TRANSPORTATION SCIENCE
Vol. 42, No. 1, February 2008, pp. 105–123
ISSN 0041-1655 | EISSN 1526-5447 | 08 | 4201 | 0105

DOI 10.1287/trsc.1070.0191

Sensitivity of Static Traffic User Equilibria with Perturbations in Arc Cost Function and Travel Demand

Shu Lu
Department of Statistics and Operations Research, University of North Carolina at Chapel Hill,
Chapel Hill, North Carolina 27599, shulu@email.unc.edu

This paper deals with sensitivity analysis of static traffic user equilibrium problems. We apply some recently developed sensitivity analysis techniques for generalized equations to analyze the behavior of the equilibrium arc flow of such a problem when both the arc cost function and the travel demand vary. Our methods permit calculation of semiderivatives under general conditions and of derivatives under more restrictive conditions; we calculate the semiderivatives by solving a linear traffic user equilibrium problem and the derivatives by matrix multiplication together with the solution of a linear equation the dimension of which is at most the number of arcs. Three numerical examples show how to use these results in practice.

Key words: traffic user equilibrium; sensitivity analysis; perturbations
History: Received: June 2006; revision received: April 2007; accepted: May 2007.

Introduction

In this paper we apply some recently developed sensitivity analysis techniques for generalized equations to the traffic user equilibrium problem. This application requires two steps. In the first step, we establish a result regarding perturbations of polyhedral convex sets. It says that under a kind of constraint nondegeneracy condition, a slightly perturbed set coincides locally with an affine translation of the unperturbed set. We then provide a sensitivity theorem for variational inequalities over perturbed sets with such a local translation property. In the second step, we apply the results obtained in the first step to the traffic user equilibrium problem. We show that the constraint nondegeneracy condition holds when the base travel demand is strictly positive. This leads to a key result in this paper—that if the base travel demand is strictly positive, then the set of feasible arc flows under a slightly perturbed travel demand locally coincides with an affine translation of the original feasible set. We then apply the sensitivity theorem for variational inequalities to study the Lipschitz continuity, semidifferentiability, and differentiability of the equilibrium arc flow.

The strict positivity assumption on the base travel demand is not as strong as it sounds and is in fact a natural requirement for the problem we study. It is an implicit assumption in other similar analyses. Stating this assumption explicitly enables us to introduce a special partition of the route set and then rewrite the critical cone in a more concise expression than has heretofore been available. This expression is convenient for checking the assumptions and computing the derivatives when they exist. We compute the derivatives simply by matrix multiplication with the solution of a linear system the dimension of which is at most the number of arcs and which may in practice be much less than the number of arcs, thanks to the concise expression of the critical cone.

This paper consists of five sections. The introductory section describes the notation we will use, reviews prior work on this problem, and explains the contribution of the present paper. Section 1 discusses the underlying theory for generalized equations and shows how to apply it to the particular case of variational inequalities. Section 2 uses the techniques from §1 to analyze sensitivity of traffic user equilibria. Section 3 provides three numerical examples illustrating the implementation of this method. Section 4 concludes the paper with a résumé and brief discussion of what we have done.

We use $\mathbb{R}^n$, $\mathbb{R}^n_+$, and $\mathbb{R}^n_{++}$ to denote the real n-dimensional space, its nonnegative orthant, and its strictly positive orthant, respectively. A polyhedral convex set in $\mathbb{R}^n$ refers to a set defined by finitely many linear inequalities of the form $\langle a, x\rangle \le b$, where $a \in \mathbb{R}^n$, $b \in \mathbb{R}$, and $\langle \cdot, \cdot \rangle$ denote the standard inner product. We use $\operatorname{cl} C$ and $\operatorname{int} C$ to denote the closure and the interior of a set C in $\mathbb{R}^n$, respectively. For a convex set C in $\mathbb{R}^n$, we use the symbols $T_C(z)$ and $N_C(z)$ to denote the tangent and normal cones to C at a point $z \in C$, defined to be

$$T_C(z) = \operatorname{cl}\{x \in \mathbb{R}^n \mid \text{for some } \lambda > 0,\ z + \lambda x \in C\},$$

and

$$N_C(z) = \{x \in \mathbb{R}^n \mid \langle x, z' - z\rangle \le 0 \text{ for all } z' \in C\},$$

where for $z \notin C$ we stipulate that $T_C(z) = \varnothing$ and $N_C(z) = \varnothing$.

A function f from a subset X of $\mathbb{R}^n$ to $\mathbb{R}^m$ is *Lipschitz continuous with modulus* λ if $\|f(x) - f(x')\| \le \lambda\|x - x'\|$ for any $x, x' \in X$; here $\|\cdot\|$ denotes the Euclidean norm. It is *semidifferentiable* at a point $x^0 \in \operatorname{int} X$ if there is a continuous, positively homogeneous function $df(x^0)$: $\mathbb{R}^n \to \mathbb{R}^m$ having the property that $f(x^0 + h) = f(x^0) + df(x^0)(h) + o(h)$; we call $df(x^0)(h)$ the *semiderivative* of f at x^0 for the direction h. If $df(x^0)$ happens to be linear, then f is differentiable. Note that semidifferentiability implies directional differentiability: If f is semidifferentiable at x^0, then the semiderivative $df(x^0)(h)$ is simply the directional derivative of f at x^0 for the direction h. In general, directional differentiability is a weaker property than semidifferentiability; however, for Lipschitz continuous functions, these two concepts are equivalent (see Shapiro 1990). Finally, a function f from a subset X of $\mathbb{R}^n$ to $\mathbb{R}^n$ is *strictly monotone* on X if $\langle f(x) - f(x'), x - x'\rangle > 0$ for any two distinct points $x, x' \in X$, and is *strongly monotone* on X if there exists a strictly positive scalar λ such that $\langle f(x) - f(x'), x - x'\rangle \ge \lambda\|x - x'\|^2$ for any two points $x, x' \in X$.

The traffic user equilibrium problem that we consider starts with a network consisting of a set of nodes $\mathcal{N}$, a set of arcs $\mathcal{A}$, and a set of origin-destination (OD) pairs $\mathcal{W} \subset \mathcal{N}^2$. Each OD pair $w \in \mathcal{W}$ is connected by a set of routes $\mathcal{P}_w$, each member of which is a set of sequentially connected arcs. Let $\mathcal{P} = \bigcup_{w\in\mathcal{W}} \mathcal{P}_w$ denote the set of all routes, and let $\alpha = |\mathcal{A}|$, $\omega = |\mathcal{W}|$, and $\pi = |\mathcal{P}|$ denote the cardinalities of $\mathcal{A}$, $\mathcal{W}$, and $\mathcal{P}$, respectively. Let the matrix $(\Lambda = [\Lambda_{wp}]) \in \mathbb{R}^{\omega\times\pi}$ denote the OD-route incidence matrix in which $\Lambda_{wp} = 1$ if route $p \in \mathcal{P}_w$ and $\Lambda_{wp} = 0$ otherwise, and the matrix $(\Delta = [\Delta_{ap}]) \in \mathbb{R}^{\alpha\times\pi}$ denote the arc-route incidence matrix; here $\Delta_{ap} = 1$ if arc a is in route p and $\Delta_{ap} = 0$ otherwise.

We will use column vectors $(d = [d_w]) \in \mathbb{R}^{\omega}$, $(q = [q_p]) \in \mathbb{R}^{\pi}$, and $(x = [x_a]) \in \mathbb{R}^{\alpha}$ to denote the travel demand (also called the OD flow), the route flow, and the arc flow, respectively. Given the travel demand d, a route flow q is *feasible* if it is nonnegative with $\Lambda q = d$, and an arc flow x is *feasible* if there exists a feasible route flow q with $\Delta q = x$. Let f: $\mathbb{R}^{\alpha} \to \mathbb{R}^{\alpha}$ denote the arc cost function; elements of the vector $f(x)$ give the cost on each arc $a \in \mathcal{A}$ under the arc flow x, while elements of $c(x) := \Delta^T f(x)$ give the cost on each route $p \in \mathcal{P}$. Finally, let elements of the vector $\psi(x) = [\psi_w(x)] \in \mathbb{R}^{\omega}$ denote the minimum cost between each OD pair $w \in \mathcal{W}$; that is, $\psi_w(x) := \min_{p\in\mathcal{P}_w} c_p(x)$.

Given the travel demand d and the arc cost function f, a feasible route flow q is an *equilibrium route flow* if it satisfies

$$0 \le c_p(\Delta q) - \psi_w(\Delta q) \perp q_p \ge 0 \quad \text{for each } w \in \mathcal{W} \text{ and } p \in \mathcal{P}_w. \tag{0.1}$$

We say that a route $p \in \mathcal{P}_w$ is *used* if $q_p > 0$, but it is *user optimal* if $c_p(\Delta q) = \psi_w(\Delta q)$. In these terms (0.1) just means that each used route is user optimal. It is possible that a user optimal route is not used.

A feasible arc flow x is an *equilibrium arc flow* if there exists an equilibrium route flow q with $\Delta q = x$. Smith (1979) showed that x is an equilibrium arc flow if and only if it solves the following problem, identified as a variational inequality (VI) problem in Dafermos (1980):

$$x \in S \quad \text{and} \quad \langle f(x), x' - x\rangle \ge 0 \quad \text{for each } x' \in S, \tag{0.2}$$

where S is the set of feasible arc flows,

$$S = \{x \in \mathbb{R}^{\alpha} \mid \text{for some } q \in \mathbb{R}^{\pi}_{+},\ \Lambda q = d,\ \Delta q = x\}. \tag{0.3}$$

The arguments in Smith (1979) also showed that if x is an equilibrium arc flow, then each $q \in \mathbb{R}^{\pi}_{+}$ with $\Lambda q = d$ and $\Delta q = x$ is an equilibrium route flow.

If we specify the travel demand d as part of the exogenously given data, then the problem we have described is the *fixed-demand traffic user equilibrium problem,* which we study in this paper. To study the behavior of the fixed-demand problem under perturbations, we treat the travel demand d as a parameter of this problem and let it take values in an open subset D of $\mathbb{R}^{\omega}$. In addition, we introduce another parameter $u \in \mathbb{R}^m$ into the arc cost function f, let it take values in an open subset U of $\mathbb{R}^m$, and associate an arc cost function $f(u, \cdot)$ with each $u \in U$.

For a given parameter $(u, d) \in U \times D$, the set of feasible route flows is given by

$$H(d) = \{q \in \mathbb{R}^{\pi}_{+}\colon \Lambda q = d\}, \tag{0.4}$$

the set of feasible arc flows is given by

$$\begin{aligned} S(d) &= \{x \in \mathbb{R}^{\alpha} \mid \text{for some } q \in \mathbb{R}^{\pi}_{+},\ \Lambda q = d,\ \Delta q = x\} \\ &= \Delta H(d), \end{aligned} \tag{0.5}$$

and the equilibrium arc flow is a solution of the problem

$$x \in S(d) \quad \text{and} \quad \langle f(u, x), x' - x\rangle \ge 0 \quad \text{for each } x' \in S(d), \tag{0.6}$$

which by the definition of normal cones is equivalent to

$$0 \in f(u, x) + N_{S(d)}(x). \tag{0.7}$$

The sensitivity analysis of this paper studies the solution properties of (0.7) near a base point (u^0, d^0, x^0), where u^0 is a point in U, d^0 is a point in D, and x^0 is a solution of (0.7) for the pair (u^0, d^0). We assume that there is an open neighborhood X of x^0 in $\mathbb{R}^{\alpha}$ such that the parametric arc cost function f is continuously differentiable on the set $U \times X$. As x^0 solves

(0.7), it must belong to the set $S(d^0)$ and there must exist some $q^0 \in H(d^0)$ with $x^0 = \Delta q^0$. This q^0 is then an equilibrium route flow, according to the remark following (0.3). We assume that we have obtained one such q^0 and will use it in the analysis. It will become clear that the arbitrary choice of q^0 does not affect the results of the analysis. Finally, for notational simplicity, let $c^0 = c(x^0)$ and $\psi^0 = \psi(x^0)$ denote the route cost and OD cost at the equilibrium, respectively.

Sensitivity analysis of traffic user equilibrium problems started when the mathematical formulations of these problems were proposed; see Beckmann, Mcguire, and Winsten (1956, §3.2) for a study on how the equilibrium flows change qualitatively in response to data changes. Later, Hall (1978) and Dafermos and Nagurney (1984) conducted more complete studies on the continuity and direction of change of the traffic pattern in a parametric problem.

Tobin and Friesz (1988), Cho, Smith, and Friesz (2000), and Yang and Bell (2005) developed approaches to calculate derivatives of the equilibrium arc flow with respect to perturbation parameters in the arc cost function and the travel demand. The underlying tool used in these papers was the classical implicit function theorem. To overcome the difficulty that the equilibrium route flow is generally nonunique, these authors constructed restricted problems to which they could apply the implicit function theorem, then showed that for purposes of sensitivity analysis, the original problems were equivalent to the restricted problems. These papers made various assumptions about (local) strong monotonicity of the arc cost function with respect to the arc flow, which then implied continuity of the parametric equilibrium arc flow. (Tobin and Friesz 1988; Yang and Bell 2005 explicitly stated such strong monotonicity assumptions; Cho, Smith, and Friesz 2000 used the condition that the Jacobian matrix of the arc cost function at the unperturbed arc flow is positive definite, which implies local strong monotonicity, as the cost function is by assumption continuously differentiable.) They also made assumptions about strict complementarity of the unperturbed traffic pattern. For example, Cho, Smith, and Friesz (2000) and Yang and Bell (2005) assumed the existence of an equilibrium route flow in which each user optimal route has positive flow. This assumption made it possible to transform the equilibrium conditions into a set of equations, to which they could apply the implicit function theorem.

Qiu and Magnanti (1989) studied continuity and directional differentiability of the solution to a VI problem when the function defining this problem is perturbed and the polyhedral convex set defining it is fixed; they then applied their results to traffic user equilibrium problems with fixed or elastic demands. Although they used the term Lipschitz continuity in their paper, the property to which they applied that term in their Definition 2.1 is not Lipschitz continuity as it is now generally understood in variational analysis, but rather what is now called *calmness*, as defined, e.g., in Rockafellar and Wets (1998, p. 399). In fact, their definition requires slightly more than calmness, as it requires the solution of the unperturbed problem to be unique. Their key assumptions included the requirement that the arc cost function (and the negative of the demand function in the elastic-demand case) be locally strongly monotone on the affine hull of the critical cone of the VI expressing the equilibrium condition.

Yen (1995) proved local existence, uniqueness, and Lipschitz continuity of the arc flow under perturbation of the arc cost function and travel demand by assuming the arc cost function to be locally Lipschitz and locally strongly monotone. To obtain this result, he showed Lipschitz continuity of the metric projection of a point onto a polyhedral convex set P with respect to perturbations in the right-hand sides of the linear inequalities defining P.

Outrata (1997) considered the sensitivity of the solution to a VI over a fixed polyhedral convex set when the function defining the VI is perturbed, and he used that result in solving a class of mathematical programs with equilibrium constraints. He used assumptions including strong monotonicity of the function defining the VI with respect to the feasible set and a strict copositivity condition on the derivative of that function with respect to a subspace containing the tangent cone. Under these assumptions he showed existence and Lipschitz continuity of solutions, together with some directional differentiability results. As an application he considered the sensitivity of the equilibrium traffic pattern under perturbations of the arc cost function.

Patriksson and Rockafellar (2003) studied the local uniqueness, Lipschitz continuity, and semidifferentiability of the elastic-demand traffic user equilibria under perturbations of the arc cost function and the travel demand function. By assuming the negative of the travel demand function to be strictly monotone in the OD cost, they formulated the problem as a VI in the (x, d) space. Other assumptions in that paper included the requirement that the arc cost function and the negative of the inverse demand function be locally strongly monotone on the affine hull of the critical cone. Recently, Josefsson and Patriksson (2007) specialized the analysis to the case of separable link cost and demand functions, with the latter also being invertible.

Patriksson (2004) dealt with sensitivity of elastic-demand traffic user equilibria and included the fixed-demand case as a special case, with perturbations in both the arc cost function and the travel demand considered. The proof for the central result in that paper, Theorem 5 on characterization of the directional differentiability, is incorrect; see Robinson (2006).

In summary, although elastic-demand equilibria include fixed-demand equilibria as a special case, the sensitivity analysis methods developed in Patriksson and Rockafellar (2003); Josefsson and Patriksson (2007); and §4.2 of Qiu and Magnanti (1989) for elastic-demand equilibria do not apply to the fixed-demand case, because those authors needed the assumption that the negative of the demand function be strongly (or strictly) monotone with respect to the OD cost, which is not true in the fixed-demand case when the demand does not depend on the OD cost. Outrata (1997) and §4.1 of Qiu and Magnanti (1989) studied sensitivity of the fixed-demand equilibria under perturbations of the arc cost function only. They omitted perturbations of the travel demand and hence dealt with a family of variational inequalities defined on a fixed set, which is very different from the problem we consider. Tobin and Friesz (1988); Cho, Smith, and Friesz (2000); Yang and Bell (2005); and Yen (1995) dealt with the fixed-demand equilibria with perturbations of both the arc cost function and the travel demand; among those studies, Yen (1995) obtained the solution Lipschitz continuity but did not study semidifferentiability or differentiability, while the former three studied solution differentiability under certain strong monotonicity and strict complementarity assumptions. We will show that under weaker assumptions, the solution is Lipschitz continuous and semidifferentiable, that differentiability holds if an additional condition is satisfied, and this additional condition is in fact equivalent to the strict complementarity assumed in Cho, Smith, and Friesz (2000) and Yang and Bell (2005). (We could not compare this with the strict complementarity assumption in Tobin and Friesz 1988, as the latter appears to contain some deficiencies; see Josefsson and Patriksson 2007.) However, our method does not require finding a solution satisfying strict complementarity. Given any initial solution, we check the condition we need by solving a single linear programming (LP) problem. If it is solvable, then we have differentiability and can compute the derivatives fairly easily. In fact, the solution of this LP can easily be used to generate a route flow satisfying strict complementarity. Although our method does not need such a route flow, this might be interesting on its own, as it is unclear from the earlier papers how to find a route flow satisfying strict complementarity easily.

1. Techniques

In this section we introduce a sensitivity theorem for generalized equations and then show a way of applying it to variational inequalities defined over polyhedral convex sets.

1.1. Sensitivity of Generalized Equations

Let a *multifunction* G from a topological space X to a topological space Y refer to an assignment for each $x \in X$ of a set $G(x) \subset Y$. The *graph* of G is a subset of $X \times Y$, defined to be $\operatorname{gph} G := \{(x, y) \mid y \in G(x)\}$. The *domain* of G is a subset of X, defined to be $\operatorname{dom} G := \{x \mid G(x) \neq \varnothing\}$. If X and Y are both subsets of some Euclidean spaces, then we say that G is a *graph-convex polyhedral multifunction* if $\operatorname{gph} G$ is a polyhedral convex set and that G is a *polyhedral multifunction* if $\operatorname{gph} G$ is the union of finitely many polyhedral convex sets. We say two multifunctions G and H from X to Y *coincide locally* at some point $(x^0, y^0) \in X \times Y$ if $(x^0, y^0) \in \operatorname{gph} G \cap \operatorname{gph} H$ and there exists a neighborhood N of (x^0, y^0) such that $N \cap \operatorname{gph} G = N \cap \operatorname{gph} H$.

Suppose that U and X are neighborhoods of points u^0 and x^0, respectively, in some Euclidean spaces. We say that a function $f\colon U \times X \to \mathbb{R}^n$ is *strictly differentiable in* (u, x) *at* (u^0, x^0) if $f(\cdot, \cdot)$ has a Fréchet derivative $df(u^0, x^0)$ at (u^0, x^0) and, moreover, for any positive ϵ there exist neighborhoods U_ϵ of u^0 in U and X_ϵ of x^0 in X such that whenever (u, x) and (u', x') belong to $U_\epsilon \times X_\epsilon$ then

$$\begin{aligned}\|f(u', x') - f(u, x) - df(u^0, x^0)(u' - u, x' - x)\| \\ \leq \epsilon \|(u', x') - (u, x)\|.\end{aligned}$$

This property holds if f is continuously differentiable in a neighborhood of (u^0, x^0) (see Rockafellar and Wets 1998, Corollary 9.19). We say that f is *strictly differentiable in* x *at* (u^0, x^0) *uniformly on* U if $f(u^0, \cdot)$ has a Fréchet derivative $d_x f(u^0, x^0)$ at x^0 and, moreover, for any positive ϵ there exist neighborhoods U_ϵ of u^0 in U and X_ϵ of x^0 in X such that whenever $u \in U_\epsilon$ and $x, x' \in X_\epsilon$ then

$$\|f(u, x') - f(u, x) - d_x f(u^0, x^0)(x' - x)\| \leq \epsilon \|x' - x\|.$$

A sufficient but not necessary condition for this property to hold is that f be strictly differentiable in (u, x) at (u^0, x^0).

The following theorem on sensitivity of generalized equations is extracted from Lu and Robinson (2007, Theorem 5.1). It contains two different parameters, u and d, which play different roles: u affects the function f, and d affects the multifunction G. No assumption exists on how they are related: For example, either could be a function of the other, or both could be functions of some third parameter.

Theorem 1.1. *Let G be a polyhedral multifunction from $\mathbb{R}^k \times \mathbb{R}^n$ to $\mathbb{R}^n$, and let (d^0, x^0) be a point of $\mathbb{R}^k \times \mathbb{R}^n$. Let D_0 be a convex subset of $\mathbb{R}^k$ containing d^0, U_0 be a neighborhood of a point u^0 in $\mathbb{R}^m$, X_0 be a polyhedral* *convex neighborhood of* x^0 *in* $\mathbb{R}^n$, *and* f *be a Lipschitz continuous function from* $U_0 \times X_0$ *to* $\mathbb{R}^n$ *that is strictly differentiable in* x *at* x^0 *uniformly on* U_0. *For* $x \in X_0$ *define*

$$L(x) := f(u^0, x^0) + d_x f(u^0, x^0)(x - x^0).$$

Assume the following:

1. *The values* u^0, d^0, *and* x^0 *satisfy the generalized equation*

$$0 \in f(u, x) + G(d, x). \tag{1.1}$$

2. *There is a neighborhood* Ω_0 *of the origin in* $\mathbb{R}^n$ *such that for each* $(d, w) \in D_0 \times \Omega_0$ *the linearized generalized equation* $w \in L(\cdot) + G(d, \cdot)$ *has a unique solution in* X_0.

Then there are neighborhoods U' *of* u^0 *in* U_0, D' *of* d^0 *in* D_0, *and* X' *of* x^0 *in* $\mathbb{R}^n$ *and a single-valued Lipschitz continuous function* x: $U' \times D' \to X'$, *such that for each* $(u, d) \in U' \times D'$ *the point* $x(u, d)$ *is the unique solution in* X' *of* (1.1).

If we assume further that f *is strictly differentiable in* (u, x) *at* (u^0, x^0) *and that* D_0 *is a neighborhood of* d^0, *then the function* $x(\cdot, \cdot)$ *is semidifferentiable at* (u^0, d^0), *and its semiderivative* $dx(u^0, d^0)(r, s)$ *coincides locally at the origin of* $\mathbb{R}^m \times \mathbb{R}^k \times \mathbb{R}^n$ *with*

$$[L(\cdot) + G(d^0 + s, \cdot)]^{-1}(-d_u f(u^0, x^0)(r)) - x^0. \tag{1.2}$$

The term $[L(\cdot) + G(d^0 + s, \cdot)]^{-1}(-d_u f(u^0, x^0)(r))$ above is just the solution of the linearized generalized equation $-d_u f(u^0, x^0)(r) \in L(\cdot) + G(d^0 + s, \cdot)$. The first hypothesis of Theorem 1.1 is routine, so its application would need to concentrate on establishing the second assumption. The following subsection will show one way to do this in the context of variational inequalities.

1.2. Sensitivity of Variational Inequalities

We apply Theorem 1.1 to variational inequalities defined over perturbed polyhedral convex sets for which a constraint nondegeneracy condition holds. For a convex subset C of $\mathbb{R}^n$, we use the symbol Π_C to denote the Euclidean projector on C and f_C to denote the *normal map* induced by a function f: $\mathbb{R}^n \to \mathbb{R}^n$, defined to be

$$f_C(z) = f(\Pi_C(z)) + z - \Pi_C(z).$$

The proposition below considers multifunctions H from $\mathbb{R}^k$ to $\mathbb{R}^l$ and S from $\mathbb{R}^k$ to $\mathbb{R}^n$ defined by

$$H(d) = \{q \in \mathbb{R}^l \mid Vd + Wq \le s\}, \quad S(d) = \Delta H(d) \tag{1.3}$$

for each $d \in \mathbb{R}^k$, where V, W, and Δ are $j \times k$, $j \times l$, and $n \times l$ matrices, respectively, and $s \in \mathbb{R}^j$. It shows that under a constraint nondegeneracy condition, the sets $H(d)$ and $S(d)$ locally coincide with affine translations of the sets $H(d^0)$ and $S(d^0)$ when d is near the base point d^0.

For a subset I of $\{1, 2, \ldots, j\}$, we define V_I and W_I to be the submatrices of V and W consisting of those rows with indices in I; we use the same convention for the subvector s_I of s. For brevity we also speak of an affine function from a subset D of $\mathbb{R}^k$ to $\mathbb{R}^l$ when we mean a function from D to $\mathbb{R}^l$ that can be extended to an affine function from $\mathbb{R}^k$ to $\mathbb{R}^l$.

Proposition 1.2. *Define multifunctions* H *and* S *by* (1.3). *Suppose that* $q^0 \in H(d^0)$ *and* $x^0 = \Delta q^0$. *Let* D *be a subset of* $\mathbb{R}^k$ *containing* d^0. *Define an index set* $A = \{i: V_i d^0 + W_i q^0 = s_i\}$. *If the constraint nondegeneracy condition* $\operatorname{Range} W_A \supset V_A(d^0 - D)$ *holds, then there exists an affine function* $\tilde{q}$: $D \to \mathbb{R}^l$ *satisfying* $W_A \tilde{q}(d) = V_A(d^0 - d)$ *and* $\tilde{q}(d^0) = 0$. *For each such function* $\tilde{q}$, *there exist neighborhoods* D_2 *of* d^0 *in* D, Q_2 *of* q^0 *in* $\mathbb{R}^l$, *and* X_2 *of* x^0 *in* $\mathbb{R}^n$, *such that*

$$H(d) \cap Q_2 = [H(d^0) + \tilde{q}(d)] \cap Q_2 \tag{1.4}$$

and

$$S(d) \cap X_2 = [S(d^0) + \Delta\tilde{q}(d)] \cap X_2 \tag{1.5}$$

for each $d \in D_2$.

Proof. First, note that the condition $\operatorname{Range} W_A \supset V_A(d^0 - D)$ implies that $\operatorname{Range} W_A \supset V_A(d^0 - \operatorname{aff} D)$, where $\operatorname{aff} D$ is the affine hull of D. So there is no loss of generality in assuming D to be an affine set.

Suppose $\operatorname{Range} W_A \supset V_A(d^0 - D)$. By linear algebra there exists an affine function $\tilde{q}$: $D \to \mathbb{R}^l$ satisfying $W_A \tilde{q}(d) = V_A(d^0 - d)$ and $\tilde{q}(d^0) = 0$. Let $cA = \{1, 2, \ldots, j\} \backslash A$; the definition of A then implies that $V_{cA} d^0 + W_{cA} q^0 < s_{cA}$. Choose neighborhoods D_1 of d^0 in D and Q_2 of q^0 in $\mathbb{R}^l$ such that

$$\begin{aligned} V_{cA} d + W_{cA} q < s_{cA} \quad \text{and} \\ V_{cA} d^0 + W_{cA}(q - \tilde{q}(d)) < s_{cA} \end{aligned} \tag{1.6}$$

for each $(d, q) \in D_1 \times Q_2$.

Then for each $d \in D_1$ we have

$$\begin{aligned} H(d) \cap Q_2 &= \{q \mid Wq \le s - Vd\} \cap Q_2 \\ &= \{q \mid W_A q \le s_A - V_A d\} \cap Q_2 \\ &= \{q \mid W_A(q - \tilde{q}(d)) \le s_A - V_A d^0\} \cap Q_2 \\ &= \{q \mid W(q - \tilde{q}(d)) \le s - V d^0\} \cap Q_2 \\ &= \{q \mid q \in H(d^0) + \tilde{q}(d)\} \cap Q_2, \end{aligned}$$

where the second and fourth equations come from the first and second inequalities in (1.6), respectively. This shows that (1.4) holds for $d \in D_1$. Next we will show the existence of neighborhoods D_2 of d^0 in D_1 and X_2 of x^0 in $\mathbb{R}^n$ such that (1.5) holds.

Define a multifunction $\bar{H}$ from $\mathbb{R}^n \times \mathbb{R}^k$ to $\mathbb{R}^l$ by

$$\bar{H}(x, d) = \{q \in \mathbb{R}^l: Vd + Wq \le s, \ \Delta q = x\}.$$

Clearly, $\operatorname{gph}\bar H$ is a polyhedral convex set containing (x^0, d^0, q^0). Then there exists a Lipschitz continuous function $\bar h$ defined on $\operatorname{dom}\bar H$ such that (i) $\bar h(x, d) \in \bar H(x, d)$ for each $(x, d) \in \operatorname{dom}\bar H$; and (ii) $\bar h(x^0, d^0) = q^0$. For example, one could use

$$\bar h(x, d) = \Pi_{\bar H(x, d)}(q^0) = \{q \in \mathbb{R}^l \mid q^0 - q \in N_{\bar H(x, d)}(q)\},$$

which by Robinson (2007, Proposition 2.4) is a Lipschitz continuous function on $\operatorname{dom}\bar H$.

Next, define another multifunction $\tilde H$ from $\mathbb{R}^n \times D$ to $\mathbb{R}^l$ by

$$\tilde H(x, d) = \{q \in H(d^0) + \tilde q(d) \colon \Delta q = x\}$$

and note that $\operatorname{gph}\tilde H$ is a polyhedral convex set containing (x^0, d^0, q^0) too. Similar to the construction of $\bar h$, we construct a Lipschitz continuous function $\tilde h$ on $\operatorname{dom}\tilde H$ such that (i) $\tilde h(x, d) \in \tilde H(x, d)$ for each $(x, d) \in \operatorname{dom}\tilde H$; and (ii) $\tilde h(x^0, d^0) = q^0$.

Now choose neighborhoods D_2 of d^0 in D_1 and X_2 of x^0 in $\mathbb{R}^n$ such that $\bar h(x, d) \in Q_2$ for each $(x, d) \in (X_2 \times D_2) \cap \operatorname{dom}\bar H$ and $\tilde h(x, d) \in Q_2$ for each $(x, d) \in (X_2 \times D_2) \cap \operatorname{dom}\tilde H$. We show that these choices of D_2 and X_2 satisfy the requirement here.

Let $d \in D_2$ and $x \in S(d) \cap X_2$. The definition of $S(d)$ ensures that $(x, d) \in \operatorname{dom}\bar H$, so that $\bar h(x, d)$ is well defined with $x = \Delta\bar h(x, d)$ and $\bar h(x, d) \in \bar H(x, d) \subset H(d)$. The way we have defined D_2 and X_2 implies that $\bar h(x, d) \in Q_2$. We thus have $x \in X_2 \cap \Delta[H(d) \cap Q_2]$. By (1.4), this means that

$$x \in X_2 \cap \Delta[(H(d^0) + \tilde q(d)) \cap Q_2]. \tag{1.7}$$

It follows that x belongs to the set $X_2 \cap \Delta[H(d^0) + \tilde q(d)]$, which is just the set $X_2 \cap [S(d^0) + \Delta\tilde q(d)]$. This shows that $S(d) \cap X_2 \subset X_2 \cap [S(d^0) + \Delta\tilde q(d)]$.

Conversely, let $d \in D_2$ and $x \in X_2 \cap [S(d^0) + \Delta\tilde q(d)]$, so that x belongs to the set $\Delta[H(d^0) + \tilde q(d)]$. Then $(x, d) \in \operatorname{dom}\tilde H$, so $\tilde h(x, d)$ is well defined with $x = \Delta\tilde h(x, d)$ and $\tilde h(x, d) \in (H(d^0) + \tilde q(d)) \cap Q_2$. It follows that (1.7) holds again. By (1.4), we have $x \in X_2 \cap \Delta[H(d) \cap Q_2]$, and then $x \in X_2 \cap S(d)$. This shows that $X_2 \cap [S(d^0) + \Delta\tilde q(d)] \subset S(d) \cap X_2$ and thereby completes the proof. □

The constraint nondegeneracy condition in Proposition 1.2 is closely related to the *linear independence constraint qualification* (LICQ) in nonlinear programming. The index set A as defined is just the active index set for the base point q^0 under the parameter d^0. If the LICQ holds at q^0 in the sense that the matrix W_A is of full row rank, then $\operatorname{Range} W_A$ certainly contains the set $V_A(d^0 - D)$, and the constraint nondegeneracy condition follows.

As Proposition 1.2 deals with perturbed polyhedral convex sets, Robinson (2003) considered set perturbations involving nonlinear constraints. If that were applied to the situation here, Robinson (2003, Theorem 3.1) would say that the perturbed sets $H(d)$ are locally diffeomorphic to $H(d^0)$ whenever the matrix W_A is of full row rank; it does not provide direct information on how $S(d)$ varies.

One more remark on the constraint nondegeneracy condition follows. Proposition 1.2 shows that it suffices for the local translation property of H to hold; in fact, it is also necessary for this if the set D is a neighborhood of d^0. We can show this by an argument similar to the proof of Robinson (2003, Theorem 3.2), but we omit such a proof because we will not use this fact.

Here is the theorem on sensitivity of variational inequalities that we will use in §2:

Theorem 1.3. *Let S be a graph-convex polyhedral multifunction from $\mathbb{R}^k$ to $\mathbb{R}^n$ with $(d^0, x^0) \in \operatorname{gph} S$. Let D be a convex subset of $\mathbb{R}^k$ containing d^0, X be a neighborhood of x^0 in $\mathbb{R}^n$, U be a neighborhood of a point u^0 in $\mathbb{R}^m$, and f be a Lipschitz continuous function from $U \times X$ to $\mathbb{R}^n$, which is strictly differentiable in x at x^0 uniformly on U. Define $L \colon \mathbb{R}^n \to \mathbb{R}^n$ by*

$$L(x) := f(u^0, x^0) + d_x f(u^0, x^0)(x - x^0).$$

Assume the following:

1. *The values u^0, d^0, and x^0 satisfy the variational inequality*
$$0 \in f(u, x) + N_{S(d)}(x). \tag{1.8}$$
2. *There are neighborhoods D_2 of d^0 in D and X_2 of x^0 in $\mathbb{R}^n$ and an affine function $\tilde x \colon D \to \mathbb{R}^n$ with $\tilde x(d^0) = 0$, such that $S(d) \cap X_2 = [S(d^0) + \tilde x(d)] \cap X_2$ for any $d \in D_2$.*
3. *The normal map L_K induced by the function L and the critical cone*
$$K := \{z \in T_{S(d^0)}(x^0) \mid \langle f(u^0, x^0), z\rangle = 0\}$$
is a global homeomorphism.

Then there are neighborhoods U' of u^0 in U, D' of d^0 in D, and X' of x^0 in $\mathbb{R}^n$, and a single-valued, Lipschitz continuous function $x \colon U' \times D' \to X'$, such that for each $(u, d) \in U' \times D'$ the point $x(u, d)$ is the unique solution in X' of (1.8).

If we assume further that f is strictly differentiable in (u, x) at (u^0, x^0) and that D is a neighborhood of d^0, then the function $x(\cdot, \cdot)$ is semidifferentiable at (u^0, d^0) with its semiderivative $dx(u^0, d^0)(r, s)$ for each $(r, s) \in \mathbb{R}^m \times \mathbb{R}^k$ given by

$$\Pi_K \circ (L_K)^{-1}[-d_u f(u^0, x^0)(r) - d_x f(u^0, x^0)(\tilde x(d^0 + s) + x^0) + f(u^0, x^0)] + \tilde x(d^0 + s). \tag{1.9}$$

Proof. Let $y^0 = x^0 - f(u^0, x^0)$. By the first assumption we have $-f(u^0, x^0) \in N_{S(d^0)}(x^0)$, and it follows that $\Pi_{S(d^0)}(y^0) = x^0$ and $L_{S(d^0)}(y^0) = 0$. The rest of this proof consists of four steps.

In Step 1, we use the third assumption and the relationship between variational inequalities and normal maps to find a convex neighborhood Ω_1 of the origin in $\mathbb{R}^n$ and a polyhedral convex neighborhood X_1 of x^0 in $\mathbb{R}^n$, such that the variational inequality $w \in L(\cdot) + N_{S(d^0)}(\cdot)$ has a unique, Lipschitz continuous solution in X_1 as w varies in Ω_1. We will denote that solution by $F(w)$.

It is shown in Robinson (1992, Theorem 5.2) that the normal map L_K is a global homeomorphism if and only if the normal map $L_{S(d^0)}$ is a local homeomorphism at y^0. The latter means, according to remarks at the beginning of §5 of that paper, that $L_{S(d^0)}$ is a local homeomorphism from some neighborhood Y_1 of y^0 in $\mathbb{R}^n$ onto its image $L_{S(d^0)}(Y_1)$, which is a neighborhood of the origin of $\mathbb{R}^n$. By the definition of local homeomorphism, the quantity $(L_{S(d^0)})^{-1}(\cdot) \cap Y_1$ is a single-valued, continuous function on $L_{S(d^0)}(Y_1)$. As

$$x^0 - L(x^0) + 0 = x^0 - f(u^0, x^0) = y^0$$

and

$$\Pi_{S(d^0)}[(L_{S(d^0)})^{-1}(0) \cap Y_1] = \Pi_{S(d^0)}(y^0) = x^0,$$

there exist a convex neighborhood Ω_1 of the origin in $L_{S(d^0)}(Y_1)$ and a polyhedral convex neighborhood X_1 of x^0 in $\mathbb{R}^n$ such that whenever $(x, w) \in X_1 \times \Omega_1$,

$$x - L(x) + w \in Y_1 \quad \text{and} \quad \Pi_{S(d^0)}[(L_{S(d^0)})^{-1}(w) \cap Y_1] \in X_1.$$

Define a multifunction F from $\mathbb{R}^n$ to $\mathbb{R}^n$ by

$$F(w) = [L(\cdot) + N_{S(d^0)}(\cdot)]^{-1}(w) \cap X_1, \tag{1.10}$$

which consists of solutions to the variational inequality $w \in L(\cdot) + N_{S(d^0)}(\cdot)$ in X_1. Let $w \in \Omega_1$. The point $\Pi_{S(d^0)}[(L_{S(d^0)})^{-1}(w) \cap Y_1]$ belongs to X_1 and solves the equation $w \in L(\cdot) + N_{S(d^0)}(\cdot)$, so it belongs to $F(w)$. On the other hand, each point x in $F(w)$ satisfies $w \in L(x) + N_{S(d^0)}(x)$; writing $z = x - L(x) + w$, we have

$$x = \Pi_{S(d^0)}(z), \quad L_{S(d^0)}(z) = w \quad \text{and} \quad z \in Y_1.$$

So z is just the point $(L_{S(d^0)})^{-1}(w) \cap Y_1$, and x is just $\Pi_{S(d^0)}[(L_{S(d^0)})^{-1}(w) \cap Y_1]$. It follows that the multifunction F is single valued on Ω_1 with $F(w) = \Pi_{S(d^0)}[(L_{S(d^0)})^{-1}(w) \cap Y_1]$ and $F(0) = x^0$. Because X_1 is a polyhedral convex set, by Robinson (2007, Lemma 2.3), F is a polyhedral multifunction; an application of Robinson (2007, Corollary 2.2) then shows that F is Lipschitz continuous on Ω_1. This completes the proof of Step 1.

In Step 2, we use the second assumption to find neighborhoods Ω'_3 of the origin in Ω_1, D'_3 of d^0 in D_2, and X'_3 of x^0 in $\mathbb{R}^n$, such that as (d, w) varies in $D'_3 \times \Omega'_3$, the variational inequality $w \in L(\cdot) + N_{S(d)}(\cdot)$ has a unique, Lipschitz continuous solution in X'_3, which is given by $F[w - L(\tilde{x}(d)) + L(0)] + \tilde{x}(d)$.

Choose neighborhoods Ω'_3 of the origin in Ω_1, D'_3 of d^0 in D_2, and X'_3 of x^0 in $\mathbb{R}^n$, such that

1. X'_3 is convex, open, and contained in $X \cap X_1 \cap X_2$;
2. $w - L(\tilde{x}(d)) + L(0) \in \Omega_1$ for each $w \in \Omega'_3$ and $d \in D'_3$;
3. $x - \tilde{x}(d) \in X_1$ for each $x \in X'_3$ and $d \in D'_3$;
4. $F[w - L(\tilde{x}(d)) + L(0)] + \tilde{x}(d) \in X'_3$ for each $w \in \Omega'_3$ and $d \in D'_3$.

Let $w \in \Omega'_3$ and $d \in D'_3$. To say that

$$x \in [L(\cdot) + N_{S(d)}(\cdot)]^{-1}(w) \cap X'_3 \tag{1.11}$$

is equivalent to saying that x belongs to X'_3 and satisfies $w \in L(x) + N_{S(d)}(x)$. As X'_3 is open and convex, we have $N_{S(d)}(x) = N_{S(d) \cap X'_3}(x)$ for x in $S(d) \cap X'_3$. It follows that (1.11) holds if and only if $w \in L(x) + N_{S(d) \cap X'_3}(x)$. But $S(d) \cap X'_3 = (S(d^0) + \tilde{x}(d)) \cap X'_3$ because X'_3 is a subset of X_2, so (1.11) is in turn equivalent to $w \in L(x) + N_{[S(d^0) + \tilde{x}(d)] \cap X'_3}(x)$. Again, as X'_3 is open and convex, the latter holds if and only if

$$w \in L(x) + N_{S(d^0) + \tilde{x}(d)}(x) \quad \text{and} \quad x \in X'_3. \tag{1.12}$$

Note that $N_{S(d^0) + \tilde{x}(d)}(x) = N_{S(d^0)}(x - \tilde{x}(d))$ and the definition of L implies that $L(x) - L(\tilde{x}(d)) + L(0) = L(x - \tilde{x}(d))$, so we may rewrite (1.12) as

$$\begin{aligned} &w - L(\tilde{x}(d)) + L(0) \\ &\qquad \in L(x - \tilde{x}(d)) + N_{S(d^0)}(x - \tilde{x}(d)) \quad \text{and} \quad x \in X'_3, \end{aligned}$$

or equivalently

$$\begin{aligned} x - \tilde{x}(d) \in [L(\cdot) + N_{S(d^0)}(\cdot)]^{-1}[w - L(\tilde{x}(d)) + L(0)] \\ \text{and} \quad x \in X'_3. \end{aligned} \tag{1.13}$$

The way we defined X'_3 and D'_3 ensures that $x - \tilde{x}(d) \in X_1$ for each $(x, d) \in X'_3 \times D'_3$. By the definition of the multifunction F in (1.10), (1.13) is equivalent to

$$x - \tilde{x}(d) \in F[w - L(\tilde{x}(d)) + L(0)] \quad \text{and} \quad x \in X'_3. \tag{1.14}$$

The fact $(d, w) \in D'_3 \times \Omega'_3$ implies that $w - L(\tilde{x}(d)) + L(0) \in \Omega_1$, that $F[w - L(\tilde{x}(d)) + L(0)]$ is a singleton and that $F[w - L(\tilde{x}(d)) + L(0)] + \tilde{x}(d) \in X'_3$. Accordingly, (1.14) is equivalent to

$$x = F[w - L(\tilde{x}(d)) + L(0)] + \tilde{x}(d). \tag{1.15}$$

We have thereby shown that (1.11) is equivalent to (1.15) whenever $(d, w) \in D'_3 \times \Omega'_3$. In other words, on $D'_3 \times \Omega'_3$ we have

$$\begin{aligned} &[L(\cdot) + N_{S(d)}(\cdot)]^{-1}(w) \cap X'_3 \\ &\qquad = F[w - L(\tilde{x}(d)) + L(0)] + \tilde{x}(d), \end{aligned} \tag{1.16}$$

which is a single-valued, Lipschitz continuous function of (d, w). This proves Step 2.

In Step 3, we find a neighborhood Ω_3 of the origin in $\mathbb{R}^n$, a convex neighborhood D_3 of d^0 in D, and a polyhedral convex neighborhood X_3 of x^0 in X, such that as (d, w) varies in $D_3 \times \Omega_3$, the variational inequality $w \in L(\cdot) + N_{S(d)}(\cdot)$ has a unique, Lipschitz continuous solution in X_3. Then we apply Theorem 1.1 to obtain the existence, uniqueness, Lipschitz continuity, and semidifferentiability of the variational inequality (1.8).

Let X_3 be a polyhedral convex neighborhood of x^0 in X_3'; in particular, X_3 is a subset of X. As the quantity $[L(\cdot) + N_{S(d)}(\cdot)]^{-1}(w) \cap X_3'$ is Lipschitz continuous with respect to (d, w) in $\Omega_3' \times D_3'$, we can choose a neighborhood Ω_3 of the origin in Ω_3' and a convex neighborhood D_3 of d^0 in D_3' such that whenever $(d, w) \in D_3 \times \Omega_3$,

$$[L(\cdot) + N_{S(d)}(\cdot)]^{-1}(w) \cap X_3' \in X_3. \quad (1.17)$$

Combining (1.16) and (1.17), for each $(d, w) \in D_3 \times \Omega_3$ we have

$$\begin{aligned}[L(\cdot) + N_{S(d)}(\cdot)]^{-1}(w) \cap X_3 \\ = F[w - L(\tilde{x}(d)) + L(0)] + \tilde{x}(d), \quad (1.18)\end{aligned}$$

which is a single-valued, Lipschitz continuous function of (d, w).

We are then ready to apply Theorem 1.1 with the choices Ω_3 for Ω_0, D_3 for D_0, U for U_0, X_3 for X_0, and the specialization

$$G(d, x) = N_{S(d)}(x) \quad (1.19)$$

for the multifunction G; by Robinson (2007, Lemma 2.3), the choice (1.19) of G is a polyhedral multifunction. It follows from this theorem that there exist neighborhoods U' of u^0 in U, D' of d^0 in D_3, and X' of x^0 in $\mathbb{R}^n$, and a single-valued, Lipschitz continuous function $x\colon U' \times D' \to X'$, such that for each $(u, d) \in U' \times D'$ the point $x(u, d)$ is the unique solution in X' of (1.8). Moreover, if f is strictly differentiable in (u, x) at (u^0, x^0) and D_3 is a neighborhood of d^0, then the function $x(\cdot, \cdot)$ is semidifferentiable at (u^0, d^0), and the graph of its semiderivative $dx(u^0, d^0)(r, s)$ coincides on a neighborhood $R_0 \times S_0 \times T_0$ of the origin in $\mathbb{R}^m \times \mathbb{R}^k \times \mathbb{R}^n$ with the graph of the function

$$[L(\cdot) + N_{S(d^0+s)}(\cdot)]^{-1}[-d_u f(u^0, x^0)(r)] \cap X_3 - x^0. \quad (1.20)$$

As D_3 is a neighborhood of d^0 in D, we have obtained all conclusions of the present theorem except the formula in (1.9), which we will show in the next step.

In Step 4, we show that (1.9) follows from (1.20).

Recall that $L(x^0) = f(u^0, x^0)$ and $-L(x^0) \in N_{S(d^0)}(x^0)$. According to Robinson (1992, Proposition 5.1), there is a neighborhood $V \times W$ of the origin in $\mathbb{R}^n \times \mathbb{R}^n$ such that

$$\begin{aligned}&[(x^0, -L(x^0)) + V \times W] \cap \operatorname{gph} N_{S(d^0)} \\ &\quad = (x^0, -L(x^0)) + [V \times W \cap \operatorname{gph} N_K]. \quad (1.21)\end{aligned}$$

Because F is a Lipschitz continuous function on Ω_1 with $F(0) = x^0$, there exists a neighborhood Ω of the origin in Ω_1 such that each $w \in \Omega$ satisfies $F(w) - x^0 \in V$ and $L(x^0) - L(F(w)) + w \in W$. Choose a neighborhood $R \times S$ of the origin in $R_0 \times S_0$, such that $-d_u f(u^0, x^0)(r) \in \Omega_3$, $d^0 + s \in D_3$, and $-d_u f(u^0, x^0)(r) - L(\tilde{x}(d^0 + s)) + L(0) \in \Omega$ whenever $(r, s) \in R \times S$.

Let $w \in \Omega$. The definition of Ω implies that

$$[F(w), w - L(F(w))] \in (x^0, -L(x^0)) + V \times W,$$

and the definition of F implies that

$$w - L(F(w)) \in N_{S(d^0)}(F(w)).$$

It follows that

$$\begin{aligned}[F(w), w - L(F(w))] &\in [(x^0, -L(x^0)) + V \times W] \cap \operatorname{gph} N_{S(d^0)} \\ &= (x^0, -L(x^0)) + [V \times W \cap \operatorname{gph} N_K],\end{aligned}$$

so $w - L(F(w)) + L(x^0) \in N_K(F(w) - x^0)$ and then

$$\Pi_K[F(w) - x^0 + w - L(F(w)) + L(x^0)] = F(w) - x^0. \quad (1.22)$$

This implies that

$$L_K[F(w) - x^0 + w - L(F(w)) + L(x^0)] = w + L(0). \quad (1.23)$$

As L_K is a global homeomorphism by assumption, (1.22) and (1.23) imply that

$$F(w) = \Pi_K \circ (L_K)^{-1}(w + L(0)) + x^0. \quad (1.24)$$

Now for each $(r, s) \in R \times S$, we have

$$\begin{aligned}&[L(\cdot) + N_{S(d^0+s)}(\cdot)]^{-1}[-d_u f(u^0, x^0)(r)] \cap X_3 \\ &= F[-d_u f(u^0, x^0)(r) - L(\tilde{x}(d^0+s)) + L(0)] + \tilde{x}(d^0+s) \\ &= \Pi_K \circ (L_K)^{-1}[-d_u f(u^0, x^0)(r) - L(\tilde{x}(d^0+s)) + 2L(0)] \\ &\qquad + x^0 + \tilde{x}(d^0+s) \\ &= \Pi_K \circ (L_K)^{-1}[-d_u f(u^0, x^0)(r) - d_x f(u^0, x^0) \\ &\qquad \cdot (\tilde{x}(d^0+s) + x^0) + f(u^0, x^0)] + x^0 + \tilde{x}(d^0+s),\end{aligned}$$

where the first and second equations come from (1.18) and (1.24), respectively, and the third is by definition of L. Combining this with (1.20), we see that the semiderivative $dx(u^0, d^0)(r, s)$ coincides locally at the origin of $\mathbb{R}^m \times \mathbb{R}^k \times \mathbb{R}^n$ with the function in (1.9). As K is a convex cone, the function Π_K is positive homogeneous on $\mathbb{R}^n$. Using facts that L and $\tilde{x}$ are both affine functions with $L(0) = f(u^0, x^0) - d_x f(u^0, x^0)x^0$ and

$\tilde{x}(d^0)=0$, one can check that (1.9) is a positive homogeneous function of (r,s). Since $dx(u^0,d^0)(r,s)$ is also positive homogeneous, it coincides with (1.9) everywhere. This shows (1.9) and completes the proof. □

The first assumption of Theorem 1.3 is routine and is just a copy of the first assumption of Theorem 1.1. The second assumption requires a kind of local translation property of the multifunction S, which by Proposition 1.2 would follow from a constraint nondegeneracy condition. The third assumption requires the normal map induced by the critical cone to be a homeomorphism; this is a standard assumption used in analysis of variational inequalities. The latter two assumptions combine to imply the second assumption in Theorem 1.1. In the next section we will show that for traffic user equilibrium problems, these assumptions hold under some simple conditions. The formula (1.9) for semiderivatives may look complicated; however, for each vector $a\in\mathbb{R}^n$ the quantity $\Pi_K\circ(L_K)^{-1}(a)$ is nothing but the solution of the linear variational inequality $a\in L(\cdot)+N_K(\cdot)$. Indeed, for traffic user equilibrium problems this is just the solution of a linearized traffic user equilibrium; we will go into detail about this in §2.4.

2. Sensitivity of Traffic User Equilibria

In this section we apply results from §1.2 to the parametric traffic user equilibrium problem introduced at the beginning of this paper. As in Theorem 1.3, we use the set

$$K=\{z\in T_{S(d^0)}(x^0)\mid\langle f(u^0,x^0),z\rangle=0\}\tag{2.1}$$

to denote the critical cone and the function $L\colon\mathbb{R}^\alpha\to\mathbb{R}^\alpha$ to denote the linear approximation of f with respect to x at (u^0,x^0),

$$L(x):=f(u^0,x^0)+d_xf(u^0,x^0)(x-x^0).\tag{2.2}$$

We also use $\tilde{L}\in\mathbb{R}^{\alpha\times\alpha}$ to denote the matrix representing the linear operator $d_xf(u^0,x^0)$.

The choice (0.5) of S is a graph-convex polyhedral multifunction, so the traffic user equilibrium problem considered here belongs to the category of parametric variational inequalities dealt with by Theorem 1.3. We will utilize the special structure of this problem to replace assumptions in that theorem by specific conditions that are easy to check and develop practical methods to compute the semiderivatives or derivatives of the parametric equilibrium arc flow.

The rest of this section consists of six parts. Section 2.1 shows that the strict positivity of the base demand d^0 implies the local translation property required in Theorem 1.3. Section 2.2 investigates the structure of the critical cone K. Following that, §2.3 shows how to check the homeomorphism condition required in Theorem 1.3. Sections 2.4 and 2.5 then discuss how to compute the semiderivatives or derivatives when they exist. Section 2.6 summarizes these results in a theorem.

2.1. Constraint Nondegeneracy Condition

The following lemma shows that the strict positivity of the base demand d^0 implies the constraint nondegeneracy condition in Proposition 1.2 and then the local translation property required in Theorem 1.3.

Lemma 2.1. *Assume the notation for the traffic user equilibrium problem from the Introduction. Suppose $d^0>0$; then there exist neighborhoods X_2 of x^0 in $\mathbb{R}^\alpha$, D_2 of d^0 in D, and an affine function $\tilde{q}\colon D\to\mathbb{R}^\pi$ as defined in* (2.7) *such that $\tilde{q}(d^0)=0$ and $S(d)\cap X_2=(S(d^0)+\Delta\tilde{q}(d))\cap X_2$ for any $d\in D_2$.*

Proof. Let I_ω and I_π denote the identity matrices in $\mathbb{R}^{\omega\times\omega}$ and $\mathbb{R}^{\pi\times\pi}$, respectively. Rewriting the set $H(d)$ defined in (0.4) as

$$H(d)=\left\{q\in\mathbb{R}^\pi\;\middle|\;\begin{bmatrix}-I_\omega\\ I_\omega\\ 0\end{bmatrix}d+\begin{bmatrix}\Lambda\\ -\Lambda\\ -I_\pi\end{bmatrix}q\le\begin{bmatrix}0\\0\\0\end{bmatrix}\right\},\tag{2.3}$$

we are in the setting of Proposition 1.2 with the choices $[-I_\omega,I_\omega,0]^T$ for the matrix V, $[\Lambda,-\Lambda,-I_\pi]^T$ for the matrix W, and $[0,0,0]^T$ for the vector s. The way we have defined q^0 ensures that $x^0=\Delta q^0$ and $q^0\in H(d^0)$; the latter means that q^0 is nonnegative with

$$d^0=\Lambda q^0,\quad\text{or equivalently,}$$
$$d^0_w=\sum_{p\in\mathcal{P}_w}q^0_p\quad\text{for each } w\in\mathcal{W}.\tag{2.4}$$

The definition of $H(d)$ in (2.3) consists of three groups of inequalities. By (2.4) the first two groups are active for the pair (d^0,q^0). If we denote the set of all unused routes by

$$\mathcal{P}^0=\{p\in\mathcal{P}\colon q^0_p=0\},\tag{2.5}$$

then an inequality in the third group is active for q^0 if and only if it corresponds to a route in $\mathcal{P}^0$.

Suppose $d^0>0$. By (2.4), for each OD pair w there exists at least one route $p\in\mathcal{P}_w$ with $q^0_p>0$; choose one such route for each w and denote it by $p(w)$, and let

$$\mathcal{P}^1=\{p(w),\,w\in\mathcal{W}\}\tag{2.6}$$

be the collection of all such routes. The definition of $\mathcal{P}^1$ ensures that $\mathcal{P}\backslash\mathcal{P}^1\supset\mathcal{P}^0$. Then, for each $d\in\mathbb{R}^\omega$, define $\tilde{q}(d)\in\mathbb{R}^\pi$ by

$$[\tilde{q}(d)]_p=\begin{cases}d_w-d^0_w & \text{if } p=p(w)\text{ for some } w\in\mathcal{W},\\ 0 & \text{otherwise.}\end{cases}\tag{2.7}$$

The function $\tilde{q}$ as defined is an affine function of d with $\tilde{q}(d^0) = 0$. Moreover, for each $d \in \mathbb{R}^\omega$, we have $\Lambda\tilde{q}(d) = d - d^0$ and $[\tilde{q}(d)]_p = 0$ for each route $p \in \mathscr{P}\backslash\mathscr{P}^1 \supset \mathscr{P}^0$. If we define the index set A as in Proposition 1.2, what we have shown implies that $W_A\tilde{q}(d) = V_A(d^0 - d)$ for each $d \in \mathbb{R}^\omega$. It follows that the constraint nondegeneracy condition in Proposition 1.2 holds here, and an application of that proposition would provide us neighborhoods X_2 and D_2 such that the required properties hold. $\square$

In the rest of this paper, we make the key assumption that $d^0 > 0$. This is not as strong as it sounds; what we actually do is to assume that the demand between all OD pairs with zero unperturbed demand will continue to be zero, and we then exclude those OD pairs from the model. Indeed, if the equilibrium arc flow $x(u, d)$ is semidifferentiable or differentiable at (u^0, d^0), then it has to be well defined in a neighborhood of (u^0, d^0) in $\mathbb{R}^m \times \mathbb{R}^\omega$. In particular, there must exist a neighborhood of d^0 in $\mathbb{R}^\omega$ such that $S(d) \neq \varnothing$ for each d in this neighborhood, and this means that $d^0 > 0$. Hence, this strict positivity assumption is in any case necessary for the semidifferentiability or differentiability of the equilibrium arc flow, and it is an implicit assumption in papers studying these properties, including Tobin and Friesz (1988); Cho, Smith, and Friesz (2000); and Yang and Bell (2005). For sensitivity analysis of elastic-demand traffic user equilibrium problems, it is not necessary to assume the travel demand at the base equilibrium to be strictly positive (see Qiu and Magnanti 1992; Patriksson and Rockafellar 2003; and Josefsson and Patriksson 2007).

2.2. Critical Cone

This subsection investigates the structure of the critical cone K for the traffic user equilibrium problem under the assumption $d^0 > 0$.

In (2.5) and (2.6) we defined route sets $\mathscr{P}^0$ and $\mathscr{P}^1$, with the former consisting of all unused routes and the latter consisting of exactly one used route for each OD pair. We may further divide the set $\mathscr{P}^0$ into two subsets $\mathscr{P}^{00}$ and $\mathscr{P}^{01}$, such that $\mathscr{P}^{00}$ consists of all unused routes that are user optimal and $\mathscr{P}^{01}$ consists of all unused routes that are not user optimal. We also define $\mathscr{P}^2$ to consist of all used routes not in $\mathscr{P}^1$. In this way, we have partitioned the entire route set by

$$\mathscr{P} = \mathscr{P}^{00} \cup \mathscr{P}^{01} \cup \mathscr{P}^1 \cup \mathscr{P}^2, \tag{2.8}$$

where

$$\begin{aligned}
\mathscr{P}^{00} &= \bigcup_{w\in\mathscr{W}} \{p \in \mathscr{P}_w\colon\ q^0_p = 0,\ c^0_p = \psi^0_w\},\\
\mathscr{P}^{01} &= \bigcup_{w\in\mathscr{W}} \{p \in \mathscr{P}_w\colon\ q^0_p = 0,\ c^0_p > \psi^0_w\},\\
\mathscr{P}^{1} &= \bigcup_{w\in\mathscr{W}} \{p \in \mathscr{P}_w\colon\ q^0_p > 0,\ p = p(w)\},\\
\mathscr{P}^{2} &= \bigcup_{w\in\mathscr{W}} \{p \in \mathscr{P}_w\colon\ q^0_p > 0,\ p \neq p(w)\}.
\end{aligned} \tag{2.9}$$

Clearly, the definition of $\mathscr{P}^1$ above is equivalent to its original definition in (2.6). Denote the cardinalities of $\mathscr{P}^{00}$, $\mathscr{P}^{01}$, $\mathscr{P}^1$, and $\mathscr{P}^2$ by π_{00}, π_{01}, π_1, and π_2, respectively, and note that $\pi_1 = \omega$ by the way we defined $\mathscr{P}^1$. For the purpose of explicit explanation, we treat $\mathscr{P}^{00}$, $\mathscr{P}^{01}$, $\mathscr{P}^1$, and $\mathscr{P}^2$ as ordered sets. We enumerate the set $\mathscr{W}$ and then order routes in $\mathscr{P}^1$ such that the jth route in $\mathscr{P}^1$ connects the jth OD pair for each j from 1 to ω.

Accordingly, we partition the matrices Λ and Δ by column as

$$\begin{bmatrix} \Lambda \\ \Delta \end{bmatrix} = \begin{bmatrix} \Lambda_{00} & \Lambda_{01} & \Lambda_1 & \Lambda_2 \\ \Delta_{00} & \Delta_{01} & \Delta_1 & \Delta_2 \end{bmatrix}, \tag{2.10}$$

where matrices Λ_{00}, Λ_{01}, Λ_1, and Λ_2 contain columns in Λ corresponding to routes in $\mathscr{P}^{00}$, $\mathscr{P}^{01}$, $\mathscr{P}^1$, and $\mathscr{P}^2$, respectively, and similarly for matrices Δ_{00}, Δ_{01}, Δ_1, and Δ_2. By the ordering we made for routes in $\mathscr{P}^1$, we have arranged that $\Lambda_1 = I_\omega$.

For each $d \in \mathbb{R}^\omega$ we divide the route flow $\tilde{q}(d)$ in (2.7) into four components:

$$\tilde{q}(d) = (\tilde{q}_{00}(d), \tilde{q}_{01}(d), \tilde{q}_1(d), \tilde{q}_2(d)).$$

The definition of $\tilde{q}(d)$ implies that $\tilde{q}_{00}(d)$, $\tilde{q}_{01}(d)$, and $\tilde{q}_2(d)$ are null vectors, so we have $\Lambda\tilde{q}(d) = \Lambda_1\tilde{q}_1(d)$ and $\Delta\tilde{q}(d) = \Delta_1\tilde{q}_1(d)$. It then follows from the equations $\Lambda_1 = I_\omega$ and $\Lambda\tilde{q}(d) = d - d^0$ that $\tilde{q}_1(d) = d - d^0$. Consequently, we have

$$\Delta\tilde{q}(d) = \Delta_1(d - d^0). \tag{2.11}$$

By the definition of $H(d)$ in (2.3) and Rockafellar and Wets (1998, Theorem 6.46), we have

$$T_{H(d^0)}(q^0) = \left\{ q \in \mathbb{R}^\pi \;\middle|\; \begin{array}{ll} \Lambda q = 0 & \\ q_p \geq 0 & \text{if } p \in \mathscr{P}^{00} \cup \mathscr{P}^{01} \\ q_p \text{ free} & \text{if } p \in \mathscr{P}^1 \cup \mathscr{P}^2 \end{array} \right\}. \tag{2.12}$$

Because $S(d^0) = \Delta H(d^0)$ and $x^0 = \Delta q^0$, we have $T_{S(d^0)}(x^0) = \Delta T_{H(d^0)}(q^0)$ by Rockafellar and Wets (1998, Theorem 6.43). Then, by its definition, the critical cone K can be written as

$$\begin{aligned}
K &= \big\{z \in \mathbb{R}^\alpha \mid \langle f(u^0, x^0), z\rangle = 0, z = \Delta q \text{ for some } q \in \mathbb{R}^\pi \\
&\qquad\qquad \text{such that } \Lambda q = 0 \text{ and } q_p \geq 0 \text{ for } p \in \mathscr{P}^{00} \cup \mathscr{P}^{01}\big\} \\
&= \big\{z \in \mathbb{R}^\alpha \mid z = \Delta q \text{ for some } q \in \mathbb{R}^\pi \text{ such that } \Lambda q = 0, \\
&\qquad\qquad \langle \Delta^T f(u^0, x^0), q\rangle = 0 \text{ and } q_p \geq 0 \text{ for } p \in \mathscr{P}^{00} \cup \mathscr{P}^{01}\big\},
\end{aligned}$$

where $\Delta^T f(u^0, x^0) = c^0$ is just the route cost at the equilibrium. The fact that q^0 is an equilibrium route flow implies that

$$c^0_p - \psi^0_w \begin{cases} = 0, & \text{if } p \in \mathscr{P}^{00} \cup \mathscr{P}^1 \cup \mathscr{P}^2; \\ > 0, & \text{if } p \in \mathscr{P}^{01}. \end{cases} \tag{2.13}$$

Hence, if $q \in \mathbb{R}^{\pi}$ satisfies $\Lambda q = 0$ and $q_p \geq 0$ for each $p \in \mathscr{P}^{00} \cup \mathscr{P}^{01}$, then

$$\begin{aligned}\langle c^0, q\rangle &= \sum_{w\in\mathscr{W}}\sum_{p\in\mathscr{P}_w} c_p^0 q_p = \sum_{w\in\mathscr{W}}\sum_{p\in\mathscr{P}_w}[\psi_w^0 q_p + (c_p^0-\psi_w^0)q_p]\\ &= \sum_{w\in\mathscr{W}}\left\{\psi_w^0\sum_{p\in\mathscr{P}_w} q_p + \sum_{p\in\mathscr{P}_w}(c_p^0-\psi_w^0)q_p\right\}\\ &= \sum_{w\in\mathscr{W}}\sum_{p\in\mathscr{P}_w}(c_p^0-\psi_w^0)q_p = \sum_{w\in\mathscr{W}}\sum_{p\in\mathscr{P}_w\cap\mathscr{P}^{01}}(c_p^0-\psi_w^0)q_p,\end{aligned}$$

which is equal to zero if and only if $q_p = 0$ for each $p \in \mathscr{P}^{01}$.

We can then rewrite K as

$$\begin{aligned}K = \Bigg\{z\in\mathbb{R}^{\alpha}\,\Bigg|\,\begin{bmatrix}0\\ z\end{bmatrix} &= \begin{bmatrix}\Lambda_{00}\\ \Delta_{00}\end{bmatrix}q_{00} + \begin{bmatrix}\Lambda_{01}\\ \Delta_{01}\end{bmatrix}q_{01}\\ &\quad + \begin{bmatrix}\Lambda_1\\ \Delta_1\end{bmatrix}q_1 + \begin{bmatrix}\Lambda_2\\ \Delta_2\end{bmatrix}q_2 \text{ for some}\\ &\quad (q_{00}, q_{01}, q_1, q_2) \in \mathbb{R}_+^{\pi_{00}} \times \{0\}^{\pi_{01}} \times \mathbb{R}^{\pi_1} \times \mathbb{R}^{\pi_2}\Bigg\}.\end{aligned} \tag{2.14}$$

Because $\Lambda_1 = I_{\omega}$, we can further simplify this expression to

$$\begin{aligned}K = \big\{z\in\mathbb{R}^{\alpha} \mid z &= (\Delta_{00}-\Delta_1\Lambda_{00})q_{00} + (\Delta_2-\Delta_1\Lambda_2)q_2\\ &\text{for some } (q_{00}, q_2)\in\mathbb{R}_+^{\pi_{00}}\times\mathbb{R}^{\pi_2}\big\}.\end{aligned} \tag{2.15}$$

Let A be a matrix the columns of which form a basis for the column space of $\Delta_2 - \Delta_1\Lambda_2$, and let B contain columns in $\Delta_{00} - \Delta_1\Lambda_{00}$ that are not in the column space of A. If we denote the number of columns in A and B by γ and β, respectively, then we can rewrite (2.15) as

$$K = \big\{z\in\mathbb{R}^{\alpha} \mid z = Au + Bv \text{ for some } u\in\mathbb{R}^{\gamma},\, v\in\mathbb{R}_+^{\beta}\big\}. \tag{2.16}$$

We will use both expressions of K in (2.14) and (2.16) in following subsections; we use (2.16) in §§2.3 and 2.5 to check the homeomorphism condition required by Theorem 1.3 and compute the derivatives of the equilibrium arc flow and use (2.14) in §2.4 to compute the semiderivatives.

Finally, recall that we chose q^0 by arbitrarily selecting one out of $H(d^0)$ that satisfies $x^0 = \Delta q^0$. This q^0 affects the way we partitioned $\mathscr{P}$ and therefore the expressions of K. However, whichever q^0 we choose does not affect the analysis results, because the definition of the critical cone K shows that it does not truly depend on q^0.

2.3. Homeomorphism Condition

By definition of the function L in (2.2) and the cone K in (2.1), L is an affine transformation and K is a polyhedral convex cone. As is shown in Robinson (1995), the normal map L_K induced by L for K is a piecewise affine map of $\mathbb{R}^{\alpha}$, the homeomorphism property in this case is equivalent to the bijectivity property, and L_K is a homeomorphism if and only if it is coherently oriented; that is, the determinants of the affine transformations with which L_K coincides in the n-cells of the normal manifold of K all have the same nonzero sign. However, it is not easy to check directly whether the coherent orientation condition holds, because in practice we cannot afford to enumerate all the n-cells of the normal manifold given the expressions for K in (2.14) and (2.16).

The definition of the matrix $\tilde{L}$ just below (2.2) implies that $L(x) = \tilde{L}x + b$ for a certain vector $b \in \mathbb{R}^{\alpha}$. Part of the proof of Facchinei and Pang (2003, Proposition 1.5.11) shows that the normal map L_K is bijective if and only if, for each $a \in \mathbb{R}^{\alpha}$, the following VI problem has a unique solution z:

$$0 \in \tilde{L}z - a + N_K(z). \tag{2.17}$$

Hence, to find conditions under which L_K is a homeomorphism, we just need to find conditions for (2.17) to have a unique solution for each $a \in \mathbb{R}^{\alpha}$. We give three different conditions, each of which suffices for this to hold.

First, according to Theorem 2.3.3 and the remark after Definition 2.3.1 in Facchinei and Pang (2003), if $\tilde{L}$ is strictly monotone on K, then the problem (2.17) has a unique solution for any $a \in \mathbb{R}^{\alpha}$. Note that $\tilde{L}$ is strictly monotone on K if and only if it is strictly monotone on the subspace $K - K$. Given the expression of K in (2.16), the subspace $K - K$ is just the column space of the matrix $[A, B]$. Let $\tilde{A}$ be a matrix the columns of which form a basis for the column space of $[A, B]$; then $\tilde{L}$ is strictly monotone on K if and only if the matrix $\tilde{A}^T\tilde{L}\tilde{A}$ is positive definite.

Second, given the expression of K in (2.16), according to the Karush-Kuhn-Tucker conditions for variational inequalities (see Facchinei and Pang 2003, Proposition 1.3.4), z solves (2.17) if and only if there exist $\theta \in \mathbb{R}^{\gamma}$, $\tau \in \mathbb{R}^{\beta}$, and $\lambda \in \mathbb{R}^{\alpha}$ such that $(z, \theta, \tau, \lambda)$ solves the following mixed complementarity (MCP) problem:

$$\begin{aligned}&0 = \tilde{L}z - \lambda - a \perp z \text{ free}\\ &0 = A^T\lambda \perp \theta \text{ free}\\ &0 \leq B^T\lambda \perp \tau \geq 0\\ &0 = z - A\theta - B\tau \perp \lambda \text{ free.}\end{aligned} \tag{2.18}$$

Eliminate λ from (2.18) to transform it to the following MCP problem

$$\begin{aligned}&0 = A^T\tilde{L}z - A^Ta \perp \theta \text{ free}\\ &0 \leq B^T\tilde{L}z - B^Ta \perp \tau \geq 0\\ &0 = z - A\theta - B\tau \perp z \text{ free}\end{aligned} \tag{2.19}$$

which can be rewritten as

$$-\begin{bmatrix} z-A\theta-B\tau \\ A^T\tilde{L}z-A^Ta \\ B^T\tilde{L}z-B^Ta \end{bmatrix} \in N_{\mathbb{R}^\alpha\times\mathbb{R}^\gamma\times\mathbb{R}^\beta_+}\left(\begin{bmatrix} z \\ \theta \\ \tau \end{bmatrix}\right), \tag{2.20}$$

or equivalently,

$$\begin{bmatrix} 0 \\ A^Ta \\ B^Ta \end{bmatrix} \in \begin{bmatrix} I & -A & -B \\ A^T\tilde{L} & 0 & 0 \\ B^T\tilde{L} & 0 & 0 \end{bmatrix}\begin{bmatrix} z \\ \theta \\ \tau \end{bmatrix} + N_{\mathbb{R}^\alpha\times\mathbb{R}^\gamma\times\mathbb{R}^\beta_+}\left(\begin{bmatrix} z \\ \theta \\ \tau \end{bmatrix}\right). \tag{2.21}$$

According to Theorem 3.1 in Robinson (1980), (2.21) has a unique solution for any choice of $a\in\mathbb{R}^\alpha$ if the following two conditions both hold:

Condition 1.1 (C1.1). The matrix

$$\begin{bmatrix} I & -A \\ A^T\tilde{L} & 0 \end{bmatrix}$$

is nonsingular, or equivalently, the matrix $A^T\tilde{L}A$ is nonsingular;

Condition 1.2 (C1.2). The matrix

$$\begin{bmatrix} B^T\tilde{L} & 0 \end{bmatrix}\begin{bmatrix} I & -A \\ A^T\tilde{L} & 0 \end{bmatrix}^{-1}\begin{bmatrix} B \\ 0 \end{bmatrix} = B^T\tilde{L}B - B^T\tilde{L}A(A^T\tilde{L}A)^{-1}A^T\tilde{L}B$$

has positive principal minors.

Therefore, if C1.1 and C1.2 hold, (2.20) has a unique solution for any $a\in\mathbb{R}^\alpha$; consequently, (2.17) has a unique solution for any $a\in\mathbb{R}^\alpha$.

Third, if we can express the polyhedral cone K by a system of linear equations and inequalities

$$K=\{z\in\mathbb{R}^\alpha \mid Mz\le 0,\ Ez=0\}, \tag{2.22}$$

then, by using the same theorem as above, we can show that L_K is a homeomorphism if the following conditions hold:

Condition 2.1 (C2.1). The matrix

$$\begin{bmatrix} \tilde{L} & E^T \\ -E & 0 \end{bmatrix}$$

is nonsingular, or equivalently, the matrix $[\tilde{L}(E^T)^\perp, E^T]$ is nonsingular, where the matrix $(E^T)^\perp$ contains columns perpendicular to the column space of E^T;

Condition 2.2 (C2.2). The matrix

$$\begin{bmatrix} M & 0 \end{bmatrix}\begin{bmatrix} \tilde{L} & E^T \\ -E & 0 \end{bmatrix}^{-1}\begin{bmatrix} M^T \\ 0 \end{bmatrix}$$

has positive principal minors.

2.4. Semidifferentiability

Continue to assume the notation and hypotheses given in the Introduction for the traffic user equilibrium problem, and suppose that $d^0>0$ and L_K is a homeomorphism. By hypothesis, f is continuously differentiable on $U\times X$, so it is Lipschitz continuous on a (possibly smaller) neighborhood of (u^0,x^0) in $\mathbb{R}^m\times\mathbb{R}^\alpha$ (see Rockafellar and Wets 1998, Theorem 9.7); therefore, there is no loss of generality in assuming f to be Lipschitz continuous on $U\times X$. By the remark following the definition of strict differentiability in §1.1, the continuous differentiability of f implies that it is strictly differentiable in (u,x) at (u^0,x^0). Further, Lemma 2.1 shows that under the assumption $d^0>0$, the second assumption of Theorem 1.3 holds with the affine function $\tilde{x}\colon D\to\mathbb{R}^\alpha$ defined by

$$\tilde{x}(d)=\Delta\tilde{q}(d)=\Delta_1(d-d^0), \tag{2.23}$$

where $\tilde{q}$ is defined in (2.7) and the second equality is from (2.11). Finally, the set D is open by hypothesis, and by considering an open and convex subset of it, we may assume without loss of generality that D is convex.

Now, an application of Theorem 1.3 generates neighborhoods U' of u^0 in U, D' of d^0 in D, X' of x^0 in $\mathbb{R}^\alpha$, and a single-valued, Lipschitz continuous function $x\colon U'\times D'\to X'$, such that for each $(u,d)\in U'\times D'$ the point $x(u,d)$ is the unique solution in X' of (0.7). Moreover, the function $x(\cdot,\cdot)$ is semidifferentiable at (u^0,d^0) with its semiderivative for each $(r,s)\in\mathbb{R}^m\times\mathbb{R}^\omega$ given by

$$\begin{aligned} &dx(u^0,d^0)(r,s)\\ &\quad=\Pi_K\circ(L_K)^{-1}\big[-d_uf(u^0,x^0)(r)\\ &\qquad\qquad -\tilde{L}(\Delta_1 s+x^0)+f(u^0,x^0)\big]+\Delta_1 s, \end{aligned} \tag{2.24}$$

where we replaced $\tilde{x}(d^0+s)$ and $d_xf(u^0,x^0)$ in (1.9) by $\Delta_1 s$ and $\tilde{L}$, respectively. For notational simplicity, we write

$$\begin{aligned} a(r,s) &= -d_uf(u^0,x^0)(r)\\ &\quad -\tilde{L}(\Delta_1 s+x^0)+f(u^0,x^0) \end{aligned} \tag{2.25}$$

and

$$z(r,s)=\Pi_K\circ(L_K)^{-1}\circ a(r,s). \tag{2.26}$$

The rest of this subsection discusses how to compute $z(r,s)$. According to the equivalence between normal maps and corresponding variational inequalities shown in Robinson (1995), $z(r,s)$ solves the VI problem

$$0\in L(z(r,s))-a(r,s)+N_K(z(r,s)) \tag{2.27}$$

and in fact is its unique solution under the assumption that L_K is a homeomorphism.

We can rewrite K in (2.14) by

$$K=[\Delta_{00},\Delta_{01},\Delta_1,\Delta_2]H, \tag{2.28}$$

where H is a set of route flows:

$$H=\{q=(q_{00},q_{01},q_1,q_2)\colon q_{00}\in\mathbb{R}^{\pi_{00}}_{+},\, q_{01}\in\{0\}^{\pi_{01}},\, q_1\in\mathbb{R}^{\pi_1},\, q_2\in\mathbb{R}^{\pi_2},\, \Lambda_{00}q_{00}+\Lambda_{01}q_{01}+\Lambda_1 q_1+\Lambda_2 q_2=0\}. \tag{2.29}$$

The problem given by (2.27)–(2.29) is a traffic user equilibrium over the same network we started with; the arc cost function for it is given by the affine function $L(\cdot)-a(r,s)$, the travel demand is zero for each OD pair, and the route flow is fixed at zero for routes in $\mathcal{P}^{01}$ and free for routes in $\mathcal{P}^1\cup\mathcal{P}^2$. We can solve it using algorithms designed for traffic user equilibria.

Another way to compute $z(r,s)$ is by solving the linear MCP problem in (2.30). Similar to the transformation from (2.17) to (2.19), one can show that $z(r,s)$ solves (2.27) if and only if there exist $\theta(r,s)\in\mathbb{R}^\gamma$ and $\tau(r,s)\in\mathbb{R}^\beta$ such that $(z(r,s),\theta(r,s),\tau(r,s))$ satisfies:

$$\begin{aligned}
&0=A^TL(z(r,s))-A^Ta(r,s)\perp\theta(r,s)\text{ free}\\
&0\le B^TL(z(r,s))-B^Ta(r,s)\perp\tau(r,s)\ge 0\\
&0=z(r,s)-A\theta(r,s)-B\tau(r,s)\perp z(r,s)\text{ free.}
\end{aligned}\tag{2.30}$$

2.5. Differentiability

In this subsection, we start by showing that when the critical cone K is a subspace, the semiderivative $dx(u^0,d^0)(r,s)$ given in (2.24) is a linear function with respect to (r,s), so the function x is differentiable at (u^0,d^0). Following that, we show how to check whether K is a subspace.

Suppose for now that K is a subspace; then it is just the column space of the matrix $[A,B]$ in view of (2.16). Let $\tilde{A}$ be a matrix the columns of which form a basis for the column space of $[A,B]$, and denote the number of columns of $\tilde{A}$ by κ. Then $z(r,s)$ solves (2.27) if and only if $z(r,s)=\tilde{A}\theta(r,s)$ for some $\theta(r,s)\in\mathbb{R}^\kappa$ satisfying

$$\tilde{A}^T[\tilde{L}(\tilde{A}\theta(r,s)-x^0)+f(u^0,x^0)-a(r,s)]=0, \tag{2.31}$$

where we used (2.2) to replace $L(\cdot)$ in (2.27) by $\tilde{L}(\cdot-x^0)+f(u^0,x^0)$.

Under the assumption that L_K is a homeomorphism, (2.27) has a unique solution $z(r,s)$ for each (r,s) in $\mathbb{R}^m\times\mathbb{R}^\omega$, so (2.31) also has a unique solution $\theta(r,s)$ for each such (r,s). Therefore, the matrix $\tilde{A}^T\tilde{L}\tilde{A}$ is nonsingular, and $z(r,s)$ is given by

$$z(r,s)=\tilde{A}(\tilde{A}^T\tilde{L}\tilde{A})^{-1}\tilde{A}^T[\tilde{L}x^0-f(u^0,x^0)+a(r,s)].$$

Then by (2.24) and (2.25), we have

$$\begin{aligned}
&dx(u^0,d^0)(r,s)\\
&\quad=\tilde{A}(\tilde{A}^T\tilde{L}\tilde{A})^{-1}\tilde{A}^T[-d_uf(u^0,x^0)(r)-\tilde{L}(\Delta_1 s)]+\Delta_1 s,
\end{aligned}$$

which is a linear function of (r,s). This shows that x is differentiable at (u^0,d^0), with Jacobian matrices $x_u(u^0,d^0)\in\mathbb{R}^{\alpha\times m}$ and $x_d(u^0,d^0)\in\mathbb{R}^{\alpha\times\omega}$ given by

$$x_u(u^0,d^0)=\tilde{A}(\tilde{A}^T\tilde{L}\tilde{A})^{-1}\tilde{A}^T[-d_uf(u^0,x^0)] \tag{2.32}$$

and

$$x_d(u^0,d^0)=\tilde{A}(\tilde{A}^T\tilde{L}\tilde{A})^{-1}\tilde{A}^T[-\tilde{L}\Delta_1]+\Delta_1. \tag{2.33}$$

Because the matrix $\tilde{A}$ has α rows and at most α columns, the dimension of the matrix $\tilde{A}^T\tilde{L}\tilde{A}$ is at most $\alpha\times\alpha$. In a large network, the number of arcs is usually much smaller than the number of OD pairs or the number of routes. In practice, we do not compute the inverted matrix $(\tilde{A}^T\tilde{L}\tilde{A})^{-1}$ to compute $x_u(u^0,d^0)$ and $x_d(u^0,d^0)$; instead, we just solve a system of linear equations.

We showed how to calculate the derivatives of the arc flow with respect to the parameter (u,d) when the critical cone K is a subspace. Below we show how to check whether K is a subspace.

First, in view of (2.16), K is trivially a subspace if the matrix B is empty. This is the fastest way to check the subspace property, but it is not complete.

Next, we characterize the subspace property by the solution property of an LP problem. Being a convex cone, K is a subspace if and only if the origin belongs to its relative interior $\operatorname{ri}K$ (see Rockafellar 1970 for the definition of relative interiors of convex sets). Given the expression of K in (2.15), an application of Rockafellar (1970, Corollary 6.6.2, Theorem 6.6) shows that

$$\operatorname{ri}K=\{z\in\mathbb{R}^\alpha\mid z=(\Delta_{00}-\Delta_1\Lambda_{00})q_{00}+(\Delta_2-\Delta_1\Lambda_2)q_2\ \text{for some } (q_{00},q_2)\in\mathbb{R}^{\pi_{00}}_{++}\times\mathbb{R}^{\pi_2}\}.$$

Accordingly, K is a subspace if and only if there exist $q_{00}\in\mathbb{R}^{\pi_{00}}_{++}$ and $q_2\in\mathbb{R}^{\pi_2}$ such that

$$(\Delta_{00}-\Delta_1\Lambda_{00})q_{00}+(\Delta_2-\Delta_1\Lambda_2)q_2=0. \tag{2.34}$$

When such (q_{00},q_2) exists, then $(\lambda q_{00},\lambda q_2)$ for each strictly positive scalar λ still satisfies (2.34). By choosing λ to be sufficiently large, we may arrange that all elements of λq_{00} be no less than 1. Hence, K is a subspace if and only if the following LP problem with a dummy objective function is solvable:

$$\min_{q_{00}\in\mathbb{R}^{\pi_{00}},\,q_2\in\mathbb{R}^{\pi_2}}\quad 0 \tag{2.35}$$

$$\text{s.t.}\quad (\Delta_{00}-\Delta_1\Lambda_{00})q_{00}+(\Delta_2-\Delta_1\Lambda_2)q_2=0 \tag{2.36}$$

$$(q_{00})_i\ge 1\quad\text{for each } i=1,\ldots,\pi_{00}. \tag{2.37}$$

This provides a complete and practical way to check if K is a subspace.

We can relate this subspace property to the strict complementarity assumption in Cho, Smith, and Friesz (2000) and Yang and Bell (2005), namely the existence of an equilibrium route flow in which each user optimal route has positive flow. Generally, if a point x in $\mathbb{R}^n$ satisfies a variational inequality $0 \in f(x) + N_S(x)$, with f being a function from $\mathbb{R}^n$ to $\mathbb{R}^n$ and S being a convex set in $\mathbb{R}^n$, then the critical cone $T_S(x) \cap \{f(x)\}^\perp$ is a subspace if and only if $-f(x) \in \operatorname{ri} N_S(x)$ (see, e.g., Robinson 1987, Lemma 2.1). If S is defined by finitely many nonlinear constraints and satisfies a certain constraint qualification, then the condition $-f(x) \in \operatorname{ri} N_S(x)$ can be rewritten as some strict complementarity condition; see related discussions in Dunn (1987) and Marcotte and Dussault (1989). The following proposition shows that under the special structure of the traffic user equilibrium problem, the critical cone K at the equilibrium arc flow x^0 is a subspace if and only if the strict complementarity assumption in Cho, Smith, and Friesz (2000) and Yang and Bell (2005) holds.

Proposition 2.2. *K is a subspace if and only if there exists a route flow $q^* \in H(d^0)$ with $x^0 = \Delta q^*$ such that for each $w \in \mathscr{W}$, each $p \in \mathscr{P}_w$ with $c_p^0 = \psi_w^0$ satisfies $q_p^* > 0$.*

Proof. For the "if" part, suppose that such a route flow q^* exists. We then choose $q^0 := q^*$ and partition the route set $\mathscr{P}$ by (2.8)–(2.9). The strict complementarity condition implies that none of the user optimal routes is unused, so the set $\mathscr{P}^{00}$ is empty. It is clear that K is a subspace in view of (2.15).

For the "only if" part, suppose that K is a subspace. Consequently, the LP problem (2.35)–(2.37) has a solution $(q_{00}, q_2) \in \mathbb{R}^{\pi_{00}} \times \mathbb{R}^{\pi_2}$. We show the existence of q^* by constructing it from (q_{00}, q_2) and the initial route flow q^0, which we write as $(q_{00}^0, q_{01}^0, q_1^0, q_2^0)$. Let λ be a positive scalar, and define

$$q^* = \begin{bmatrix} q_{00}^* \\ q_{01}^* \\ q_1^* \\ q_2^* \end{bmatrix} = \begin{bmatrix} q_{00}^0 + \lambda q_{00} \\ q_{01}^0 \\ q_1^0 - \lambda(\Lambda_{00} q_{00} + \Lambda_2 q_2) \\ q_2^0 + \lambda q_2 \end{bmatrix}. \qquad (2.38)$$

It then follows from the equality $\Lambda_1 = I_\omega$ that $\Lambda q^* = \Lambda q^0$ and from the equality (2.36) that $\Delta q^* = \Delta q^0$. The way we defined $\mathscr{P}^{00}$, $\mathscr{P}^{01}$, $\mathscr{P}^1$, and $\mathscr{P}^2$ implies that $q_{00}^0 = 0$, $q_{01}^0 = 0$, $q_1^0 > 0$, and $q_2^0 > 0$. By (2.37) we have $q_{00} > 0$, so $q_{00}^* > 0$. Further, by choosing λ to be sufficiently small, we can arrange that $q_1^* > 0$ and $q_2^* > 0$. This proves the "only if" part. □

Proposition 2.1 makes precise what the subspace property means for the traffic user equilibrium problem, and shows an interesting connection between the methodology here and those in earlier work. It also provides a way to construct a route flow q^* satisfying strict complementarity from any initial route flow q^0 and the solution (q_{00}, q_2) of the LP problem; one may simply define q^* by (2.38) and choose a positive λ such that $q_1^0 - \lambda(\Lambda_{00} q_{00} + \Lambda_2 q_2) > 0$ and $q_2^0 + \lambda q_2 > 0$.

Finally, when K is a subspace, or equivalently, when there exists a route flow q^* satisfying strict complementarity, we may choose $q^0 = q^*$ and partition the route set $\mathscr{P}$ by (2.8)–(2.9), as in the proof of last proposition. Then (2.14) becomes

$$K = \left\{ z \in \mathbb{R}^\alpha \;\middle|\; \begin{bmatrix} 0 \\ z \end{bmatrix} = \begin{bmatrix} \Lambda_1 \\ \Delta_1 \end{bmatrix} q_1 + \begin{bmatrix} \Lambda_2 \\ \Delta_2 \end{bmatrix} q_2 \text{ for some } (q_1, q_2) \in \mathbb{R}^{\pi_1} \times \mathbb{R}^{\pi_2} \right\}. \qquad (2.39)$$

In particular, if an arc is not on each of the used routes, then both rows in Δ_1 and Δ_2 corresponding to this arc are zero vectors. This means that we have in fact excluded such arcs from the analysis by restricting to the cone K, while in Tobin and Friesz (1988); Cho, Smith, and Friesz (2000); and Yang and Bell (2005), these arcs were excluded from the model in an explicit manner. This shows another interesting connection between the methodology here and those in earlier work.

2.6. Summary

The following theorem summarizes the development in §2 for sensitivity of traffic user equilibria.

Theorem 2.3. *Assume the notation and hypotheses given in the Introduction for the traffic user equilibrium problem. Define an affine function L: $\mathbb{R}^\alpha \to \mathbb{R}^\alpha$ by* (2.2) *and a cone K in $\mathbb{R}^\alpha$ by* (2.1). *Let $\tilde{L} \in \mathbb{R}^{\alpha\times\alpha}$ denote the matrix representing the linear operator $d_x f(u^0, x^0)$.*

Suppose first that $d^0 > 0$; we may then partition the route set $\mathscr{P}$ by (2.8)–(2.9) *and the matrices Λ and Δ by* (2.10). *The cone K satisfies* (2.14) *and* (2.16), *where in* (2.16) *A is a matrix the columns of which form a basis for the column space of $\Delta_2 - \Delta_1\Lambda_2$, and B is a matrix that consists of columns in $\Delta_{00} - \Delta_1\Lambda_{00}$ that are not in the column space of A. Let $\tilde{A}$ be a matrix the columns of which form a basis for the column space of $[A, B]$.*

Suppose further that the normal map L_K is a homeomorphism; then there exist neighborhoods U' of u^0 in U, D' of d^0 in D, X' of x^0 in $\mathbb{R}^\alpha$, and a single-valued, Lipschitz continuous function x: $U' \times D' \to X'$, such that for each $(u, d) \in U' \times D'$ the point $x(u, d)$ is the unique solution in X' of (0.7). *Moreover, the function $x(\cdot, \cdot)$ is semidifferentiable at (u^0, d^0) with $dx(u^0, d^0)(r, s) = z(r, s) + \Delta_1 s$ for each $(r, s) \in \mathbb{R}^m \times \mathbb{R}^\omega$, where $z(r, s)$ is the solution of the linear traffic user equilibrium problem* (2.27).

Any one of the following conditions suffices to ensure that L_K is a homeomorphism:

1. *The matrix $\tilde{A}^T \tilde{L} \tilde{A}$ is positive definite*
2. *Conditions* 1.1 *and* 1.2 *hold*
3. *Conditions* 2.1 *and* 2.2 *hold.*

If in addition K is a subspace, then the function $x(\cdot, \cdot)$ *is differentiable at* (u^0, d^0) *with Jacobian matrices* $x_u(u^0, d^0)$ *and* $x_d(u^0, d^0)$ *given by* (2.32) *and* (2.33). *K is a subspace if and only if the LP problem* (2.35)–(2.37) *is solvable.*

In what follows we compare the methodology in this paper with those in papers studying the same problem.

One key assumption we made here is that $d^0 > 0$. As discussed at the end of §2.1, this is an implicit assumption in Tobin and Friesz (1988); Cho, Smith, and Friesz (2000); and Yang and Bell (2005); Yen (1995) did not need this, as it did not study solution semidifferentiability or differentiability.

Another key assumption here is that L_K is a homeomorphism, and we listed three sufficient conditions for this to hold. The first of those is easy to check and is weaker than the various versions of strong monotonicity assumptions in Tobin and Friesz (1988); Cho, Smith, and Friesz (2000); Yang and Bell (2005); and Yen (1995) (because each of these strong monotonicity assumptions would require the matrix $\tilde{L}$ itself to be positive definite). But it still leaves out cases in which the analysis here actually applies. The second is also practical, and it covers some cases left out by the first condition, such as the example in §3.1. Application of the third condition requires transforming K to the form of (2.22), and this might not be easy to do in practice; we listed this condition for the sake of completeness.

Finally, we check differentiability by solving an LP problem (2.35)–(2.37); if it is solvable, then we have differentiability. By Proposition 2.2, that problem is solvable if and only if the strict complementarity assumption in Cho, Smith, and Friesz (2000) and Yang and Bell (2005) holds. However, our method does not require finding a solution satisfying strict complementarity; any initial route flow can be used for our analysis. Besides, the solution of that LP problem even provides a way to find a route flow satisfying strict complementarity; this is a new result on its own.

In summary, this paper studies semidifferentiability of the equilibrium arc flow and includes the differentiable cases as special cases. Even for differentiability, the assumptions used here are weaker than those in earlier papers; the example in §3.1 would not be diagnosed as differentiable by the methods in Cho, Smith, and Friesz (2000) or Yang and Bell (2005), though as shown here it actually is.

The computational load of the present method is also lower than that of earlier methods. We present two approaches to compute the semiderivatives: the first is by solving a linear traffic user equilibrium problem over the original network, which will be easy to implement for practitioners using transportation software; the second is by solving a linear MCP problem, which will be convenient for people using general mathematical programming software. Computation of the derivatives is by matrix multiplication together with the solution of a linear system whose dimension is at most of the number of arcs, while the gradient formulas in Cho, Smith, and Friesz (2000) and Yang and Bell (2005) require solving linear systems the dimensions of which are at least the number of arcs and may in practice be much more than the number of arcs.

3. Examples

We provide three numerical examples to show how to use the proposed method, how the conditions for application work, and how the network size affects the analysis.

3.1. Example 1

The network shown in Figure 1 consists of four arcs ($\alpha = 4$), two OD pairs ($\omega = 2$), and four routes ($\pi = 4$). We treat the travel demand $d \in \mathbb{R}^2$ as a perturbation parameter and introduce another perturbation parameter $u \in \mathbb{R}^4$ into the arc cost function.

The incidence matrices in this problem are

$$\Lambda = \begin{array}{c} \\ \text{OD1} \\ \text{OD2} \end{array} \begin{array}{c} \begin{array}{cccc} \text{route 1} & \text{route 2} & \text{route 3} & \text{route 4} \end{array} \\ \left(\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array} \right) \end{array},$$

and

$$\Delta = \begin{array}{c} \\ \text{arc 1} \\ \text{arc 2} \\ \text{arc 3} \\ \text{arc 4} \end{array} \begin{array}{c} \begin{array}{cccc} \text{route 1} & \text{route 2} & \text{route 3} & \text{route 4} \end{array} \\ \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) \end{array}.$$

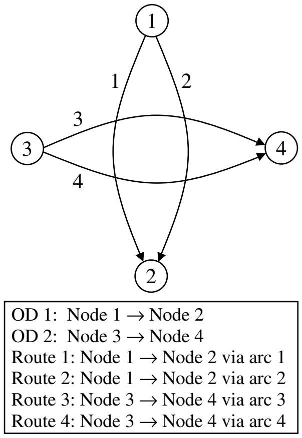

Figure 1 **Example Network**

The parametric arc cost function is given by

$$f(u,x)=\begin{bmatrix}1&0.5&1&0\\0.5&1&0&1\\1&0&1&0.5\\0&1&0.5&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}+\begin{bmatrix}20\\20\\20\\20\end{bmatrix}+\begin{bmatrix}u_1\\u_2\\u_3\\u_4\end{bmatrix}.$$

Given base parameters $u^0=[0,0,0,0]^T$ and $d^0=[20,20]^T$, we show how to conduct sensitivity analysis for this problem step by step.

Step 1. Check if $d^0>0$. If not, exclude OD pairs w with $d^0_w=0$ from the model.

In this example $d^0>0$ clearly holds.

Step 2. Solve the problem (0.7) under (u^0,d^0) to obtain an equilibrium arc flow x^0 and a route flow q^0 in $H(d^0)$ with $x^0=\Delta q^0$. Compute route cost c^0 and OD cost ψ^0.

For this problem, it is easy to check that $x^0=[10,10,10,10]^T$ is an equilibrium arc flow with the arc cost $f(u^0,x^0)=[45,45,45,45]^T$. Here the feasible route flow corresponding to x^0 is unique and given by $q^0=[10,10,10,10]^T$. Accordingly, $c^0=[45,45,45,45]^T$ and $\psi^0=[45,45]^T$.

Step 3. Partition the route set by (2.8)–(2.9), and the matrices Λ and Δ by (2.10). Let A be a matrix the columns of which form a basis for the column space of $\Delta_2-\Delta_1\Lambda_2$, B be a matrix that consists of columns in $\Delta_{00}-\Delta_1\Lambda_{00}$ that are not in the column space of A, $\tilde{A}$ be a matrix the columns of which form a basis for the column space of $[A,B]$, and $\tilde{L}$ be the matrix representing the linear operator $d_xf(u^0,x^0)$. Compute $\tilde{A}^T\tilde{L}\tilde{A}$.

Following (2.9), we partition the route set $\mathcal{P}$ into subsets $\mathcal{P}^{00}=\mathcal{P}^{01}=\varnothing$, $\mathcal{P}^1=\{1,3\}$, and $\mathcal{P}^2=\{2,4\}$. Note that the choices of $\mathcal{P}^1$ and $\mathcal{P}^2$ are not unique, as there are multiple choices for the function $p(w)$ in (2.9). Correspondingly, we partition the matrices Λ and Δ as

$$\begin{bmatrix}\Lambda\\\Delta\end{bmatrix}=\begin{bmatrix}\Lambda_1&\Lambda_2\\\Delta_1&\Delta_2\end{bmatrix}=\left[\begin{array}{cc|cc}1&0&1&0\\0&1&0&1\\\hline 1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{array}\right],$$

with the matrices Δ_{00}, Δ_{01}, Λ_{00}, and Λ_{01} empty. Accordingly, the matrix B is empty, and matrices A, $\tilde{A}$, $\tilde{L}$, and $\tilde{A}^T\tilde{L}\tilde{A}$ are given by

$$A=\tilde{A}=\begin{bmatrix}-1&0\\1&0\\0&-1\\0&1\end{bmatrix},\quad \tilde{L}=\begin{bmatrix}1&0.5&1&0\\0.5&1&0&1\\1&0&1&0.5\\0&1&0.5&1\end{bmatrix}$$

$$\text{and}\quad \tilde{A}^T\tilde{L}\tilde{A}=\begin{bmatrix}1&2\\2&1\end{bmatrix}. \tag{3.1}$$

Step 4. Check if either of the following two conditions hold: (a) The matrix $\tilde{A}^T\tilde{L}\tilde{A}$ is positive definite; (b) C1.1 and C1.2 both hold. If either of these hold, conclude the existence, local uniqueness, and Lipschitz continuity of the equilibrium arc flow $x(u,d)$, as (u,d) varies locally around (u^0,d^0), and proceed to next step. If neither of these hold, terminate the analysis without any conclusion about properties of the parametric equilibrium arc flow.

The matrix $\tilde{A}^T\tilde{L}\tilde{A}$ given in (3.1) is not positive definite, so we need to check C1.1 and C1.2. In this example, the matrix $A^T\tilde{L}A$ is equal to $\tilde{A}^T\tilde{L}\tilde{A}$, which is nonsingular, so C1.1 holds. The matrix $B^T\tilde{L}B-B^T\tilde{L}A(A^T\tilde{L}A)^{-1}A^T\tilde{L}B$ is empty, so C1.2 trivially holds. As a result, the equilibrium arc flow $x(u,d)$ is a locally unique and Lipschitz continuous function of (u,d) as (u,d) varies locally around (u^0,d^0).

Step 5. Check the subspace property by first checking if the matrix B is empty. If so, go to Step 6; otherwise try to solve the LP problem (2.35)–(2.37). If it is solvable, go to Step 6; otherwise, go to Step 7.

In this example the matrix B is empty, so the subspace property holds.

Step 6. Complete the analysis with the conclusion that the function $x(\cdot,\cdot)$ is differentiable at (u^0,d^0) with Jacobian matrices given by (2.32) and (2.33).

For this example the Jacobian matrices computed from (2.32) and (2.33) are

$$x_u(u^0,d^0)=\begin{bmatrix}1/3&-1/3&-2/3&2/3\\-1/3&1/3&2/3&-2/3\\-2/3&2/3&1/3&-1/3\\2/3&-2/3&-1/3&1/3\end{bmatrix},$$

$$x_d(u^0,d^0)=\begin{bmatrix}1/2&0\\1/2&0\\0&1/2\\0&1/2\end{bmatrix}.$$

Step 7. Complete the analysis with the conclusion that the function $x(\cdot,\cdot)$ is semidifferentiable at (u^0,d^0), with $dx(u^0,d^0)(r,s)=z(r,s)+\Delta_1 s$ for any given direction $(r,s)\in\mathbb{R}^m\times\mathbb{R}^{\omega}$. Computation of $z(r,s)$ can be done by solving the linear traffic user equilibrium problem (2.27)–(2.29) or by solving the linear MCP (2.30).

Sensitivity analysis for the present example ends with Step 6, so it is unnecessary to proceed to Step 7.

We conclude this subsection with a set of remarks on the Jacobian matrices we obtain. Elements of the matrix $x_d(u^0,d^0)$ show the changes of the equilibrium arc flow in response to changes in the travel demand. The result fits one's expectation that by symmetry the arcs connecting the same OD pair will share the change in the travel demand equally. Elements of the matrix $x_u(u^0, d^0)$ show the changes of the equilibrium arc flow in response to changes in the free-flow arc costs. For example, the first column shows the change of the arc flow when the free-flow cost on arc 1 is slightly increased. It may look strange that the flow on arc 1 increases as the free-flow cost of this arc increases. It is the special arc cost function here that causes this result. Note that the cost on arc 1 increases as a result of the increase of the free-flow cost and the flow on arc 1. But this increase is partly counteracted by the decrease of the flow on arcs 2 and 3. On the other hand, as a net effect of the decrease of the flow on arc 2 and the increase of the flow on arc 1 and arc 4, the cost on arc 2 increases by the same amount as arc 1, so that the costs on arcs 1 and 2 are again equal at the new equilibrium.

3.2. Example 2

We use the Sioux Falls test network (see http://www.bgu.ac.il/~bargera/tntp/ for the network configuration), which contains 76 arcs ($\alpha = 76$) and 24 nodes. When unperturbed, positive travel demand exists between 528 OD pairs. We include these OD pairs in our model ($\omega = 528$). We treat the travel demand $d \in \mathbb{R}^{528}$ as a perturbation parameter and introduce another perturbation parameter $u \in \mathbb{R}^{76}$ into the arc cost function:

$$f_a(u, x) = (FFT_a + u_a) \times \left[1 + 0.15\left(\frac{x_a}{CAP_a}\right)^4\right] \quad \text{for each } a \in \mathcal{A}, \quad (3.2)$$

where FFT_a and CAP_a are the free-flow cost and the capacity of arc a, respectively. Setting d^0 to be the unperturbed travel demand and u^0 to be zero, we follow the same steps as in Example 1.

Step 1. Here $d^0 > 0$ holds by the setup of the model.

Step 2. We obtain the unperturbed equilibrium arc flow x^0 by applying the origin-based traffic assignment algorithm (OBA) proposed in Bar-Gera (2002), which automatically generates a complete list of all the user optimal routes. Reports from this algorithm also provide the route cost c^0, the OD cost ψ^0, and a route flow q^0. There are 770 user optimal routes, of which 63 are unused, between those 528 OD pairs; we include them in our model ($\pi = 770$). It is safe for us to exclude the routes that are not user optimal from the model, because if we had included them in the model, we would put them into the set $\mathcal{P}^{01}$ and then leave them out of all calculations.

Step 3. A partition of the route set following (2.9) produces an empty $\mathcal{P}^{01}$, a subset $\mathcal{P}^{00}$ containing 63 unused routes, a subset $\mathcal{P}^{1}$ containing 528 used routes (one for each OD pair), and a subset $\mathcal{P}^{2}$ containing 179 used routes. We find the matrix A to be in $\mathbb{R}^{76\times 29}$ and the matrix B to be empty, so the matrix $\tilde{A}$ is equal to A. The matrix $\tilde{A}^T\tilde{L}\tilde{A}$ is of dimension 29×29.

Step 4. As matrix B is empty, C1.2 trivially holds. We only need to check C1.1. The matrix $A^T\tilde{L}A$ is equal to $\tilde{A}^T\tilde{L}\tilde{A}$ in this example and is nonsingular; thus, C1.1 holds. We conclude that the equilibrium arc flow $x(u, d)$ is a locally unique and Lipschitz continuous function of (u, d), as (u, d) varies locally around (u^0, d^0).

Step 5. Matrix B is empty, so the subspace property holds. Note that Proposition 2.2 then implies the existence of a route flow q^* satisfying strict complementarity, although q^0 itself does not satisfy it.

Step 6. The function $x(\cdot, \cdot)$ is differentiable at (u^0, d^0). In calculating the partial derivatives using (2.32) and (2.33), the principal computation involved is the calculation of LU factors of $\tilde{A}^T\tilde{L}\tilde{A}$ followed by back substitution. As the matrix to be factored is 29×29, this workload is satisfactory, considering the dimensions of the network dealt with.

The analysis is now complete; a few remarks follow. Elements of matrices $x_u(u^0, d^0)$ and $x_d(u^0, d^0)$ reflect the changes of the equilibrium arc flow in response to changes in the free-flow arc costs and the travel demand, respectively. Those matrices are too large to be presented here. However, it may be of some interest to comment on the values of their entries. Elements of $x_d(u^0, d^0)$ range from −0.9 to 1.4, with about 77% of them lying between −0.1 and 0.1. All diagonal elements in $x_u(u^0, d^0)$ have nonpositive values; that is, the flow on each arc decreases or remains at the same level when only the free-flow cost of this arc increases.

Finally, we present an example comparison of actual perturbed arc flows to the first-order approximation obtained using the partial derivatives. For this comparison we changed the travel demand from Node 3 to Node 10 from 300 to 310 and the free-flow cost on arc 1 from 6 to 6.2 and reapplied the traffic assignment algorithm to obtain the actual perturbed equilibrium arc flows.

For each arc, there are two relative errors of interest. The first is the difference of the true perturbed flow and the first-order approximation, divided by the true perturbed flow: This is the error of approximation relative to the true perturbed flow. These relative errors ranged from −0.01% to 0.01%, so the first-order approximation was very accurate.

One could also look at that same difference divided by the true difference (between the perturbed and unperturbed flows); this is the error in the *flow change* computed by approximation, relative to the true flow change. These errors ranged from −2.3% to 2.1%. They are larger than the relative errors in flow because the flows are much larger than the differences.

3.3. Example 3

We use the Anaheim network (see http://www.bgu.ac.il/~bargera/tntp/ again for network data) as another example. It contains 914 arcs ($\alpha = 914$) and 416 nodes. When unperturbed, positive travel demand exists between 1,406 OD pairs. We include these OD pairs in our model ($\omega = 1{,}406$). Again, we treat the travel demand $d \in \mathbb{R}^{1{,}406}$ as a parameter and use the parametric arc cost function (3.2), where $u \in \mathbb{R}^{914}$ is another parameter. The travel demand data posted on that website turns out to be too low for the problem to be interesting, so we multiply it by a factor 1.2. Then, setting d^0 to be the unperturbed travel demand and u^0 to be zero, we follow the same procedure as in Example 2.

Step 1. Here $d^0 > 0$ holds by the setup of the model.

Step 2. We obtain x^0, c^0, ψ^0, and q^0 by applying OBA. There are 1,710 user optimal routes (of which 86 are unused); we include them in our model ($\pi = 1{,}710$).

Step 3. A partition of the route set produces an empty $\mathcal{P}^{01}$, a subset $\mathcal{P}^{00}$ containing 86 unused routes, a subset $\mathcal{P}^1$ containing 1,406 used routes (one for each OD pair), and a subset $\mathcal{P}^2$ containing 218 used routes. We find the matrix A to be in $\mathbb{R}^{914\times 42}$ and the matrix B to be empty, so the matrix $\tilde{A}$ is equal to A. The matrix $\tilde{A}^T\tilde{L}\tilde{A}$ is of dimension 42×42.

Step 4. As matrix B is empty, C1.2 trivially holds. The matrix $A^T\tilde{L}A$ is equal to $\tilde{A}^T\tilde{L}\tilde{A}$ in this example and is nonsingular. So C1.1 holds. We conclude that the equilibrium arc flow $x(u, d)$ is a locally unique and Lipschitz continuous function of (u, d) as (u, d) varies locally around (u^0, d^0).

Step 5. Matrix B is empty, so the subspace property holds. Proceed to Step 6.

Step 6. The function $x(\cdot,\cdot)$ is differentiable at (u^0, d^0), and the principal computation involved in calculating the partial derivatives is the calculation of LU factors of the 42×42-dimensional matrix $\tilde{A}^T\tilde{L}\tilde{A}$, followed by back substitution.

To test the result, we changed d^0 to d^1 by increasing the travel demand from Node 1 to Node 2 from 1,639.08 to 1645.08 and reapplied OBA to obtain the perturbed equilibrium arc flow x^1. We then computed $\|x^0\| = 11{,}460$, $\|x^1 - x^0\| = 22.25$, and $\|x^1 - x^0 - x_d(u^0, d^0)(d^1 - d^0)\| = 0.004$. This shows that the first-order approximation provided by the partial derivatives was accurate.

We also changed u^0 to u^2 by increasing the free-flow time of arc 10 from 1 to 1.001 and reapplied OBA to obtain the perturbed equilibrium arc flow x^2. We then computed $\|x^2 - x^0\| = 54.20$ and $\|x^2 - x^0 - x_u(u^0, d^0)(u^2 - u^0)\| = 0.063$. Again, the approximation provided by the partial derivatives was accurate.

4. Conclusion

By applying the sensitivity analysis techniques for generalized equations to the traffic user equilibrium problem, we provide mild and easily checkable conditions for the semidifferentiability and differentiability of the equilibrium arc flow. We calculate the semiderivatives by solving a linear traffic user equilibrium and the derivatives by matrix multiplication together with the solution of a linear system whose dimension is at most the number of arcs.

The sensitivity analysis results provide first-order approximations of new equilibrium solutions and thus help to develop insights about the equilibrium problem. We can use the values of the partial derivatives to identify the most critical input parameters and to quantify the impact of model input errors, or embed the sensitivity analysis process in bilevel programs to optimize design variables such as road price and traffic control parameters. We can also embed the sensitivity analysis process in OD estimation algorithms.

Acknowledgments

The author thanks Professor Stephen M. Robinson for his support, guidance, advice, and overall contribution to this research. The author thanks Professor David Boyce for his comments and suggestions, and Professor Hillel Bar-Gera, who updated the origin-based traffic assignment algorithm to accommodate the requirements of the numerical examples on the Sioux Falls and Anaheim test networks. Finally, the author thanks two anonymous referees and the associate editor for comments that greatly improved this paper. This research was sponsored in part by the National Science Foundation under Grant DMS-0305930, and in part by the Air Force Research Laboratory under Agreement FA9550-04-1-0192.

References

Bar-Gera, H. 2002. Origin-based algorithm for the traffic assignment problem. *Transportation Sci.* **36** 398–417.

Beckmann, M., C. B. Mcguire, C. B. Winsten. 1956. *Studies in the Economics of Transportation*. Yale University Press, New Haven, CT.

Cho, H.-J., T. E. Smith, T. L. Friesz. 2000. A reduction method for local sensitivity analyses of network equilibrium arc flow. *Transportation Res. Part B* **34** 31–51.

Dafermos, S. 1980. Traffic equilibrium and variational inequalities. *Transportation Sci.* **14** 42–54.

Dafermos, S., A. Nagurney. 1984. Sensitivity analysis for the asymmetric network equilibrium problem. *Math. Programming* **28** 174–184.

Dunn, J. C. 1987. On the convergence of projected gradient processes to singular critical points. *J. Optim. Theory Appl.* **55** 203–216.

Facchinei, F., J.-S. Pang. 2003. *Finite-Dimensional Variational Inequalities and Complementarity Problems*. Springer Series in Operations Research, Springer-Verlag, New York.

Hall, M. A. 1978. Properties of the equilibrium state in transportation networks. *Transportation Sci.* **12** 208–216.

Josefsson, M., M. Patriksson. 2007. Sensitivity analysis of separable traffic equilibrium equilibria with application to bilevel optimization in network design. *Transportation Res. Part B* **41** 4–31.

Lu, S., S. M. Robinson. 2008. Variational inequalities over perturbed polyhedral convex sets. *Math. Oper. Res.* Forthcoming.

Marcotte, P., J.-P. Dussault. 1989. A sequential linear programming algorithm for solving monotone variational inequalities. *SIAM J. Control Optim.* **27** 1260–1278.

Outrata, J. V. 1997. On a special class of mathematical programs with equilibrium constraints. P. Gritzmann, R. Horst, E. Sachs, R. Tichatschke, eds. *Recent Advances in Optimization: Proceedings of the Eighth French-German Conference on Optimization*, Vol. 452. *Lecture Notes in Economics and Mathematical Systems*. Springer-Verlag, Berlin, 246–260.

Patriksson, M. 2004. Sensitivity analysis of traffic equilibria. *Transportation Sci.* **38** 258–281.

Patriksson, M., R. T. Rockafellar. 2003. Sensitivity analysis of aggregated variational inequality problems, with application to traffic equilibrium. *Transportation Sci.* **37** 56–68.

Qiu, Y., T. L. Magnanti. 1989. Sensitivity analysis for variational inequalities defined on polyhedral sets. *Math. Oper. Res.* **14** 410–432.

Qiu, Y., T. L. Magnanti. 1992. Sensitivity analysis for variational inequalities. *Math. Oper. Res.* **17** 61–76.

Robinson, S. M. 1980. Strongly regular generalized equations. *Math. Oper. Res.* **5** 43–62.

Robinson, S. M. 1992. Normal maps induced by linear transformations. *Math. Oper. Res.* **17** 691–714.

Robinson, S. M. 1995. Sensitivity analysis of variational inequalities by normal-map techniques. F. Giannessi, A. Maugeri, eds. *Variational Inequalities and Network Equilibrium Problems*. Plenum Press, New York, 257–269.

Robinson, S. M. 1987. Local structure of feasible sets in nonlinear programming, part III: Stability and sensitivity. *Math. Programming Study* **30** 45–66.

Robinson, S. M. 2003. Constraint nondegeneracy in variational analysis. *Math. Oper. Res.* **28** 201–232.

Robinson, S. M. 2006. Strong regularity and the sensitivity analysis of traffic equilibria: A comment. *Transportation Sci.* **40** 540–542.

Robinson, S. M. 2007. Solution continuity in monotone polyhedral variational inequalities. *SIAM J. Optim.* **18** 1046–1060.

Rockafellar, R. T. 1970. *Convex Analysis*. Princeton University Press, Princeton, NJ.

Rockafellar, R. T., R. J-B. Wets. 1998. *Variational Analysis. Grundlehren der mathematischen Wissenschaften*, Vol. 317. Springer-Verlag, Berlin.

Shapiro, A. 1990. On concepts of directional differentiability. *J. Optim. Theory Appl.* **66** 477–487.

Smith, M. J. 1979. The existence, uniqueness and stability of traffic equilibria. *Transportation Res. Part B* **13** 295–304.

Tobin, R. L., T. L. Friesz. 1988. Sensitivity analysis for equilibrium network flow. *Transportation Sci.* **22** 242–250.

Yang, H., M. G. H. Bell. 2005. Sensitivity analysis of network traffic equilibria revisited: The corrected approach. http://ihome.ust.hk/∼cehyang/paper/Yang-Bell-2005-04-30.pdf.

Yen, N. D. 1995. Lipschitz continuity of solutions of variational inequalities with a parametric polyhedral constraint. *Math. Oper. Res.* **20** 695–708.