [1], Fulton [2], Oda [8]). The terminology "normal fan" used here follows Ziegler [16], which defined the normal fan of a nonempty polytope P to be the cones of those linear functions which are maximal on a fixed face of P. (As used here, the term *polytope* denotes a bounded polyhedral convex set.) This definition is in essence equivalent to ours, but we consider general polyhedral convex sets, not only polytopes.

The structure of the normal fan of a polyhedral convex set reflects many geometric properties of that set. As implied by Corollary 2 below, if two polyhedral convex sets have the same normal fan, then we can represent them using the same linear constraints except for different right hand sides; they have the same collection of active index sets with these representations, and therefore have similar face structures. Hence, we can study geometric properties of a polyhedral convex set by studying the structure of its normal fan, and study how two polyhedral convex sets are related by studying the relationship between their normal fans.

Many properties for normal fans of polytopes extend easily to general polyhedral convex sets; Section 2 discusses some of those that are of interest here. However, certain aspects of normal fans of general polyhedral convex sets are not relevant in the study of polytopes. For instance, a proper subfan of the normal fan of a polytope is the normal fan of another polyhedral convex set if its underlying set is convex, but it cannot be the normal fan of another polytope (see Theorem 1 and the discussions following it). For this reason, subfans of normal fans are ignored in the study of polytopes. But they are of substantial interest here, because this subfan relation between the normal fans of two sets reflects a simplification relation between their face structures, and a reduction relation between constraints defining them. Specifically, suppose that I_0 is a subset of I, and define a new set $P_0 := P(b, I_0, J)$. Under certain conditions (see Proposition 3), the normal fan of P_0 is a subfan of that of P, and the set P_0 provides a local approximation and simplification for P. For that reason, Section 3 presents results about subfans and their properties. These results provide a unified framework for objects that are useful in sensitivity analysis of optimization problems and variaitonal inequalities, and they might also be useful in some vertex enumeration or column generalization algorithms; see the discussion after Proposition 3.

2 Normal fans

In this section, we start with the definition of a fan in $\mathbb{R}^n$, and then show that the normal fan of a polyhedral convex set is indeed a fan. Following that, we list some properties of normal fans. Finally, we discuss a situation in which the normal fan of a polyhedral convex set refines that of another.

As is defined in [16], a *fan* in $\mathbb{R}^n$ is a family of nonempty polyhedral convex cones in $\mathbb{R}^n$,

$$\mathscr{N} = \{N_1, N_2, \cdots, N_k\},$$

such that (i) every nonempty face of a cone in $\mathscr{N}$ also belongs to $\mathscr{N}$, and (ii) the intersection of any two cones in $\mathscr{N}$ is a face of both. The *underlying set* $|\mathscr{N}|$ of $\mathscr{N}$ is the union of all its cones. A *subfan* of $\mathscr{N}$ is a subset of $\mathscr{N}$ that is itself a fan.

The following properties hold for a fan.

Lemma 1 *Let $\mathscr{N}$ be a fan in $\mathbb{R}^n$. Then*

(a) If two cones N_1 and N_2 in $\mathscr{N}$ satisfy $(\operatorname{ri} N_1) \cap N_2 \neq \emptyset$, then $N_1 \subset N_2$.
(b) The relative interiors of the cones in $\mathscr{N}$ form a partition of $|\mathscr{N}|$ (that is, they are disjoint and their union is $|\mathscr{N}|$).
(c) All cones in $\mathscr{N}$ have a common lineality space.

Proof Part (a) follows from [11, Theorem 18.1] and the definition of fan, and part (b) is a consequence of part (a) and the definition of fan. For part (c), let N_1 and N_2 be two cones in $\mathscr{N}$. Then $N_1 \cap N_2$ is a face of N_1, and as this face contains the origin it is nonempty. Any nonempty face of N_1 is a cone containing $\operatorname{lin} N_1$, so $\operatorname{lin} N_1 \subset N_1 \cap N_2$. It follows that $\operatorname{lin} N_1 \subset N_2$, so $\operatorname{lin} N_1 \subset \operatorname{lin} N_2$. By symmetry we have $\operatorname{lin} N_1 = \operatorname{lin} N_2$. □

At the beginning of this paper we defined the normal fan of a polyhedral convex set P to be the collection of its normal cones. If we denote it by $\mathscr{N}(P)$, then we have

$$\mathscr{N}(P) = \{N_P(x) : x \in P\} = \{N_P(F) : F \in \mathscr{F}(P) \setminus \{\emptyset\}\}, \tag{10}$$

where the second equality comes from the fact that the relative interiors of nonempty faces of P form a partition of P. Moreover, if $P = P(b,I,J)$ as defined in (2), then

$$\mathscr{N}(P) = \big\{\operatorname{pos}\{a_i, i \in I'\} + \operatorname{span}\{a_j, j \in J\} : I' \in \mathfrak{I}(b,I,J)\big\} \tag{11}$$

due to the one-to-one correspondence between $\mathfrak{I}(b,I,J)$ and $\mathscr{F}(P) \setminus \{\emptyset\}$.

We illustrate normal fans with the following example, to which we shall return periodically in what follows.

Example 1 Let $n = 2$ and $m = 4$, with

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ -1 & -1 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad b' = \begin{bmatrix} 0 \\ 0 \\ -0.5 \\ 1 \end{bmatrix}.$$

Let $I = \{1,2,3,4\}$ and $J = \emptyset$. Then

$$\mathfrak{I}(b,I,J) = \{\emptyset, \{1\}, \{1,2,3\}, \{2\}, \{2,4\}, \{4\}, \{1,4\}\},$$
$$\mathfrak{I}(b',I,J) = \{\emptyset, \{1\}, \{1,3\}, \{3\}, \{2,3\}, \{2\}, \{2,4\}, \{4\}, \{1,4\}\}.$$

Normal fans of P and P' are then given by

$$\mathscr{N}(P) = \{\operatorname{pos}\{a_i, i \in I'\} : I' \in \mathfrak{I}(b,I,J)\},$$
$$\mathscr{N}(P') = \{\operatorname{pos}\{a_i, i \in I'\} : I' \in \mathfrak{I}(b',I,J)\}.$$

The following proposition contains a list of properties of $\mathscr{N}(P)$.

Proposition 1 *Let $P = P(b,I,J)$ be a nonempty polyhedral convex set. Then the following properties hold:*

(a) Let I' be an element of $\mathfrak{I}(b,I,J)$, define $F = P(b, I \setminus I', J \cup I')$ and $N = N_P(F)$. Let N' be a nonempty face of N. Then there exists a face F' of P such that $F' \supset F$ and $N_P(F') = N'$.

(b) *Let I_1 and I_2 be two elements of $\Im(b,I,J)$. Define $F_k = P(b,I\setminus I_k, J\cup I_k)$ and $N_k = N_P(F_k)$ for $k=1,2$. Then $I_1\cap I_2$ belongs to $\Im(b,I,J)$ too; if we write $F = P(b,I\setminus (I_1\cap I_2), J\cup(I_1\cap I_2))$, then we have*

$$N_1\cap N_2 = N_P(F), \tag{12}$$

which is a face of N_1 and N_2. Moreover, $a_i \notin N_1\cap N_2$ for each i in $I_1\setminus I_2$ or $I_2\setminus I_1$.

(c) *Let I_k, F_k and N_k be as defined in part (b) for $k=1,2$. Then*

$$F_1 \supset F_2 \Leftrightarrow I_1\subset I_2 \Leftrightarrow N_1\subset N_2, \tag{13}$$

$$F_1 = F_2 \Leftrightarrow I_1 = I_2 \Leftrightarrow N_1 = N_2. \tag{14}$$

(d) *Let I_1 belong to $\Im(b,I,J)$, $F_1 = P(b,I\setminus I_1, J\cup I_1)$ and $N_1 = N_P(F_1)$. Let i be an element of $\cup_{I'\in\Im(b,I,J)} I'$. Then*

$$i\in I_1 \Leftrightarrow a_i\in N_1.$$

(e) *Let $\operatorname{rc}P$ be the recession cone of P; then the underlying set $|\mathscr{N}(P)|$ of $\mathscr{N}(P)$ is given by*

$$|\mathscr{N}(P)| = \bigcup_{F\in\mathscr{F}(P)\setminus\{\emptyset\}} N_P(F) = \bigcup_{x\in P} N_P(x) = (\operatorname{rc}P)^\circ = \operatorname{pos}\{a_i, i\in I\} + \operatorname{span}\{a_j, j\in J\}.$$

We illustrate part (b) of this proposition in the context of Example 1. Let $I_1 = \{1,2,3\}$ and $I_2 = \{2,4\}$, which are two elements of $\Im(b,I,J)$. Define F_1, F_2, N_1 and N_2 as in part (b). Then $F_1 = \{0\}$, $F_2 = \{(1,0)\}$, $N_1 = \mathbb{R}^2_-$, and $N_2 = \operatorname{pos}\{(0,-1),(1,1)\}$. Clearly, $I_1\cap I_2 = \{2\}$ is also an element of $\Im(b,I,J)$. If we let F be the face of P defined by $I_1\cap I_2$, then F is the line segment connecting the origin and the point $(1,0)$, and $N_P(F) = \{0\}\times\mathbb{R}_-$, which is exactly the intersection of N_1 and N_2, and is in fact the common face of them.

Proof For part (a), see the second part of [14, Lemma 2.4.2]. To prove part (b), note that by (6) we have

$$N_k = \operatorname{pos}\{a_i,\ i\in I_k\} + \operatorname{span}\{a_j, j\in J\} \tag{15}$$

for $k=1,2$, while the fact that $I_k\in\Im(b,I,J)$ implies the existence of $\bar{x}_k$ for $k=1,2$ such that

$$\langle a_i,\bar{x}_k\rangle < b_i,\ i\in I\setminus I_k, \quad \langle a_j,\bar{x}_k\rangle = b_j,\ j\in J\cup I_k. \tag{16}$$

Using the point $(\bar{x}_1+\bar{x}_2)/2$, it is easy to check that $I_1\cap I_2\in\Im(b,I,J)$. Moreover, (16) implies

$$\langle a_i,\bar{x}_2-\bar{x}_1\rangle \begin{cases} =0, & i\in J\cup(I_1\cap I_2),\\ <0, & i\in \quad I_1\setminus I_2,\\ >0, & i\in \quad I_2\setminus I_1.\end{cases} \tag{17}$$

It follows from this and (15) that

$$\langle y,\bar{x}_2-\bar{x}_1\rangle \le 0 \text{ for each } y\in N_1 \text{ and } \langle y,\bar{x}_2-\bar{x}_1\rangle \ge 0 \text{ for each } y\in N_2. \tag{18}$$

Now we obtain a chain of set inclusions,

$$N_1\cap N_2 \subset N_1\cap(\bar{x}_2-\bar{x}_1)^\perp \subset \operatorname{pos}\{a_i,\ i\in I_1\cap I_2\} + \operatorname{span}\{a_j, j\in J\} \subset N_1\cap N_2,$$

where the first, second and third inclusions come from (18), (17) and (15) respectively. Here and thereafter, for a vector $v \in \mathbb{R}^n$ we denote by $v^\perp$ the subspace of $\mathbb{R}^n$ perpendicular to the subspace spanned by the vector v. It follows that

$$N_1 \cap (\bar{x}_2 - \bar{x}_1)^\perp = N_1 \cap N_2 = \operatorname{pos}\{a_i,\ i \in I_1 \cap I_2\} + \operatorname{span}\{a_j, j \in J\}. \tag{19}$$

But $N_1 \cap (\bar{x}_2 - \bar{x}_1)^\perp$ is just $\operatorname{argmax}_{N_1} \langle \bar{x}_2 - \bar{x}_1, \cdot \rangle$ by (18), so $N_1 \cap N_2$ is a face of N_1. By symmetry it is also a face of N_2. The second equality of (19) proves (12). Moreover, it follows from (17) and the first equality of (19) that $a_i \notin N_1 \cap N_2$ for each i in $I_1 \setminus I_2$ or $I_2 \setminus I_1$.

For part (c), we have $F_1 \supset F_2 \Leftarrow I_1 \subset I_2$ by the formula for $P(b, I \setminus I_k, J \cup I_k)$ in (2), and $I_1 \subset I_2 \Rightarrow N_1 \subset N_2$ by the formula (6). We also have $F_1 \supset F_2 \Rightarrow I_1 \subset I_2$ because I_k is just the active index set for F_k. Now suppose $N_1 \subset N_2$; we show that $I_1 \subset I_2$. If there existed some $i \in I_1 \setminus I_2$, then we would have $a_i \notin N_1 \cap N_2$ by the last statement of part (b). The fact that $i \in I_1$ implies that $a_i \in N_1$, so we must have $a_i \notin N_2$. But it contradicts the hypothesis $N_1 \subset N_2$. This shows that $I_1 \setminus I_2 = \emptyset$, and therefore proves $I_1 \subset I_2 \Leftarrow N_1 \subset N_2$ and completes the proof of (13). We obtain (14) by applying (13) twice.

The $\Rightarrow$ direction of part (d) follows immediately from (6), so we only need to prove the other direction. By hypothesis there exists $I_2 \in \mathfrak{I}(b, I, J)$ such that $i \in I_2$; we then define $F_2 = P(b, I \setminus I_2, J \cup I_2)$ and $N_2 = N_P(F_2)$, and note that $a_i \in N_2$. Now if the inclusion $a_i \in N_1$ holds, then we actually have $a_i \in N_1 \cap N_2$, which implies that $i \in I_1 \cap I_2$ by the last statement of part (b). This shows that $i \in I_1 \Leftarrow a_i \in N_1$.

For part (e), the first and second equalities come from (10). To show the third equality, recall that from the decomposition theorem of polyhedral convex sets we have $P = \operatorname{rc} P + Q$ where Q is a compact polyhedral convex set. Therefore, a point $y \in \mathbb{R}^n$ belongs to $(\operatorname{rc} P)^\circ$ if and only if the set $\operatorname{argmax}_P \langle y, \cdot \rangle$ is nonempty. But the latter set is just $(N_P)^{-1}(y)$, and this proves the third equality. The fourth equality follows from the fact that $\operatorname{rc} P = P(0, I, J)$. □

Parts (a) and (b) in Proposition 1 imply that any nonempty face N' of a cone $N \in \mathscr{N}(P)$ belongs to $\mathscr{N}(P)$ and that the intersection of any two cones in $\mathscr{N}(P)$ is a face of both, so $\mathscr{N}(P)$ satisfies the two requirements for being a fan. It is therefore appropriate to call it the normal fan of P.

Recall that according to (11), $\mathscr{N}(P)$ is the collection of cones $\operatorname{pos}\{a_i, i \in I'\} + \operatorname{span}\{a_j, j \in J\}$ for I' in $\mathfrak{I}(b, I, J)$. By the second equivalence in (14), different index sets in $\mathfrak{I}(b, I, J)$ correspond to different cones in $\mathscr{N}(P)$, so there is an one-to-one correspondence between $\mathfrak{I}(b, I, J)$ and $\mathscr{N}(P)$.

The next corollary lists some additional facts about the normal fan $\mathscr{N}(P)$. Part (c) here identifies the relative interior of $N_P(F)$ for a face F with the collection of vectors y such that $F = \operatorname{argmax}_P \langle y, \cdot \rangle$. Part of [9, Proposition 2.1] says that $F = (N_P)^{-1}(y)$ for $y \in \operatorname{ri} N_P(F)$, which is the $\subset$ direction of (20). [13, Equation 2.4.3] is equivalent to (20), but was established only for polytopes.

Corollary 1 *Let $\mathscr{N}(P)$ be the normal fan of a nonempty polyhedral convex set P in $\mathbb{R}^n$. The following properties hold:*

(a) If two cones N_1 and N_2 in $\mathscr{N}(P)$ satisfy $(\operatorname{ri} N_1) \cap N_2 \neq \emptyset$, then $N_1 \subset N_2$.
(b) The relative interiors of the cones in $\mathscr{N}(P)$ form a partition of $(\operatorname{rc} P)^\circ$.
(c) Each nonempty face F of P satisfies

$$\operatorname{ri} N_P(F) = \{y : F = (N_P)^{-1}(y)\}. \tag{20}$$

To illustrate part (c) of this corollary in the context of Example 1, let $F = \{0\}$, which is a nonempty face of P. Then $N_P(F) = \mathbb{R}^2_-$, and its relative interior is the set $\{y \in \mathbb{R}^2 : y_1 < 0, y_2 < 0\}$. This is exactly the collection of y's such that $\{0\} = (N_P)^{-1}(y)$.

Proof As $\mathscr{N}(P)$ is a fan, parts (a) and (b) here follow from parts (a) and (b) of Lemma 1 respectively, using part (e) of Proposition 1.

For part (c), first let $y \in \operatorname{ri} N_P(F)$ and define $F' = (N_P)^{-1}(y)$. By (7), F' is just the set $\operatorname{argmax}_P \langle y, \cdot \rangle$, which is a face of P. By the discussion just before (8), we have $F \subset F'$; in particular, F' is nonempty, so it belongs to $\mathscr{F}(P) \setminus \{\emptyset\}$. The definition $F' = (N_P)^{-1}(y)$ ensures that $y \in N_P(F')$. So $y \in \operatorname{ri} N_P(F) \cap N_P(F')$; by part (a) this implies that $N_P(F) \subset N_P(F')$. It then follows from (13) that $F \supset F'$. We conclude that $F = F'$, and this shows the $\subset$ direction of (20). For the other direction, let y belong to the set on the right side of (20). Then $y \in N_P(x)$ for each $x \in F$, so by part (e) of Proposition 1 y belongs to $(\operatorname{rc} P)^\circ$. It then follows from part (b) that $y \in \operatorname{ri} N_P(F')$ for some nonempty face F' of P. We then have

$$F' = (N_P)^{-1}(y) = F,$$

where the first equality holds by the $\subset$ direction of (20) that we just established, and the second equality holds by the way we defined y. It follows that $y \in \operatorname{ri} N_P(F)$. This proves the $\supset$ direction of (20). $\square$

The next corollary shows that the normal fan of P determines a system of linear inequalities that represents P, along with its active index sets. Consequently, if two sets P and P' have the same normal fan, then we may express them as $P(b,I,J)$ and $P(b',I,J)$ by proper selection of a matrix A, vectors b and b', and index sets I and J, with $\mathfrak{I}(b,I,J) = \mathfrak{I}(b',I,J)$.

Corollary 2 *Let $\mathscr{N}(P)$ be the normal fan of a nonempty polyhedral convex set P in $\mathbb{R}^n$. Let L be the common lineality space of cones in $\mathscr{N}(P)$, with $L = \operatorname{span}\{a_1, \cdots, a_k\}$ for some vectors $a_1, \cdots, a_k$ in $\mathbb{R}^n$. Let $\{a_{k+1}, \cdots, a_m\}$ be the collection of unit generators of extreme rays of cones $N \cap L^\perp$ for $N \in \mathscr{N}(P)$. Then there exists a vector $b \in \mathbb{R}^m$ such that*

$$P = \{x \in \mathbb{R}^n \mid \langle a_i, x \rangle = b_i, i = 1, \cdots, k, \ \langle a_i, x \rangle \le b_i, i = k+1, \cdots, m\}. \tag{21}$$

Further, let $I = \{k+1, \cdots, m\}$ and $J = \{1, \cdots, k\}$. For each cone N in $\mathscr{N}(P)$, define I' to be the subset of I such that $\{a_i, i \in I'\}$ is the collection of unit generators of extreme rays of $N \cap L^\perp$. Then I' belongs to $\mathfrak{I}(b,I,J)$ with

$$N = \operatorname{pos}\{a_i, i \in I'\} + \operatorname{span}\{a_j, j \in J\}. \tag{22}$$

Proof For each $i = 1, \cdots, m$ define $b_i = \max_P \langle a_i, \cdot \rangle$, which is well defined because a_i belongs to some normal cone of P. This definition ensures that the inequality $\langle a_i, \cdot \rangle \le b_i$ holds for all points of P for each i. For $i = 1, \cdots, k$, both a_i and $-a_i$ belong to the normal cone of P at each point of P, so the equality $\langle a_i, \cdot \rangle = b_i$ holds for all points of P. This proves the $\subset$ direction of (21).

To prove the other direction, it is enough to prove that the points in the right hand side of (21) satisfy any valid inequality for P. For this purpose, let $\langle a', \cdot \rangle \le b'$ be an arbitrary valid inequality for P, where a' is a nonzero vector in $\mathbb{R}^n$ and b' is a scalar. Then a' belongs to $N_P(\bar{x})$ for some point $\bar{x} \in P$. But $\operatorname{lin} N_P(\bar{x}) = L$, and $N_P(\bar{x}) \cap L^\perp$ is generated by its extreme rays, so

$$N_P(\bar{x}) = L + N_P(\bar{x}) \cap L^\perp = \operatorname{span}\{a_1, \cdots, a_k\} + \operatorname{pos}\{a_i, i \in I'\} \tag{23}$$

for some $I' \subset \{k+1,\cdots,m\}$. Therefore, there exist scalars λ_i for $i=1,\cdots,k$ and $i \in I'$ such that

$$a' = \sum_{i=1}^{k} \lambda_i a_i + \sum_{i \in I'} \lambda_i a_i,$$

with $\lambda_i \geq 0$ for $i \in I'$. In addition, it follows from (23) that

$$\langle a_i, \bar{x} \rangle = \max_P \langle a_i, \cdot \rangle = b_i$$

for each $i = 1,\cdots,k$ and $i \in I'$, so that

$$b' \geq \langle a', \bar{x} \rangle = \sum_{i=1}^{k} \lambda_i b_i + \sum_{i \in I'} \lambda_i b_i.$$

Consequently, any point x that satisfies the equations and inequalities on the right hand side of (21) will satisfy $\langle a', x \rangle \leq b'$. This proves (21).

The equation (22) follows from the fact that $N = L + N \cap L^\perp$. It remains to prove $I' \in \mathfrak{I}(b,I,J)$. For this, let I'' be the element of $\mathfrak{I}(b,I,J)$ such that

$$N = \text{span}\{a_j, j \in J\} + \text{pos}\{a_i, i \in I''\};$$

we need to prove that $I' = I''$. First, for each $i \in I$, because we defined a_i to be a generator of an extreme ray of the cone $N \cap L^\perp$ for some $N \in \mathscr{N}(P)$, the cone $\text{pos}\{a_i\}$ is a face of $N \cap L^\perp$, so $L + \text{pos}\{a_i\}$ is a face of N. Hence, $L + \text{pos}\{a_i\}$ also belongs to $\mathscr{N}(P)$, so there exists $I_0 \in \mathfrak{I}(b,I,J)$ such that

$$L + \text{pos}\{a_i\} = \text{span}\{a_j, j \in J\} + \text{pos}\{a_i, i \in I_0\}.$$

By the definition of vectors a_i for $i \in I$, the only I_0 that will satisfy the equation above is $I_0 = \{i\}$. Consequently, $\{i\}$ belongs to $\mathfrak{I}(b,I,J)$. Now, if i is an element of I', then $a_i \in N$; by part (d) of Proposition 1, we have $i \in I''$. This proves $I' \subset I''$. For the other direction, let i be an element of I''; then $L + \text{pos}\{a_i\}$ is a subset of N. But we already know that $L + \text{pos}\{a_i\}$ belongs to $\mathscr{N}(P)$, so $L + \text{pos}\{a_i\}$ is a face of N. This implies that $\text{pos}\{a_i\}$ is a face of $N \cap L^\perp$, so $i \in I'$ by the definition of I'. This proves $I'' \subset I'$. ☐

The next proposition concerns a possible relation between the normal fans of two polyhedral convex sets.

Proposition 2 *Let P and P' be two polyhedral convex sets in $\mathbb{R}^n$ with $\text{rc}\, P = \text{rc}\, P'$. Suppose that for each cone N' in $\mathscr{N}(P')$ there exists a cone N_0 in $\mathscr{N}(P)$ such that $N' \subset N_0$. Then there exists a map Ψ from $\mathscr{N}(P')$ to $\mathscr{N}(P)$ such that for each cone N' in $\mathscr{N}(P')$,*

$$\text{ri}\, N' \subset \text{ri}\, \Psi(N') \quad \textit{and} \quad N' \subset \Psi(N'). \tag{24}$$

The cone $\Psi(N')$ is the smallest cone in $\mathscr{N}(P)$ containing N'. The map Ψ is surjective, inclusion preserving and satisfies

$$\text{ri}\, N = \bigcup_{N' \in \Psi^{-1}(N)} \text{ri}\, N' \quad \textit{and} \quad N = \bigcup_{N' \in \Psi^{-1}(N)} N' \tag{25}$$

for each $N \in \mathscr{N}(P)$.

The sets P and P' in Example 1 satisfy the relation discussed in Proposition 2. First, $\operatorname{rc} P = \operatorname{rc} P' = \{0\}$. Second, each index set in $\mathfrak{I}(b', I, J)$ is a subset of some index set in $\mathfrak{I}(b, I, J)$. Accordingly, each cone in $\mathscr{N}(P')$ is a subset of some cone in $\mathscr{N}(P)$. We will discuss more details when returning to this example in Section 4.3.

Proof Let N' be a cone in $\mathscr{N}(P')$; by hypothesis there exists a cone N_0 in $\mathscr{N}(P)$ with $N' \subset N_0$. As $\operatorname{ri} N'$ is a relatively open convex subset of N_0, it follows from [11, Theorem 18.2] that there exists a unique face N of N_0 such that $\operatorname{ri} N' \subset \operatorname{ri} N$. Taking closures on both sides shows that $N' \subset N$ too. As part (a) of Proposition 1 ensures that N is an element of $\mathscr{N}(P)$, we can define $\Psi(N') = N$. Defining $\Psi(N')$ in this way for each cone N' in $\mathscr{N}(P')$, we obtain a map Ψ from $\mathscr{N}(P')$ to $\mathscr{N}(P)$ as required.

To see that $\Psi(N')$ is the smallest element of $\mathscr{N}(P)$ containing N', it is enough to show that $\Psi(N')$ is included in any element of $\mathscr{N}(P)$ that contains N'. For this, let N be an element of $\mathscr{N}(P)$ with $N' \subset N$. Both $\operatorname{ri} \Psi(N')$ and N contain $\operatorname{ri} N'$ as a subset, so $\operatorname{ri} \Psi(N') \cap N \neq \emptyset$. It then follows from part (a) of Corollary 1 that $\Psi(N') \subset N$.

To see that Ψ is inclusion preserving, let N'_1 and N'_2 be two cones in $\mathscr{N}(P')$ with $N'_1 \subset N'_2$. Then $\Psi(N'_2)$ is an element of $\mathscr{N}(P)$ containing N'_2, so it contains N'_1. As $\Psi(N'_1)$ is the smallest element of $\mathscr{N}(P)$ containing N'_1, we have $\Psi(N'_1) \subset \Psi(N'_2)$.

To show (25), note that part (b) of Corollary 1 and the hypothesis $\operatorname{rc} P = \operatorname{rc} P'$ imply

$$\bigcup_{N \in \mathscr{N}(P)} \operatorname{ri} N = (\operatorname{rc} P)^\circ = (\operatorname{rc} P')^\circ = \bigcup_{N' \in \mathscr{N}(P')} \operatorname{ri} N'. \tag{26}$$

Let N be a cone in $\mathscr{N}(P)$; the definition of Ψ ensures that $\operatorname{ri} N \supset \cup_{N' \in \Psi^{-1}(N)} \operatorname{ri} N'$. For the other direction, let x be a point in $\operatorname{ri} N$. By (26), there exists a cone N' in $\mathscr{N}(P')$ such that $x \in \operatorname{ri} N'$. But $\operatorname{ri} N' \subset \operatorname{ri} \Psi(N')$ by the definition of Ψ, so x belongs to $\operatorname{ri} \Psi(N')$. Consequently, $\operatorname{ri} N$ and $\operatorname{ri} \Psi(N')$ meet at the point x. We then apply part (b) of Corollary 1 again to find that $N = \Psi(N')$, so $N' \in \Psi^{-1}(N)$ and $x \in \cup_{N' \in \Psi^{-1}(N)} \operatorname{ri} N'$. This shows that $\operatorname{ri} N \subset \cup_{N' \in \Psi^{-1}(N)} \operatorname{ri} N'$ and completes the proof of the first equation in (25). We obtain the second equation in (25) by taking closures on both sides of the first equation, noting that $\Psi^{-1}(N)$ is a finite set.

A by-product of (25) is that $\Psi^{-1}(N) \neq \emptyset$ for each cone N in $\mathscr{N}(P)$: that is, Ψ is surjective. ☐

In Proposition 2, the relative interiors of cones in $\mathscr{N}(P')$ and $\mathscr{N}(P)$ form two partitions for the same underlying set $(\operatorname{rc} P)^\circ = (\operatorname{rc} P')^\circ$, and the first equation in (25) means that the former partition is a refinement of the latter. For this reason, we say that $\mathscr{N}(P')$ *refines* $\mathscr{N}(P)$ if two polyhedral convex sets P and P' satisfy the properties in Proposition 2.

A related notion for polytopes appeared in Shephard [15], and was slightly changed by Meyer [7]; see also Kallay [5]. In the terminology used here, Meyer says that two polytopes P and P' in $\mathbb{R}^n$ satisfy $P \leq P'$ if $\dim \operatorname{argmax}_P \langle y, \cdot \rangle \leq \dim \operatorname{argmax}_{P'} \langle y, \cdot \rangle$ for each $y \in \mathbb{R}^n \setminus \{0\}$, which is in essence equivalent to the situation that $\mathscr{N}(P')$ refines $\mathscr{N}(P)$ considered here; see [7, Theorem 2.6]. Shephard [15] also defined P and P' to be *locally similar* if $\dim \operatorname{argmax}_P \langle y, \cdot \rangle = \dim \operatorname{argmax}_{P'} \langle y, \cdot \rangle$ for each $y \in \mathbb{R}^n \setminus \{0\}$; equivalent terminologies include 'strongly isomorphic', 'analogous', 'related' and 'normally equivalent' (see Schneider [13] and Grünbaum [4]).

3 Subfans and constraint reduction

This section concerns changes in the normal fan when we remove some of the constraints defining the original polyhedral convex set; in particular, we are interested in when the normal fan of the new set is a subfan of the original normal fan. To begin, we ask which subfans of the original normal fan are themselves normal fans. Corollary 3 of Theorem 1 answers this question.

Theorem 1 *Let $P = P(b,I,J)$ be a nonempty polyhedral convex set. Let $\mathscr{N}$ be a nonempty subfan of $\mathscr{N}(P)$ with its underlying set $|\mathscr{N}|$ being convex. Define a subset $\mathfrak{I}_0$ of $\mathfrak{I}(b,I,J)$ by*

$$\mathfrak{I}_0 = \left\{I' \in \mathfrak{I}(b,I,J) \mid \operatorname{pos}\{a_i, i \in I'\} + \operatorname{span}\{a_j, j \in J\} \in \mathscr{N}\right\}, \tag{27}$$

a subset I_0 of I by

$$I_0 = \bigcup_{I' \in \mathfrak{I}_0} I', \tag{28}$$

and a polyhedral convex set P_0 by

$$P_0 = P(b,I_0,J) = \{x \in \mathbb{R}^n \mid \langle a_i, x\rangle \le b_i, i \in I_0, \langle a_j, x\rangle = b_j, j \in J\}. \tag{29}$$

Then the following equations hold:

$$|\mathscr{N}| = \operatorname{pos}\{a_i, i \in I_0\} + \operatorname{span}\{a_j, i \in J\}, \tag{30}$$

$$\{I' \in \mathfrak{I}(b,I,J) \mid I' \subset I_0\} = \mathfrak{I}_0 = \mathfrak{I}(b,I_0,J), \tag{31}$$

and

$$\mathscr{N}(P_0) = \mathscr{N}. \tag{32}$$

To illustrate the present theorem with the set P in Example 1, take $\mathscr{N}$ to be

$$\mathscr{N} = \left\{\{0\},\ \mathbb{R}_- \times \{0\},\ \mathbb{R}^2_-,\ \{0\} \times \mathbb{R}_-\right\}$$

which is a subfan of $\mathscr{N}(P)$ with its underlying set convex. Defining $\mathfrak{I}_0$, I_0 and P_0 as in the theorem, we have

$$\mathfrak{I}_0 = \{\emptyset, \{1\}, \{1,2,3\}, \{2\}\},$$

$I_0 = \{1,2,3\}$ and $P_0 = P(b,I_0,J) = \mathbb{R}^2_+$. As claimed by the theorem, the equalities $\mathfrak{I}(b,I_0,J) = \mathfrak{I}_0$ and $\mathscr{N}(P_0) = \mathscr{N}$ hold. Note that the convexity assumption in the theorem is necessary for (32) to hold, because the underlying set of the normal fan of a polyhedral convex set has to be convex. In this example, if we add cones $\operatorname{pos}\{(0,-1),(1,1)\}$ and $\operatorname{pos}\{(1,1)\}$ to the set $\mathscr{N}$, then $|\mathscr{N}|$ is no more convex. If we again define $\mathfrak{I}_0$, I_0 and P_0 as in the theorem, then

$$\mathfrak{I}_0 = \{\emptyset, \{1\}, \{1,2,3\}, \{2\}, \{2,4\}, \{4\}\},$$

$I_0 = \{1,2,3,4\}$ and $P_0 = P$. The equalities $\mathfrak{I}(b,I_0,J) = \mathfrak{I}_0$ and $\mathscr{N}(P_0) = \mathscr{N}$ fail.

Proof First, it follows from the definition of $\mathfrak{I}_0$ in (27) and the one-to-one correspondence between $\mathscr{N}(P)$ and $\mathfrak{I}(b,I,J)$ that

$$\mathscr{N} = \left\{\operatorname{pos}\{a_i, i \in I'\} + \operatorname{span}\{a_j, j \in J\} : I' \in \mathfrak{I}_0\right\}. \tag{33}$$

Applying the formula (11) to P_0, we have

$$\mathscr{N}(P_0) = \left\{\operatorname{pos}\{a_i, i \in I'\} + \operatorname{span}\{a_j, j \in J\} : I' \in \mathfrak{I}(b,I_0,J)\right\}. \tag{34}$$

Now, the $\subset$ direction of (30) follows immediately from (33), because the definition of I_0 implies that $I' \subset I_0$ for each $I' \in \mathfrak{I}_0$.

To prove the other direction of (30), note that $\mathscr{N}$ is nonempty by hypothesis, so there exists at least one $I' \in \mathfrak{I}(b,I,J)$ such that $\mathrm{pos}\{a_i, i \in I'\} + \mathrm{span}\{a_j, i \in J\}$ belongs to $\mathscr{N}$. So we have

$$\mathrm{span}\{a_j, j \in J\} \subset |\mathscr{N}|. \tag{35}$$

We need to show that $\mathrm{pos}\{a_i, i \in I_0\} \subset |\mathscr{N}|$. If I_0 is empty (it is possible for this to happen) then this trivially holds. If I_0 is nonempty, then let i be one of its elements. The definition of I_0 implies the existence of $I' \in \mathfrak{I}_0$ with $i \in I'$, and the definition of $\mathfrak{I}_0$ then implies

$$a_i \in \mathrm{pos}\{a_i, i \in I'\} + \mathrm{span}\{a_j, j \in J\} \subset |\mathscr{N}|.$$

This shows that $a_i \in |\mathscr{N}|$ for each $i \in I_0$. But $|\mathscr{N}|$ is convex by hypothesis, and it is a cone because $\mathscr{N}$ is a fan. So we have $\mathrm{pos}\{a_i, i \in I_0\} \subset |\mathscr{N}|$. Combining this with (35) and using the fact that $|\mathscr{N}|$ is a convex cone, we obtain the $\supset$ direction for (30).

Next, we establish the first equality in (31). The definitions of $\mathfrak{I}_0$ and I_0 imply that $\mathfrak{I}_0 \subset \{I' \in \mathfrak{I}(b,I,J) \mid I' \subset I_0\}$. For the other direction, let I' be a subset of I_0 belonging to $\mathfrak{I}(b,I,J)$, and write $N' = \mathrm{pos}\{a_i, i \in I'\} + \mathrm{span}\{a_j, i \in J\}$. The fact that $I' \subset I_0$ then implies that $N' \subset |\mathscr{N}|$ by (30); in particular, $\emptyset \neq \mathrm{ri}\, N' \subset |\mathscr{N}|$. As $\mathscr{N}$ is a fan, the relative interiors of the cones in $\mathscr{N}$ form a partition of $|\mathscr{N}|$, so there exists a cone N'' in $\mathscr{N}$ with $(\mathrm{ri}\, N') \cap (\mathrm{ri}\, N'') \neq \emptyset$. Note that both N' and N'' belong to $\mathscr{N}(P)$: $N' \in \mathscr{N}(P)$ because $I' \in \mathfrak{I}(b,I,J)$, while $N'' \in \mathscr{N}(P)$ because $\mathscr{N}$ is a subfan of $\mathscr{N}(P)$. It then follows that $N' = N''$, so that $N' \in \mathscr{N}$. This shows that $I' \in \mathfrak{I}_0$ and therefore proves $\mathfrak{I}_0 \supset \{I' \in \mathfrak{I}(b,I,J) \mid I' \subset I_0\}$.

Now, we prove the second equality in (31). Let I' belongs to $\mathfrak{I}_0$, then it is a subset of I_0 belonging to $\mathfrak{I}(b,I,J)$. As $I' \in \mathfrak{I}(b,I,J)$, there exists a point $\bar{x}$ satisfying (3), in particular, we have

$$\langle a_i, \bar{x}\rangle < b_i, i \in I_0 \setminus I', \quad \langle a_j, \bar{x}\rangle = b_j, j \in J \cup I',$$

so that $I' \in \mathfrak{I}(b,I_0,J)$. This shows that $\mathfrak{I}_0 \subset \mathfrak{I}(b,I_0,J)$, and it follows from this and (33) and (34) that $\mathscr{N} \subset \mathscr{N}(P_0)$.

To show that $\mathfrak{I}_0 \supset \mathfrak{I}(b,I_0,J)$, let I' belong to $\mathfrak{I}(b,I_0,J)$ and write $N' = \mathrm{pos}\{a_i, i \in I'\} + \mathrm{span}\{a_j, i \in J\}$. Then $N' \in \mathscr{N}(P_0)$ by (34). The fact that $I' \in \mathfrak{I}(b,I_0,J)$ requires that $I' \subset I_0$, so we have $N' \subset |\mathscr{N}|$ by (30); in particular, $\emptyset \neq \mathrm{ri}\, N' \subset |\mathscr{N}|$. As $\mathscr{N}$ is a fan, the relative interiors of the cones in $\mathscr{N}$ form a partition of $|\mathscr{N}|$, so there exists a cone N'' in $\mathscr{N}$ with $(\mathrm{ri}\, N') \cap (\mathrm{ri}\, N'') \neq \emptyset$. By (33), there exists $I'' \in \mathfrak{I}_0$ such that

$$N'' = \mathrm{pos}\{a_i, i \in I''\} + \mathrm{span}\{a_j, j \in J\}.$$

Note that we have already shown $\mathfrak{I}_0 \subset \mathfrak{I}(b,I_0,J)$ and $\mathscr{N} \subset \mathscr{N}(P_0)$, so I'' belongs to $\mathfrak{I}(b,I_0,J)$ and N'' belongs to $\mathscr{N}(P_0)$. Then N' and N'' are two cones in $\mathscr{N}(P_0)$ whose relative interiors meet, so we have $N' = N''$, which in turn implies that $I' = I''$ by the one-to-one correspondence between $\mathscr{N}(P_0)$ and $\mathfrak{I}(b,I_0,J)$. It follows that $I' \in \mathfrak{I}_0$. This shows that $\mathfrak{I}_0 \supset \mathfrak{I}(b,I_0,J)$ and thereby completes the proof of (31). Finally, we obtain (32) by combining (33), (34) and (31). □

The following corollary follows immediately from Theorem 1.

Corollary 3 *Let P be a nonempty polyhedral convex set in $\mathbb{R}^n$. A nonempty subfan $\mathscr{N}$ of $\mathscr{N}(P)$ is itself the normal fan of a polyhedral convex set if and only if its underlying set $|\mathscr{N}|$ is convex.*

Two more observations for the situation considered in Theorem 1 are:

(a) Here each nonempty face F_0 of P_0 is an "extension" of some nonempty face F of P in the sense that $F_0 \supset F$ and $\operatorname{aff} F_0 = \operatorname{aff} F$. To see this, recall that $F_0 = P(b, I_0 \setminus I', J \cup I')$ for some index set $I' \in \mathfrak{I}(b, I_0, J)$; by (31) I' belongs to $\mathfrak{I}(b, I, J)$, so it also defines a nonempty face $F = P(b, I \setminus I', J \cup I')$ of P. We have

$$P(b, I_0 \setminus I', J \cup I') \supset P(b, I \setminus I', J \cup I')$$

and

$$\operatorname{aff} P(b, I_0 \setminus I', J \cup I') = \operatorname{aff} P(b, I \setminus I', J \cup I')$$

by formulas (2) and (4).

(b) By part (b) of Lemma 1, the relative interiors of cones in $\mathscr{N}(P)$ form a partition of $|\mathscr{N}(P)|$, and $|\mathscr{N}|$ is the union of the relative interiors of cones in $\mathscr{N}$. Accordingly, if $\mathscr{N}$ is a proper subset of $\mathscr{N}(P)$, then $|\mathscr{N}|$ is a proper subset of $|\mathscr{N}(P)|$. In this case, $|\mathscr{N}|$ must be a proper subset of $\mathbb{R}^n$. But $|\mathscr{N}|$ is a convex cone, so it must be contained in a halfspace generated by a hyperplane through the origin, that is, $|\mathscr{N}|^\circ$ must contain at least one half line. Note that $\operatorname{rc} P_0 = |\mathscr{N}|^\circ$ by (32) and part (c) of Proposition 1. Hence, if $\mathscr{N}$ is a proper subset of $\mathscr{N}(P)$, then $\operatorname{rc} P_0$ contains at least one half line, and therefore P_0 is unbounded.

Theorem 1 starts with a given subfan $\mathscr{N}$ of $\mathscr{N}(P)$, uses it to define a subset I_0 of I and a set P_0, and proves that the normal fan of P_0 is $\mathscr{N}$ if $|\mathscr{N}|$ is convex. The next proposition starts with a given subset I_0 of I, uses it to define $\mathscr{N}$ and P_0, and relates them to the setting of Theorem 1 to obtain similar conclusions.

Proposition 3 *Let $P = P(b, I, J)$ be a nonempty polyhedral convex set. Let I_0 be a subset of I; define $P_0 = P(b, I_0, J)$,*

$$\mathfrak{I}_0 = \{I' \in \mathfrak{I}(b, I, J) \mid I' \subset I_0\},$$

and

$$\mathscr{N} = \left\{\operatorname{pos}\{a_i, i \in I'\} + \operatorname{span}\{a_j, j \in J\} : I' \in \mathfrak{I}_0\right\}.$$

Then $\mathfrak{I}_0 \subset \mathfrak{I}(b, I_0, J)$ and $\mathscr{N}$ is a subfan of $\mathscr{N}(P)$. Moreover, the following statements are equivalent:

(a) $|\mathscr{N}|$ is convex and $a_i \in |\mathscr{N}|$ for each $i \in I_0$.
(b) $\mathfrak{I}(b, I_0, J) = \mathfrak{I}_0$.
(c) $\mathscr{N}(P_0) = \mathscr{N}$.
(d) $|\mathscr{N}| = \operatorname{pos}\{a_i, i \in I_0\} + \operatorname{span}\{a_j, j \in J\}$.

To apply the present proposition to the set P in Example 1, we may let $I_0 = \{1,2,3\}$. Sets P_0, $\mathfrak{I}_0$ and $\mathscr{N}$ are then given by

$$P_0 = P(b, I_0, J) = \mathbb{R}^2_+,$$

$$\mathfrak{I}_0 = \{I' \in \mathfrak{I}(b, I, J) \mid I' \subset I_0\} = \left\{\emptyset, \{1\}, \{1,2,3\}, \{2\}\right\},$$

and

$$\mathscr{N} = \left\{\{0\},\ \mathbb{R}_- \times \{0\},\ \mathbb{R}^2_-, \{0\} \times \mathbb{R}_-\right\}.$$

It follows that $|\mathscr{N}| = \mathbb{R}^2_-$ is a convex set consisting of each vector a_i for $i \in I_0$, so statement (a) in the proposition hold. It is easy to verify that statements (b), (c) and (d) hold as well.

In the example above, the set I_0 happens to be an element of $\mathfrak{I}(b, I, J)$. This does not have to hold in general for application of Proposition 3. For instance, we may apply this proposition to the set P' in Example 1 with the same set $I_0 = \{1,2,3\}$. For this case, sets P_0, $\mathfrak{I}_0$ and $\mathscr{N}$ are

$$P_0 = P(b', I_0, J) = \{x \in \mathbb{R}^2_+ \mid x_1 + x_2 \geq 0.5\},$$

$$\mathfrak{I}_0 = \{I' \in \mathfrak{I}(b', I, J) \mid I' \subset I_0\} = \big\{\emptyset, \{1\}, \{1,3\}, \{3\}, \{2,3\}, \{2\}\big\},$$

and

$$\begin{aligned}\mathscr{N} = \big\{&\{0\},\ \mathbb{R}_- \times \{0\},\ \operatorname{pos}\{(-1,0),(-1,-1)\},\ \operatorname{pos}\{(-1,-1)\},\\ &\operatorname{pos}\{(-1,-1),(0,-1)\},\ \{0\} \times \mathbb{R}_-\big\}.\end{aligned} \tag{36}$$

Again, $|\mathscr{N}| = \mathbb{R}^2_-$ is a convex set consisting of each vector a_i for $i \in I_0$, so statement (a) in the proposition hold. It follows from either an application of the proposition or a direct check that other statements in that proposition hold in this case as well.

Proof To prove $\mathfrak{I}_0 \subset \mathfrak{I}(b, I_0, J)$, let I' belong to $\mathfrak{I}_0$; then $I' \in \mathfrak{I}(b, I, J)$ and there exists a point $\bar{x}$ satisfying (3). In particular, $\bar{x}$ satisfies

$$\langle a_i, \bar{x}\rangle < b_i, i \in I_0 \setminus I', \ \ \langle a_j, \bar{x}\rangle = b_j, j \in J \cup I',$$

so $I' \in \mathfrak{I}(b, I_0, J)$.

Next we show that $\mathscr{N}$ is a subfan of $\mathscr{N}(P)$. If $\mathscr{N}$ is empty then there is nothing to prove; suppose for now that $\mathscr{N}$ is nonempty. As $\mathscr{N}$ is a subset of $\mathscr{N}(P)$, we only need to show that every nonempty face of a cone in $\mathscr{N}$ also belongs to $\mathscr{N}$. Let N be a cone in $\mathscr{N}$ with $N = \operatorname{pos}\{a_i, i \in I'\} + \operatorname{span}\{a_j, j \in J\}$ for some I' in $\mathfrak{I}_0$, and let N' be a nonempty face of N. Then N' belongs to $\mathscr{N}(P)$ because N belongs to $\mathscr{N}(P)$ and $\mathscr{N}(P)$ is a fan. So there exists $I'' \in \mathfrak{I}(b, I, J)$ with $N' = \operatorname{pos}\{a_i, i \in I''\} + \operatorname{span}\{a_j, j \in J\}$. The fact $N' \subset N$ implies that $I'' \subset I'$, according to (13). So we have $I'' \subset I' \subset I_0$. It follows that I'' belongs to $\mathfrak{I}_0$ and therefore N' belongs to $\mathscr{N}$. This shows that $\mathscr{N}$ is a fan.

Below we prove the equivalence between statements (a), (b), (c) and (d). The fact that (b) implies (c) is a simple consequence of (11). If (c) holds, then $|\mathscr{N}| = |\mathscr{N}(P_0)|$, which equals $\operatorname{pos}\{a_i, i \in I_0\} + \operatorname{span}\{a_j, j \in J\}$ by part (e) of Proposition 1. This proves that (c) implies (d). The fact that (d) implies (a) is trivial. It remains to prove that (a) implies (b).

Suppose that (a) holds. It follows from the definition of $\mathscr{N}$ and the one-to-one correspondence between $\mathscr{N}(P)$ and $\mathfrak{I}(b, I, J)$ that

$$\mathfrak{I}_0 = \big\{I' \in \mathfrak{I}(b, I, J) \mid \operatorname{pos}\{a_i, i \in I'\} + \operatorname{span}\{a_j, j \in J\} \in \mathscr{N}\big\}.$$

If we define $I_1 = \cup_{I' \in \mathfrak{I}_0} I'$, then we can apply Theorem 1 with $\mathscr{N}$, $\mathfrak{I}_0$ and I_1 corresponding to $\mathscr{N}$, $\mathfrak{I}_0$ and I_0 in that theorem, to find that

$$\mathfrak{I}(b, I_1, J) = \mathfrak{I}_0. \tag{37}$$

The definition of I_1 ensures that $I_0 \supset I_1$. If $I_0 = I_1$, then (b) trivially follows from (37). Suppose for now that I_1 is a proper subset of I_0. Let i_0 be an element in $I_0 \setminus I_1$; by hypothesis, we have $a_{i_0} \in |\mathscr{N}|$. The definition of $\mathscr{N}$ then implies the existence of $I' \in \mathfrak{I}_0$ such that

$$a_{i_0} \in \operatorname{pos}\{a_i, i \in I'\} + \operatorname{span}\{a_j, j \in J\},$$

so we can write a_{i_0} as

$$a_{i_0} = \sum_{i \in I'} \lambda_i a_i + \sum_{j \in J} \lambda_j a_j, \tag{38}$$

with $\lambda_i \geq 0$ for $i \in I'$ and $\lambda_j \in \mathbb{R}$ for $j \in J$. The fact that $I' \in \mathfrak{I}_0$ requires that $I' \in \mathfrak{I}(b,I,J)$, which implies the existence of a point $\bar{x} \in \mathbb{R}^n$ satisfying (3). The facts that $i_0 \in I_0 \setminus I_1$ and $I' \subset I_1$ imply that $i_0 \in I_0 \setminus I'$. We then have

$$b_{i_0} > \langle a_{i_0}, \bar{x} \rangle = \langle \sum_{i \in I'} \lambda_i a_i + \sum_{j \in J} \lambda_j a_j, \bar{x} \rangle = \sum_{i \in I'} \lambda_i b_i + \sum_{j \in J} \lambda_j b_j, \tag{39}$$

where the first inequality and the third equation follow from (3), and the second equation is from (38). Then, for each $x \in P(b, I_1, J)$, we have

$$\langle a_{i_0}, x \rangle = \langle \sum_{i \in I'} \lambda_i a_i + \sum_{j \in J} \lambda_j a_j, x \rangle \leq \sum_{i \in I'} \lambda_i b_i + \sum_{j \in J} \lambda_j b_j < b_{i_0},$$

where the second inequality holds because $x \in P(b, I_1, J)$ and $I' \subset I_1$, and the third inequality is from (39). This means that the set $P(b, I_1, J)$ lies in the open halfspace defined by the constraint $\langle a_{i_0}, \cdot \rangle < b_{i_0}$. Accordingly, when we add the constraint $\langle a_{i_0}, \cdot \rangle \leq b_{i_0}$ to those defining $P(b, I_1, J)$, both the set and the active index sets remain unchanged. This proves $P(b, I_1, J) = P_0$ and $\mathfrak{I}(b, I_1, J) = \mathfrak{I}(b, I_0, J)$, which in view of (37) imply that $\mathfrak{I}(b, I_0, J) = \mathfrak{I}_0$. This proves that (a) implies (b). $\square$

The set P_0 in Proposition 3 and its faces generalize objects that are useful in sensitivity analysis of optimization problems and variaitonal inequalities. For a special case of Proposition 3, let I_0 belong to $\mathfrak{I}(b, I, J)$. Then the set P_0 is the translation of a tangent cone to P, and a certain face of P_0 is the translation of a critical cone to P. Using results here, Theorem 2 of the next section will establish a relation between the critical cone and P; it says that the normal manifold of the critical cone is the localization of that of P at the point defining the critical cone. For another special case, let I_0 continue to belong to $\mathfrak{I}(b, I, J)$, and perturb the vector b slightly to get b'. Corollary 4 will show that sets $P(b', I, J)$ and $P(b', I_0, J)$ behave as P and P_0 do in Proposition 3. In [6] we used the normal manifold of a certain face of $P(b', I_0, J)$ to provide a localization of the normal manifold of $P(b', I, J)$. Such localization is useful for characterizing good behavior of variational inequalities under right hand side perturbations.

Theorem 1 and Proposition 3 may also be useful in some vertex enumeration or column generalization algorithms. For an immediate application, Proposition 3 can provide information about vertices of P_0 provided that one has enumerated vertices of the set P. Clearly, each vertex of P the active index set of which is a subset of I_0 is a vertex of P_0. If statement (a) in Proposition 3 holds, then it follows from statement (b) of this proposition that each vertex of P_0 occurs in this way, so that one can easily enumerate vertices of P_0. When statement (a) in Proposition 3 fails, then the situation will be complicated and is beyond the scope of this paper.

The following technical lemma considers a nonempty face F of the set P defined by some $I_c \in \mathfrak{I}(b, I, J)$, and provides a relationship between the collection of active index sets of F and that of P. It is useful in Section 4.

Lemma 2 *Let $P = P(b, I, J)$ be a nonempty polyhedral convex set. Let I_c belong to $\mathfrak{I}(b, I, J)$ and let $F = P(b, I \setminus I_c, J \cup I_c)$ be the face of P defined by I_c. Then*

$$\{I_c \cup I'' : I'' \in \mathfrak{I}(b, I \setminus I_c, J \cup I_c)\} = \{I' \in \mathfrak{I}(b, I, J) : I' \supset I_c\}. \tag{40}$$

In Example 1, take $I_c = \{1\}$; then the face F of P defined by I_c is $\{0\} \times [0,1]$, with

$$\mathfrak{I}(b, I \setminus I_c, J \cup I_c) = \{\{2,3\}, \emptyset, \{4\}\},$$

which satisfies (40).

Proof First, let I'' belong to $\mathfrak{I}(b, I\setminus I_c, J\cup I_c)$. Then there exists $\bar{x}\in\mathbb{R}^n$ satisfying

$$\langle a_i,\bar{x}\rangle < b_i, i\in(I\setminus I_c)\setminus I'', \quad \langle a_j,\bar{x}\rangle = b_j, j\in(J\cup I_c)\cup I'',$$

so that $I_c\cup I''$ belongs to $\mathfrak{I}(b,I,J)$. On the other hand, let I' belong to $\mathfrak{I}(b,I,J)$ with $I'\supset I_c$. Then there exists $\bar{x}\in\mathbb{R}^n$ satisfying

$$\langle a_i,\bar{x}\rangle < b_i, i\in I\setminus I', \quad \langle a_j,\bar{x}\rangle = b_j, j\in J\cup I',$$

so that $I'\setminus I_c$ belongs to $\mathfrak{I}(b, I\setminus I_c, J\cup I_c)$. □

4 Special cases

This section contains several examples in which we apply techniques in previous sections to obtain results about normal fans or normal manifolds of various sets. Section 4.1 relates the familiar tangent cone of a polyhedral convex set to a special case of the setting in Proposition 3. Following that, Section 4.2 shows that the normal manifold of the critical cone associated with a polyhedral convex set P and a point z_0 in $\mathbb{R}^n$ is the localization of the normal manifold of P at z_0. Next, Section 4.3 considers perturbations of the right hand sides of the constraints defining a polyhedral convex set P. It shows that the normal fan of a slightly perturbed set P' refines the original normal fan in the way defined in Proposition 2, and then studies how an active index set I_0 of P behaves for P'. Clearly, I_0 is not necessarily an active index set for P', so the polyhedral convex set defined by those constraints defining P' that correspond to indices in I_0 is not necessarily a translation of any tangent cone of P'. However, the conditions in Proposition 3 still hold, so the normal fan of that set is still a subfan of that of P'.

For notational purposes, we define the cone generated by a subset S of $\mathbb{R}^n$ to be the set

$$\operatorname{cone} S = \{x\in\mathbb{R}^n \mid x=\lambda s,\ \ s\in S,\ \ \lambda>0\}.$$

By [10, Exercise 1.27], if each of two convex sets S_1 and S_2 contains the origin, then

$$\operatorname{cone}(S_1+S_2) = \operatorname{cone} S_1 + \operatorname{cone} S_2. \tag{41}$$

4.1 Tangent cones

The tangent cone to a convex subset C of $\mathbb{R}^n$ at a point $x_0\in C$ is the set

$$T_C(x_0) = \operatorname{cl}\operatorname{cone}(C-x_0),$$

see [12, Theorem 6.9] or [10, Corollary 2.12] for details. For a nonempty polyhedral convex set $P=P(b,I,J)$, the tangent cone to P at a point $x_0\in P$ is given by

$$T_P(x_0) = \operatorname{cone}(P-x_0), \tag{42}$$

where we can drop the closure operation because $\operatorname{cone}(P-x_0)$ is a polyhedral convex cone and is therefore closed. Moreover, if we denote the active index set at x_0 by

$$I_0 = \{i\in I : \langle a_i,x_0\rangle = b_i\}, \tag{43}$$

then we have

$$T_P(x_0) = P(0,I_0,J). \tag{44}$$

See, e.g., [12, Theorem 6.46] for a proof of this.

The following lemma relates $T_P(x_0)$ to a special case of Proposition 3.

Lemma 3 *Let $P = P(b,I,J)$ be a nonempty polyhedral convex set, x_0 be a point in P with its active index set I_0 defined in* (43)*, and define P_0, $\mathfrak{I}_0$ and $\mathscr{N}$ as in Proposition 3. Then $P_0 = x_0 + T_P(x_0)$, I_0 belongs to $\mathfrak{I}_0$ and satisfies $I_0 = \cup_{I' \in \mathfrak{I}_0} I'$, $\mathfrak{I}(0,I_0,J) = \mathfrak{I}(b,I_0,J) = \mathfrak{I}_0$, $\mathscr{N}(T_P(x_0)) = \mathscr{N}(P_0) = \mathscr{N}$, and $|\mathscr{N}| = \operatorname{pos}\{a_i, i \in I_0\} + \operatorname{span}\{a_j, j \in J\}$.*

Proof First, it follows from the facts $P_0 = P(b,I_0,J)$, $T_P(x_0) = P(0,I_0,J)$, and $\langle a_i, x_0 \rangle = b_i$ for each $i \in I_0 \cup J$ that $P_0 = x_0 + T_P(x_0)$, $\mathfrak{I}(b,I_0,J) = \mathfrak{I}(0,I_0,J)$, and $\mathscr{N}(P_0) = \mathscr{N}(T_P(x_0))$.

Next, the definition of I_0 implies that it is an element of $\mathfrak{I}(b,I,J)$, so it belongs to $\mathfrak{I}_0$. Each other element of $\mathfrak{I}_0$ is a subset of I_0, so $I_0 = \cup_{I' \in \mathfrak{I}_0} I'$.

Finally, the definition of $\mathscr{N}$ implies that each cone in $\mathscr{N}$ is a subset of $\operatorname{pos}\{a_i, i \in I_0\} + \operatorname{span}\{a_j, j \in J\}$. The latter cone is itself an element of $\mathscr{N}$ since I_0 belongs to $\mathfrak{I}_0$. This proves $|\mathscr{N}| = \operatorname{pos}\{a_i, i \in I_0\} + \operatorname{span}\{a_j, j \in J\}$. The remaining conclusions of the present lemma then follow from those of Proposition 3. □

4.2 Critical cones

Let $P = P(b,I,J)$ be a nonempty polyhedral convex set, z_0 be a point in $\mathbb{R}^n$, x_0 be its Euclidean projection on P, and I_0 be the active index set for x_0 defined by (43). The critical cone to P associated with z_0 is defined to be

$$K = T_P(x_0) \cap (z_0 - x_0)^{\perp}.$$

For notational simplicity, write $T = T_P(x_0) = P(0,I_0,J)$, whose collection of active index sets is given by

$$\mathfrak{I}(0,I_0,J) = \{I' \in \mathfrak{I}(b,I,J) \mid I' \subset I_0\}, \tag{45}$$

according to Lemma 3.

In the following lemma, we express K in the form of $P(0, I_0 \setminus I_c, J \cup I_c)$ for some subset I_c of I_0, and then find a relation between the collection of active index sets for K and that of P.

Lemma 4 *Let P, z_0, x_0, I_0, K and T be as above. There exists a unique element I_c of $\mathfrak{I}(0,I_0,J)$ such that $K = P(0, I_0 \setminus I_c, J \cup I_c)$, with*

$$z_0 - x_0 \in \operatorname{ri} N_T(K) = \operatorname{ri}[\operatorname{pos}\{a_i, i \in I_c\} + \operatorname{span}\{a_j, j \in J\}]. \tag{46}$$

Moreover,

$$\begin{aligned} &\{I_c \cup I'' : I'' \in \mathfrak{I}(0, I_0 \setminus I_c, J \cup I_c)\} \\ =&\{I' \in \mathfrak{I}(b,I,J) : I_0 \supset I' \supset I_c\}. \end{aligned} \tag{47}$$

In Example 1, we may take $z_0 = (-1,0)$; then x_0 is the origin of $\mathbb{R}^2$, with $I_0 = \{1,2,3\}$ and $T = \mathbb{R}^2_+$. The critical cone K is $\{0\} \times \mathbb{R}_+$, and $\{1\}$ is the unique choice for I_c such that $K = P(0, I_0 \setminus I_c, J \cup I_c)$. The collection of active index sets for K is

$$\mathfrak{I}(0, I_0 \setminus I_c, J \cup I_c) = \{\{2,3\}, \emptyset\}.$$

Proof The definition of x_0 implies that the vector $z_0 - x_0$ belongs to $N_P(x_0)$, which is the polar cone of T. Hence, we can rewrite the cone K as

$$K = \operatorname*{argmax}_{T} \langle z_0 - x_0, \cdot \rangle, \tag{48}$$

which is a nonempty face of T. But T is just $P(0, I_0, J)$, and its active index sets are given by (45). By the one-to-one correspondence between active index sets and nonempty faces, there exists a unique element I_c of $\mathfrak{I}(0, I_0, J)$ such that $K = P(0, I_0 \setminus I_c, J \cup I_c)$.

On the other hand, by the statement in (7) we may further rewrite K in (48) as $K = (N_T)^{-1}(z_0 - x_0)$. It then follows from part (c) of Corollary 1 that (46) holds.

Since K is the face of T defined by the set $I_c \in \mathfrak{I}(0, I_0, J)$, an application of Lemma 2 gives

$$\{I_c \cup I'' : I'' \in \mathfrak{I}(0, I_0 \setminus I_c, J \cup I_c)\} = \{I' \in \mathfrak{I}(0, I_0, J) : I' \supset I_c\}.$$

This and (45) together imply (47). □

The following theorem provides a relation between the normal manifold of K and that of P. By definition, the normal manifold of P is the collection $\{C_P(F) : F \in \mathscr{F}(P) \setminus \emptyset\}$, where $C_P(F)$ is as given in (9), so the expression in (49) is just the *localization* of the normal manifold of P at z_0, in the sense of [14, Section 2.2.2].

Theorem 2 *The normal manifold of K is equal to*

$$\{\operatorname{cone}(C_P(F) - z_0) : F \in \mathscr{F}(P) \setminus \emptyset,\ z_0 \in C_P(F)\}. \tag{49}$$

We continue illustrating this in the context of Example 1 for $z_0 = (-1, 0)$. There are two 2-cells in the normal manifold of P that contain z_0. One of them is the 2-cell defined by the active index set $\{1\}$, and it is equal to $\mathbb{R}_- \times \mathbb{R}_+$. The other is the 2-cell defined by $\{1, 2, 3\}$, and it is equal to $\mathbb{R}^2_-$. On the other hand, as $K = \{0\} \times \mathbb{R}_+$, its normal manifold consists of the two halfspaces divided by the x_1 axis, which is exactly the localization of the normal manifold of P at z_0.

Proof First, we prove

$$\begin{aligned} &\{I' \in \mathfrak{I}(b, I, J) : I_0 \supset I' \supset I_c\} \\ =&\{I' \in \mathfrak{I}(b, I, J) : z_0 \in C_P[P(b, I \setminus I', J \cup I')]\}. \end{aligned} \tag{50}$$

For the $\subset$ direction of (50), let I' belong to $\mathfrak{I}(b, I, J)$ with $I_0 \supset I' \supset I_c$. Then it follows from (43) and (13) that

$$x_0 \in P(b, I \setminus I_0, J \cup I_0) \subset P(b, I \setminus I', J \cup I'). \tag{51}$$

On the other hand, since I_c is an element of $\mathfrak{I}(0, I_0, J)$, by (45) it is an element of $\mathfrak{I}(b, I, J)$ as well. It then follows from (46) and (13) that

$$z_0 - x_0 \in N_P[P(b, I \setminus I_c, J \cup I_c)] \subset N_P[P(b, I \setminus I', J \cup I')]. \tag{52}$$

Adding (51) and (52) together gives that $z_0 \in C_P[P(b, I \setminus I', J \cup I')]$. This proves the $\subset$ direction of (50).

For the other direction of (50), let I' belong to $\mathfrak{I}(b, I, J)$ with $z_0 \in C_P[P(b, I \setminus I', J \cup I')]$. Then there exist $x \in P(b, I \setminus I', J \cup I')$ and $y \in N_P[P(b, I \setminus I', J \cup I')]$ such that $z_0 = x + y$. As y belongs to $N_P(x)$ by (8), x is just the Euclidean projection of z_0 on P. It follows that

$$(x_0 = x) \in P(b, I \setminus I', J \cup I') \tag{53}$$

and

$$(z_0 - x_0 = y) \in N_P[P(b, I \setminus I', J \cup I')]. \tag{54}$$

It follows from (53) and the definition of I_0 that $I' \subset I_0$. By (54) and (46) we have

$$N_P[P(b, I \setminus I', J \cup I')] \cap \operatorname{ri} N_P[P(b, I \setminus I_c, J \cup I_c)] \neq \emptyset,$$

which implies that $N_P[P(b, I \setminus I', J \cup I')] \supset N_P[P(b, I \setminus I_c, J \cup I_c)]$ by part (a) of Corollary 1. This in turn implies that $I' \supset I_c$ by (13). So each I' in the right hand side of (50) satisfies $I_0 \supset I' \supset I_c$. This completes the proof of (50).

Next, note that (50) and (47) together imply

$$\begin{aligned} &\{I_c \cup I'' : I'' \in \mathfrak{I}(0, I_0 \setminus I_c, J \cup I_c)\} \\ &= \{I' \in \mathfrak{I}(b, I, J) : z_0 \in C_P[P(b, I \setminus I', J \cup I')]\}. \end{aligned} \tag{55}$$

Hence, to prove the present theorem, it suffices to prove

$$C_K[P(0, (I_0 \setminus I_c) \setminus I'', J \cup I_c \cup I'')] = \operatorname{cone}\{C_P[P(b, I \setminus (I_c \cup I''), J \cup I_c \cup I'')] - z_0\} \tag{56}$$

for each $I'' \in \mathfrak{I}(0, I_0 \setminus I_c, J \cup I_c)$.

For this purpose, let I'' belong to $\mathfrak{I}(0, I_0 \setminus I_c, J \cup I_c)$, so that by (47) the set $I_c \cup I''$ belongs to $\mathfrak{I}(b, I, J)$ with $I_0 \supset (I_c \cup I'') \supset I_c$. By (51) and (52) we have

$$x_0 \in P(b, I \setminus (I_c \cup I''), J \cup I_c \cup I''))$$

and

$$z_0 - x_0 \in N_P[P(b, I \setminus (I_c \cup I''), J \cup I_c \cup I'')].$$

An application of (41) then shows

$$\begin{aligned} &\operatorname{cone}\{C_P[P(b, I \setminus (I_c \cup I''), J \cup I_c \cup I'')] - z_0\} \\ &\quad = \operatorname{cone}\{P(b, I \setminus (I_c \cup I''), J \cup I_c \cup I'') - x_0\} \\ &\quad + \operatorname{cone}\{N_P[P(b, I \setminus (I_c \cup I''), J \cup I_c \cup I'')] - (z_0 - x_0)\}. \end{aligned} \tag{57}$$

Note that $\operatorname{cone}[P(b, I \setminus (I_c \cup I''), J \cup I_c \cup I'') - x_0]$ is just the tangent cone to the set $P(b, I \setminus (I_c \cup I''), J \cup I_c \cup I''))$ at x_0, while in the latter set the active index set for x_0 is just $I_0 \setminus (I_c \cup I'')$. By the formula in (44), we have

$$\operatorname{cone}[P(b, I \setminus (I_c \cup I''), J \cup I_c \cup I'') - x_0] = P(0, I_0 \setminus (I_c \cup I''), J \cup I_c \cup I''). \tag{58}$$

On the other hand, we have

$$N_P[P(b, I \setminus (I_c \cup I''), J \cup I_c \cup I'')] = \operatorname{pos}\{a_i, i \in I_c \cup I''\} + \operatorname{span}\{a_j, j \in J\} - (z_0 - x_0),$$

and it follows from [11, Corollary 6.6.2] that

$$\operatorname{ri}[\operatorname{pos}\{a_i, i \in I_c\} + \operatorname{span}\{a_j, j \in J\}] = \{\sum_{i \in I_c} \lambda_i a_i \mid \lambda_i > 0\} + \operatorname{span}\{a_j, j \in J\},$$

so by (46) we can write $z_0 - x_0$ as

$$z_0 - x_0 = \sum_{i \in I_c} \lambda_i a_i + \sum_{j \in J} \lambda_j a_j$$

with $\lambda_i > 0$ for each $i \in I_c$ and $\lambda_j \in \mathbb{R}$ for each $j \in J$. Consequently,

$$\begin{aligned}&\mathrm{pos}\{a_i, i \in I_c \cup I''\} + \mathrm{span}\{a_j, j \in J\} - (z_0 - x_0)\\ &= \mathrm{pos}\{a_i, i \in I''\} + \sum_{i \in I_c} (\mathrm{pos}\{a_i\} - \lambda_i a_i) + \sum_{j \in J} (\mathrm{span}\{a_j\} - \lambda_j a_j),\end{aligned}$$

where $\mathrm{pos}\{a_i, i \in I''\}$ is a polyhedral convex cone, the set $\mathrm{pos}\{a_i\} - \lambda_i a_i$ for each $i \in I_c$ is a convex set containing the origin with $\mathrm{cone}(\mathrm{pos}\{a_i\} - \lambda_i a_i) = \mathrm{span}\{a_i\}$, and the set $\mathrm{span}\{a_j\} - \lambda_j a_j$ for each $j \in J$ is just $\mathrm{span}\{a_j\}$. An application of (41) implies

$$\begin{aligned}&\mathrm{cone}[\mathrm{pos}\{a_i, i \in I_c \cup I''\} + \mathrm{span}\{a_j, j \in J\} - (z_0 - x_0)]\\ &= \mathrm{pos}\{a_i, i \in I''\} + \sum_{i \in I_c} \mathrm{span}\{a_i\} + \sum_{j \in J} \mathrm{span}\{a_j\}\\ &= \mathrm{pos}\{a_i, i \in I''\} + \mathrm{span}\{a_j, j \in J \cup I_c\}\end{aligned} \tag{59}$$

which is just $N_K[P(0, (I_0 \setminus I_c) \setminus I''), J \cup I_c \cup I'')]$.

The equation (56) follows from (57), (58) and (59). □

4.3 Set perturbation in the right hand side

Let $P = P(b, I, J)$ be a nonempty polyhedral convex set. This section applies results in Sections 2 and 3 to study some properties of sets $P(b', I, J)$ for vectors b' close to b. Extensive study exists on the stability of linear inequality systems with respect to perturbations in the right hand sides and/or the linear operators defining such systems; see, e.g., [3] for a literature review. The interest here lies in how changes of the right hand side affect the normal fan and face structure of the set. The proposition below shows that the normal fans of the perturbed sets refine that of P in the way defined in Proposition 2, and that the map Ψ defined in Proposition 2 satisfies a special formula.

Proposition 4 *There exists a neighborhood U of b in $\mathbb{R}^m$ such that the following hold for each $b' \in U$ with $P(b', I, J) \neq \emptyset$:*

(a) For each $I' \in \mathfrak{I}(b', I, J)$, there exists an element $\Phi(I')$ of $\mathfrak{I}(b, I, J)$ which is the smallest element of $\mathfrak{I}(b, I, J)$ containing I'.

(b) The normal fan of $P(b', I, J)$ refines that of P in the sense of Proposition 2. Let Ψ be the mapping from the normal fan of $P(b', I, J)$ to that of P defined in Proposition 2; then for each $I' \in \mathfrak{I}(b', I, J)$,

$$\Psi[\mathrm{pos}\{a_i, i \in I'\} + \mathrm{span}\{a_j, j \in J\}] = \mathrm{pos}\{a_i, i \in \Phi(I')\} + \mathrm{span}\{a_j, j \in J\}. \tag{60}$$

On p. 10 we already noticed that in Example 1 the set P' refines the set P in the sense of Proposition 2. In fact, they satisfy properties (a) and (b) of Proposition 4. For instance, $\{1, 2, 3\}$ is the smallest element of $\mathfrak{I}(b, I, J)$ containing $\{3\}$, so $\Phi(\{3\}) = \{1, 2, 3\}$. The cone generated by the vector $(-1, -1)$ is the element of $\mathscr{N}(P')$ corresponding to $\{3\}$, so by (60) its image under Ψ is $\mathbb{R}^2_-$.

Proof It follows from [6, Proposition 2.1 (i)] that there exists a neighborhood U of b in $\mathbb{R}^m$ such that each $b' \in U$ with $P(b', I, J) \neq \emptyset$ satisfies part (a) here.

Next we show that each $b' \in U$ with $P(b',I,J) \neq \emptyset$ also satisfies part (b). Write $P' = P(b',I,J)$ for brevity. As P and P' are both nonempty, it is easy to check that

$$\operatorname{rc} P = \operatorname{rc} P' = P(0,I,J).$$

Now, let N' be a cone in $\mathscr{N}(P')$; then there exists $I' \in \mathfrak{I}(b',I,J)$ such that

$$N' = \operatorname{pos}\{a_i, i \in I'\} + \operatorname{span}\{a_j, j \in J\}.$$

Write

$$N = \operatorname{pos}\{a_i, i \in \Phi(I')\} + \operatorname{span}\{a_j, j \in J\},$$

and observe that $N \in \mathscr{N}(P)$ as $\Phi(I') \in \mathfrak{I}(b,I,J)$. The definition of $\Phi(I')$ ensures that $I' \subset \Phi(I')$, so $N' \subset N$. This shows that each cone N' in $\mathscr{N}(P')$ is a subset of a cone N in $\mathscr{N}(P)$, so that P' refines P in the sense of Proposition 2, and we may define a map Ψ from $\mathscr{N}(P')$ to $\mathscr{N}(P)$ as in that proposition.

It remains to prove (60), which just says that $\Psi(N')$ is the element of $\mathscr{N}(P)$ that corresponds to $\Phi(I')$. According to Proposition 2, the cone $\Psi(N')$ is the smallest element of $\mathscr{N}(P)$ containing N'; by the definition of Φ, $\Phi(I')$ is the smallest element of $\mathfrak{I}(b,I,J)$ containing I'. Hence, to prove (60) it suffices to prove that the elements of $\mathfrak{I}(b,I,J)$ containing I' correspond exactly to elements of $\mathscr{N}(P)$ containing N' in the one-to-one correspondence between $\mathfrak{I}(b,I,J)$ and $\mathscr{N}(P)$.

Clearly, if I_1 is an element of $\mathfrak{I}(b,I,J)$ containing I', then the cone in $\mathscr{N}(P)$ that corresponds to I_1 must contain N'. This proves one direction of the statement above.

Now, for each element i of I', we have $i \in I' \subset \Phi(I')$, so $i \in \bigcup_{I'' \in \mathfrak{I}(b,I,J)} I''$. Hence, if N_1 is a cone in $\mathscr{N}(P)$ that contains N', then by part (d) of Proposition 1, the element of $\mathfrak{I}(b,I,J)$ that corresponds to N_1 must contain I'. This proves the other direction of the statement above, and thereby completes the proof of the present proposition. $\square$

The following corollary concerns how active index sets of $P(b,I,J)$ behave in $P(b',I,J)$. Specifically, if I_0 belongs to $\mathfrak{I}(b,I,J)$ and b' belongs to U with $P(b',I,J) \neq \emptyset$, then we will show that the hypotheses in Proposition 3 hold for I_0 and $P(b',I,J)$, and the normal fan of $P(b',I_0,J)$ is a subfan of that of $P(b',I,J)$.

Corollary 4 *Let U be the neighborhood of b as in Proposition 4, and let b' belong to U with $P' = P(b',I,J)$ nonempty. If I_0 is an element of $\mathfrak{I}(b,I,J)$, then the normal fan of $P(b',I_0,J)$ is a subfan of $\mathscr{N}(P')$ with its underlying set given by* $\operatorname{pos}\{a_i, i \in I_0\} + \operatorname{span}\{a_j, j \in J\}$*, and*

$$\mathfrak{I}(b',I_0,J) = \{I' \in \mathfrak{I}(b',I,J) \mid I' \subset I_0\}. \tag{61}$$

In Example 1, the set $I_0 = \{1,2,3\}$ is an element of $\mathfrak{I}(b,I,J)$. As already noted on p. 13, the normal fan of $P(b',I_0,J)$ is given by the set $\mathscr{N}$ in (36), which is a subfan of $\mathscr{N}(P')$.

Proof As b' belongs to U with $P' = P(b',I,J)$ nonempty, it satisfies properties (a) and (b) of Proposition 4. Let the mappings $\Phi : \mathfrak{I}(b',I,J) \to \mathfrak{I}(b,I,J)$ and $\Psi : \mathscr{N}(P') \to \mathscr{N}(P)$ be as defined in Proposition 4. In addition, define

$$\mathfrak{I}_0 = \{I' \in \mathfrak{I}(b',I,J) \mid I' \subset I_0\},$$

$$\mathscr{N} = \left\{\operatorname{pos}\{a_i, i \in I'\} + \operatorname{span}\{a_j, j \in J\} : I' \in \mathfrak{I}_0\right\},$$

and

$$N_0 = \operatorname{pos}\{a_i, i \in I_0\} + \operatorname{span}\{a_j, j \in J\}. \tag{62}$$

Note that each cone in $\mathscr{N}$ is a subset of N_0 by definitions of $\mathscr{N}$ and N_0, so we have $|\mathscr{N}| \subset N_0$. On the other hand, N_0 belongs to $\mathscr{N}(P)$ because I_0 belongs to $\mathfrak{I}(b,I,J)$, so it follows from (25) that

$$N_0 = \bigcup_{N' \in \Psi^{-1}(N_0)} N'. \tag{63}$$

Let N' be an element of $\Psi^{-1}(N_0)$. As N' belongs to $\mathscr{N}(P')$, there exists I' in $\mathfrak{I}(b',I,J)$ such that

$$N' = \operatorname{pos}\{a_i, i \in I'\} + \operatorname{span}\{a_j, j \in J\}.$$

By (60) we have

$$N_0 = \Psi(N') = \operatorname{pos}\{a_i, i \in \Phi(I')\} + \operatorname{span}\{a_j, j \in J\}.$$

Comparing this with (62) and applying (14), we obtain $\Phi(I') = I_0$. In particular, I' is a subset of I_0, so it belongs to $\mathfrak{I}_0$; consequently, N' belongs to $\mathscr{N}$. This shows that each element of $\Psi^{-1}(N_0)$ belongs to $\mathscr{N}$, and it then follows from (63) that $|\mathscr{N}| \supset N_0$.

So far we have shown that $|\mathscr{N}| = N_0$, which implies immediately that $|\mathscr{N}|$ is convex. Moreover, the definition of N_0 in (62) ensures that for each $i \in I_0$ the vector a_i belongs to N_0 , so a_i belongs to $|\mathscr{N}|$. We can then apply Proposition 3 to conclude that $\mathfrak{I}(b',I_0,J) = \mathfrak{I}_0$ and that $\mathscr{N}[P(b',I_0,J)] = \mathscr{N}$ is a subfan of $\mathscr{N}(P')$ with its underlying set given by $\operatorname{pos}\{a_i, i \in I_0\} + \operatorname{span}\{a_j, j \in J\}$. □

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