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Implications of the Constant Rank Constraint Qualification

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Abstract This paper investigates properties of a parametric set defined by finitely many equality and inequality constraints under the constant rank constraint qualification (CRCQ). We show, under the CRCQ, that the indicator function of this set is prox-regular with compatible parametrization, that the set-valued map that assigns each parameter to the set defined by that parameter satisfies a continuity property similar to the Aubin property, and that the Euclidean projector onto this set is a piecewise smooth function. We also show in the absence of parameters that the CRCQ implies the Mangasarian-Fromovitz constraint qualification to hold in some alternative expression of the set.

Keywords Parametric constraints · constant rank constraint qualification · prox-regular · set-valued map · projector

1 Introduction

Consider a parametric set

$$S(u) = \{x \in \mathbb{R}^n \mid f_i(x,u) \leq 0, \quad i \in I, \quad f_i(x,u) = 0, \quad i \in J\} \tag{1}$$

where u is a vector in $\mathbb{R}^m$, I and J are disjoint finite index sets, and $f_i(x,u)$ for each $i \in I \cup J$ is a function from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}$. Unless explicitly stated otherwise, we assume that each $f_i(x,u)$ is continuously differentiable on an open set $\bar{X} \times \bar{U}$ in $\mathbb{R}^n \times \mathbb{R}^m$. Let

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$(\bar{x}, \bar{u}) \in \bar{X} \times \bar{U}$ be a point satisfying $\bar{x} \in S(\bar{u})$; this paper focuses on the behavior around $(\bar{x}, \bar{u})$ of the multifunction S under the *constant rank constraint qualification* (CRCQ) [11,38].

To introduce the definition of the CRCQ, denote the index set of active constraints for a pair (x, u) satisfying $x \in S(u)$ by

$$I(x, u) = \{i \in I \cup J : f_i(x, u) = 0\}. \tag{2}$$

Definition 1 The CRCQ holds at $(\bar{x}, \bar{u})$ if there exist neighborhoods X of $\bar{x}$ in $\bar{X}$ and U of $\bar{u}$ in $\bar{U}$ such that for each $K \subset I(\bar{x}, \bar{u})$ the family $\{\nabla_x f_i(x, u), i \in K\}$ is of constant rank on $X \times U$.

The CRCQ condition defined above is one of the various constraint qualifications used in analysis and computation of nonlinear programs and variational conditions; see, e.g., [1], [7, Section 3.2] and [17] for reviews of constraint qualifications. Among these constraint qualifications, the *linear independence constraint qualification* (LICQ) and the *Mangasarian-Fromovitz constraint qualification* (MFCQ) [9,20] are the best known. Each of these two conditions (or their generalized versions) leads to special properties of local structure of the set $S(u)$, see [27,30]. Extensive study exists on sensitivity and stability of nonlinear programs and variational conditions under the LICQ or MFCQ conditions; see [5,8,10,13–15,18,28,31,37,39] and references therein. Roughly speaking, when combined with suitable second-order assumptions, the LICQ implies solution Lipschitz continuity and B-differentiability of parametric nonlinear programs, while the MFCQ implies solution continuity and directional differentiability but not Lipschitz continuity, see [29] for an example. Similar conclusions hold for variational conditions if one replaces the second-order assumptions by suitable coherent orientation conditions. The MFCQ/CRCQ combination is an intermediate assumption, stronger than the MFCQ but weaker than the LICQ. Ralph and Dempe showed in [26] that the solution to a parametric nonlinear program is piecewise smooth, hence Lipschitz continuous and B-differentiable, under a combination of the MFCQ, CRCQ and a second-order condition; see also [5,15,17]. Luo, Pang and Ralph provided in [19, Section 4.2] a parallel result for variational inequalities based on some results in [21]; see also [7, Section 5.4].

The objective of this paper is to provide some new insights into the behavior of the multifunction S under the CRCQ assumption alone. We are particularly interested in those aspects that are closely related to solution analysis of nonlinear programs and variational conditions; in future study we will explore how to apply results of this paper to analyze sensitivity and stability of such problems under the CRCQ.

The section below, Section 2, contains some preliminary results about the CRCQ. Following that, in Section 3 we suppress the parameter u and show that in the absence of perturbations if the CRCQ holds then the MFCQ also holds in some alternative expression of the set. In Sections 4 to 6 we bring the parameter u back and investigate several aspects regarding the behavior of $S(u)$. Section 4 shows that under the CRCQ the indicator function of $S(u)$ is prox-regular with compatible parametrization. Section 5 proves that under the CRCQ the multifunction S satisfies a continuity property similar to the Aubin property; this result extends [11, Proposition 2.5]. Finally, Section 6 combines the prox-regularity in Section 4 and the continuity in Section 5 to show that under the CRCQ the Euclidean projector on $S(u)$ is locally a single-valued piecewise smooth function.

Except where we explicitly state otherwise, we use $\|\cdot\|$ to denote the Euclidean norm and $\mathbb{B}$ to denote the closed unit ball, and all projectors and balls will be Euclidean. We use the notation f_K to denote the function consisting of f_i for indices i in a set $K \subset I \cup J$, $|K|$ to denote the cardinality of K, $\operatorname{sgn} x$ to denote the sign of a real number x: $\operatorname{sgn} x = 1$ (or $0, -1$) if $x > 0$ (or $= 0, < 0$).

2 Preliminaries

In this section, we present some technical results that will be useful in the rest of this paper. These results concern how the functions f_i for $i \in I(\bar{x}, \bar{u})$ are interrelated under the CRCQ assumption.

The proof of the following lemma is similar to the proof of the constant rank theorem (see, e.g., [4, Theorem 2.4.6]).

Lemma 1 *Assume that the CRCQ holds at $(\bar{x}, \bar{u})$. Let K_1 and K_2 be two disjoint subsets of $I(\bar{x}, \bar{u})$, such that $\nabla_x f_{K_1}(\bar{x}, \bar{u})$ is of full row rank, and that the row space of $\nabla_x f_{K_2}(\bar{x}, \bar{u})$ is contained in that of $\nabla_x f_{K_1}(\bar{x}, \bar{u})$. Let $|K_1| = k$. Partition the variable x as $x = (x_1, x_2)$, with $x_1 \in \mathbb{R}^k$ and $x_2 \in \mathbb{R}^{n-k}$, such that $\nabla_{x_1} f_{K_1}(\bar{x}, \bar{u})$ is a nonsingular $k \times k$ matrix.*

Then there exist neighborhoods X of $\bar{x}$ in $\bar{X}$, U of $\bar{u}$ in $\bar{U}$ and Y of $f_{K_1}(\bar{x}, \bar{u})$ in $\mathbb{R}^k$, and a continuously differentiable function $G : Y \times U \to \mathbb{R}^{|K_2|}$, such that $f_{K_1}(x, u) \in Y$ for each $(x, u) \in X \times U$, with

$$f_{K_2}(x, u) = G(f_{K_1}(x, u), u). \tag{3}$$

Moreover, the Jacobian matrix of G at $(f_{K_1}(x, u), u)$ is given by

$$\begin{aligned} \nabla_{y,u} G(f_{K_1}(x, u), u) = &\big[\nabla_{x_1} f_{K_2}(x, u) \nabla_{x_1} f_{K_1}^{-1}(x, u), \\ &\nabla_u f_{K_2}(x, u) - \nabla_{x_1} f_{K_2}(x, u) \nabla_{x_1} f_{K_1}^{-1}(x, u) \nabla_u f_{K_1}(x, u)\big]. \end{aligned} \tag{4}$$

Finally, if row vectors $\lambda_1 \in \mathbb{R}^k$ and $\lambda_2 \in \mathbb{R}^{|K_2|}$ satisfy

$$\lambda_1 \nabla_x f_{K_1}(x, u) + \lambda_2 \nabla_x f_{K_2}(x, u) = 0 \tag{5}$$

for some $(x, u) \in X \times U$, then

$$\lambda_2 \nabla_y G(f_{K_1}(x, u), u) = -\lambda_1. \tag{6}$$

Proof Let $K = K_1 \cup K_2$; then by hypothesis $\nabla_x f_K(\bar{x}, \bar{u})$ is of rank k. Define a function $F : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ by

$$F(x_1, x_2, u) = (f_{K_1}(x, u), x_2, u),$$

which is continuously differentiable on $\bar{X} \times \bar{U}$, with its Jacobian matrix at (x, u) being

$$\nabla F(x, u) = \begin{bmatrix} \nabla_{x_1} f_{K_1}(x, u) & \nabla_{x_2} f_{K_1}(x, u) & \nabla_u f_{K_1}(x, u) \\ 0 & I_{n-k} & 0 \\ 0 & 0 & I_m \end{bmatrix}.$$

Note that $\nabla F(\bar{x}, \bar{u})$ is nonsingular. By the inverse function theorem, there exist neighborhoods Z of $(\bar{x}, \bar{u})$ in $\bar{X} \times \bar{U}$ and W of $F(\bar{x}, \bar{u})$ in $\mathbb{R}^{n+m}$, such that F is a

local diffeomorphism from Z onto W. By shrinking Z and W if necessary, we may assume that W is a box, that $\nabla_{x_1} f_{K_1}(x,u)$ is nonsingular for each $(x,u) \in Z$, and that $\nabla_x f_K(x,u)$ is of rank k on Z.

Write each element of W as (y, x_2, u). For each $(y, x_2, u) \in W$, there exists a unique $(x,u) = (x_1, x_2, u) \in Z$ such that $(y, x_2, u) = F(x_1, x_2, u)$. We have

$$\nabla F(x,u)^{-1} = \begin{bmatrix} \nabla_{x_1} f_{K_1}^{-1} & -\nabla_{x_1} f_{K_1}^{-1} \nabla_{x_2} f_{K_1} & -\nabla_{x_1} f_{K_1}^{-1} \nabla_u f_{K_1} \\ 0 & I_{n-k} & 0 \\ 0 & 0 & I_m \end{bmatrix},$$

where we suppressed in the right hand side the argument (x,u) for notational simplicity. Now consider the map $f_K \circ F^{-1}$ from W to $\mathbb{R}^{|K|}$; we have

$$\nabla(f_K \circ F^{-1})(y, x_2, u) = \nabla f_K(x,u) \nabla F^{-1}(y, x_2, u) = \nabla f_K(x,u) \nabla F(x,u)^{-1} =$$
$$\begin{bmatrix} I_k & 0 & 0 \\ \nabla_{x_1} f_{K_2} \nabla_{x_1} f_{K_1}^{-1} & \nabla_{x_2} f_{K_2} - \nabla_{x_1} f_{K_2} \nabla_{x_1} f_{K_1}^{-1} \nabla_{x_2} f_{K_1} & \nabla_u f_{K_2} - \nabla_{x_1} f_{K_2} \nabla_{x_1} f_{K_1}^{-1} \nabla_u f_{K_1} \end{bmatrix}. \tag{7}$$

In establishing (7) we also note that

$$\begin{aligned} &\begin{bmatrix} \nabla_{x_1} f_{K_1} & \nabla_{x_2} f_{K_1} \\ \nabla_{x_1} f_{K_2} & \nabla_{x_2} f_{K_2} \end{bmatrix} \begin{bmatrix} \nabla_{x_1} f_{K_1}^{-1} & -\nabla_{x_1} f_{K_1}^{-1} \nabla_{x_2} f_{K_1} \\ 0 & I_{n-k} \end{bmatrix} \\ &= \begin{bmatrix} I_k & 0 \\ \nabla_{x_1} f_{K_2} \nabla_{x_1} f_{K_1}^{-1} & \nabla_{x_2} f_{K_2} - \nabla_{x_1} f_{K_2} \nabla_{x_1} f_{K_1}^{-1} \nabla_{x_2} f_{K_1} \end{bmatrix}, \end{aligned}$$

which is of rank k by the way we chose Z. Consequently,

$$\nabla_{x_2} f_{K_2} - \nabla_{x_1} f_{K_2} \nabla_{x_1} f_{K_1}^{-1} \nabla_{x_2} f_{K_1} = 0. \tag{8}$$

It follows from (7) and (8) that $\nabla_{x_2}(f_K \circ F^{-1})(y, x_2, u) = 0$ for each $(y, x_2, u) \in W$. Because W is a box, it is in particular convex. Consequently,

$$f_K \circ F^{-1}(y, x_2, u) = f_K \circ F^{-1}(y, x_2', u) \tag{9}$$

for each (y, x_2, u) and (y, x_2', u) in W.

Next, choose neighborhoods X of $\bar{x}$, U of $\bar{u}$ and Y of $f_{K_1}(\bar{x}, \bar{u})$, such that $X \times U \subset Z$, $Y \times \{\bar{x}_2\} \times U \subset W$, and $f_{K_1}(x,u) \in Y$ for each $(x,u) \in X \times U$. For each $(y,u) \in Y \times U$, define

$$G(y,u) = f_{K_2} \circ F^{-1}(y, \bar{x}_2, u), \tag{10}$$

which is a continuously differentiable function of (y,u) because F^{-1} is continuously differentiable on W. For each $(x,u) \in X \times U$, we have

$$\begin{aligned} f_{K_2}(x,u) &= f_{K_2} \circ F^{-1}(f_{K_1}(x,u), x_2, u) \\ &= f_{K_2} \circ F^{-1}(f_{K_1}(x,u), \bar{x}_2, u) = G(f_{K_1}(x,u), u), \end{aligned}$$

where the second equality follows from (9) because both $(f_{K_1}(x,u), x_2, u)$ and $(f_{K_1}(x,u), \bar{x}_2, u)$ belong to W. This proves (3). Moreover, the Jacobian matrix of G at $(f_{K_1}(x,u), u)$ is given by

$$\begin{aligned} \nabla_{y,u} G(f_{K_1}(x,u), u) =& \nabla_{y,u}(f_{K_2} \circ F^{-1})(f_{K_1}(x,u), \bar{x}_2, u) \\ =& \nabla_{y,u}(f_{K_2} \circ F^{-1})(f_{K_1}(x,u), x_2, u) \\ =& \begin{bmatrix} \nabla_{x_1} f_{K_2} \nabla_{x_1} f_{K_1}^{-1} & \nabla_u f_{K_2} - \nabla_{x_1} f_{K_2} \nabla_{x_1} f_{K_1}^{-1} \nabla_u f_{K_1} \end{bmatrix}, \end{aligned}$$

where the first, second and third equalities follow from (10), (9) and (7) respectively. This proves (4).

Finally, if (5) holds for some $(x, u) \in X \times U$, then

$$\begin{aligned} &\lambda_2 \ \nabla_y G(f_{K_1}(x,u), u) = \lambda_2 \ \nabla_{x_1} f_{K_2}(x,u) \ \nabla_{x_1} f_{K_1}^{-1}(x,u) \\ &= -\lambda_1 \ \nabla_{x_1} f_{K_1}(x,u) \ \nabla_{x_1} f_{K_1}^{-1}(x,u) = -\lambda_1, \end{aligned}$$

where the first and second equalities follow from (4) and (5) respectively. This proves (6). □

The following corollary applies Lemma 1 to the special case in which K_2 contains a single element.

Corollary 1 *Assume that the CRCQ holds at $(\bar{x}, \bar{u})$. Let K be a subset of $I(\bar{x}, \bar{u})$ such that $\nabla_x f_K(\bar{x}, \bar{u})$ is of full row rank. Suppose that $j \in I(\bar{x}, \bar{u}) \setminus K$ satisfies*

$$\nabla_x f_j(\bar{x}, \bar{u}) = \sum_{i \in K} \bar{\lambda}_i \nabla_x f_i(\bar{x}, \bar{u}) \tag{11}$$

for some $\bar{\lambda} \in \mathbb{R}^{|K|}$. Then there exist neighborhoods X of $\bar{x}$ in $\bar{X}$, U of $\bar{u}$ in $\bar{U}$, Y of $f_K(\bar{x}, \bar{u})$ in $\mathbb{R}^{|K|}$, with Y being convex, and a continuously differentiable function $g : Y \times U \to \mathbb{R}$, such that $f_K(x, u) \in Y$ and $f_j(x, u) = g(f_K(x, u), u)$ for each $(x, u) \in X \times U$. Moreover, for each $i \in K$ and each $(y, u) \in Y \times U$,

$$\operatorname{sgn} \frac{\partial}{\partial y_i} g(y, u) = \operatorname{sgn} \bar{\lambda}_i. \tag{12}$$

Proof Apply Lemma 1 with sets $K_1 = \{i \in K \mid \bar{\lambda}_i \neq 0\}$ and $K_2 = \{j\}$ to obtain neighborhoods X of $\bar{x}$ in $\bar{X}$, U of $\bar{u}$ in $\bar{U}$, V of $f_{K_1}(\bar{x}, \bar{u})$ in $\mathbb{R}^{|K_1|}$, and a continuously differentiable function $h : V \times U \to \mathbb{R}$ such that $f_{K_1}(x, u) \in V$ and $f_j(x, u) = h(f_{K_1}(x, u), u)$ for each $(x, u) \in X \times U$. In view of (11), it follows from (6) that

$$\frac{\partial}{\partial v_i} h(f_{K_1}(\bar{x}, \bar{u}), \bar{u}) = \bar{\lambda}_i$$

for each $i \in K_1$. The definition of K_1 implies that $\bar{\lambda}_i \neq 0$ for each $i \in K_1$; by shrinking X, U and V if necessary, we may assume that V is convex, and that

$$\operatorname{sgn} \frac{\partial}{\partial v_i} h(v, u) = \operatorname{sgn} \bar{\lambda}_i$$

for each $i \in K_1$ and each $(v, u) \in V \times U$.

Next, let $K_0 = K \setminus K_1$. Let $Y = \mathbb{R}^{|K_0|} \times V$, and write each $y \in Y$ as $y = (w, v)$ with $w \in \mathbb{R}^{|K_0|}$ and $v \in V$. Define a continuously differentiable function $g : Y \times U \to \mathbb{R}$ by

$$g(y, u) = g(w, v, u) := h(v, u).$$

For each $(x, u) \in X \times U$, we have $f_K(x, u) = (f_{K_0}(x, u), f_{K_1}(x, u)) \in Y$ and

$$f_j(x, u) = h(f_{K_1}(x, u), u) = g(f_K(x, u), u).$$

For each $(y, u) \in Y \times U$, we have

$$\operatorname{sgn} \frac{\partial}{\partial y_i} g(y, u) = \operatorname{sgn} \frac{\partial}{\partial v_i} h(v, u) = \operatorname{sgn} \bar{\lambda}_i$$

for each $i \in K_1$, and $\frac{\partial}{\partial y_i} g(y, u) = 0 = \bar{\lambda}_i$ for each $i \in K_0$. This proves (12). □

3 The CRCQ and the MFCQ in the absence of perturbations

This section concerns the relation between the CRCQ and the MFCQ. The MFCQ is defined as follows.

Definition 2 The MFCQ holds at $(\bar{x}, \bar{u})$ if

(a) there exists a vector $w \in \mathbb{R}^n$ such that

$$\begin{aligned} \langle \nabla_x f_i(\bar{x}, \bar{u}), w \rangle &< 0,\ i \in I(\bar{x}, \bar{u}) \cap I, \\ &= 0,\ i \in J; \end{aligned}$$

(b) the family $\{\nabla_x f_i(\bar{x}, \bar{u}), \quad i \in J\}$ has full rank $|J|$.

It was shown in [11] by two examples that the CRCQ is neither weaker nor stronger than the MFCQ when one takes the perturbation parameter u into account. In the rest of this section, we suppress the parameter u, and focus on the relation between the CRCQ and the MFCQ in the absence of perturbations. Let

$$S = \{x \in \mathbb{R}^n \mid f_i(x) \leq 0, i \in I, \quad f_i(x) = 0, i \in J\}, \tag{13}$$

where I and J are disjoint finite index sets, and f_i for each $i \in I \cup J$ is a continuously differentiable function from an open set $\bar{X}$ in $\mathbb{R}^n$ to $\mathbb{R}$. The main result of this section is that if the CRCQ holds at a point $\bar{x} \in S \cap \bar{X}$, then we can find an alternative expression for S by eliminating some indices from both I and J, and at the same time moving some indices from I to J, so that the set S is locally unchanged and the MFCQ holds at $\bar{x}$ under the new expression.

The following proposition shows that under a certain condition some inequality constraints in the definition of S have to hold as equations. We use $I(\bar{x})$ to denote the index set of active constraints at $\bar{x}$.

Proposition 1 *Assume that the CRCQ holds at $\bar{x}$. Let K be a subset of $I(\bar{x}) \cap I$ and J_0 be a subset of J, such that there exist a $|K|$-dimensional row vector $\lambda > 0$ and a $|J_0|$-dimensional row vector γ with*

$$\lambda \nabla f_K(\bar{x}) + \gamma \nabla f_{J_0}(\bar{x}) = 0. \tag{14}$$

Then there exists a neighborhood X of $\bar{x}$ in $\bar{X}$ such that $f_K(x) = 0$ for each $x \in X \cap S$.

Proof First, assume without loss of generality that $\nabla f_{J_0}(\bar{x})$ is of full row rank, because if it were not then we could replace J_0 by a subset of it.

Next, partition K as $K = K_1 \cup K_2$, such that the set $K_0 := K_1 \cup J_0$ satisfies (i) the matrix $\nabla f_{K_0}(\bar{x})$ is of full row rank, (ii) the row space of $\nabla f_{K_2}(\bar{x})$ is contained in that of $\nabla f_{K_0}(\bar{x})$. Partition λ as $\lambda = (\lambda_1, \lambda_2)$ corresponding to the partition of K. Then we may rewrite (14) as

$$\lambda_1 \nabla f_{K_1}(\bar{x}) + \lambda_2 \nabla f_{K_2}(\bar{x}) + \gamma \nabla f_{J_0}(\bar{x}) = 0.$$

By Lemma 1, there exist neighborhoods X of $\bar{x}$ in $\bar{X}$ and Y of $f_{K_0}(\bar{x})$ in $\mathbb{R}^{|K_0|}$, and a continuously differentiable function G from Y to $\mathbb{R}^{|K_2|}$, such that $f_{K_0}(x) \in Y$ for each $x \in X$, with

$$f_{K_2}(x) = G(f_{K_0}(x)). \tag{15}$$

Moreover, the Jacobian matrix of G at $f_{K_0}(\bar{x})$ satisfies

$$\lambda_2 \nabla G(f_{K_0}(\bar{x})) = -\left[\lambda_1 \ \gamma\right].$$

Because $\lambda_1 > 0$, the function $g(\cdot) = \lambda_2 G(\cdot)$ has negative partial derivatives at $f_{K_0}(\bar{x})$ with respect to the first $|K_1|$ variables. By shrinking X and Y if necessary, we may assume that Y is convex, and that the function $g(\cdot)$ has negative partial derivatives with respect to the first $|K_1|$ variables at each $y \in Y$.

The definitions of K_0 and K_2 imply that $f_{K_0}(\bar{x}) = 0$ and $f_{K_2}(\bar{x}) = 0$, so $G(0) = 0$ and $g(0) = 0$. Suppose that $f_{K_1}(x) \neq 0$ for some $x \in X \cap S$. It follows from the mean value theorem that

$$g(f_{K_0}(x)) = g(f_{K_0}(x)) - g(0) = g'(y)(f_{K_0}(x))$$

for some y lying between 0 and $f_{K_0}(x)$. The choice of x implies that $f_{K_1}(x) \leq 0$, $f_{K_1}(x) \neq 0$, $f_{J_0}(x) = 0$ and $f_{K_0}(x) \in Y$. The way we chose Y implies that $y \in Y$ because Y is convex, and that the first $|K_1|$ components of $g'(y)$ are negative. Consequently, $g(f_{K_0}(x)) > 0$. On the other hand,

$$g(f_{K_0}(x)) = \lambda_2 G(f_{K_0}(x)) = \lambda_2 f_{K_2}(x),$$

where the first equality holds by the definition of g and the second holds by (15). The fact $x \in S$ implies that $f_{K_2}(x) \leq 0$, and we have $\lambda_2 > 0$ by hypothesis. It follows that $g(f_{K_0}(x)) \leq 0$, a contradiction. This proves that $f_{K_1}(x) = 0$ for each $x \in X \cap S$; as a result, we also have $f_{K_2}(x) = G(0) = 0$ for each such x. □

Below is the main result of this section.

Theorem 1 *Assume that the CRCQ holds at $\bar{x}$. There exist disjoint subsets I' and I^* of I, and a subset J^* of J, such that the set*

$$S^* = \{x \in \mathbb{R}^n \mid f_i(x) \leq 0, i \in I', \quad f_i(x) = 0, i \in J^* \cup I^*\} \tag{16}$$

locally coincides with S around $\bar{x}$, and the MFCQ holds at $\bar{x} \in S^*$.

Proof Let $\mathcal{K}$ be the collection of all subsets K of $I(\bar{x}) \cap I$ such that there exist a $|K|$-dimensional row vector $\lambda > 0$ and a $|J|$-dimensional row vector γ with

$$\lambda \nabla f_K(\bar{x}) + \gamma \nabla f_J(\bar{x}) = 0.$$

Define subsets I_1 and I_2 of $I(\bar{x}) \cap I$ as

$$I_2 = \bigcup_{K \in \mathcal{K}} K \quad \text{and} \quad I_1 = (I(\bar{x}) \cap I) \setminus I_2.$$

By Proposition 1, for each $K \in \mathcal{K}$ there exists a neighborhood X_K of $\bar{x}$ in $\bar{X}$ such that $f_K(\cdot) \equiv 0$ on $X_K \cap S$. Choose a neighborhood X_1 of $\bar{x}$ in $\bar{X}$ such that $X_1 \subset X_K$ for each K and that the inactive constraints at $\bar{x}$ remain inactive in X_1. We have

$$S \cap X_1 = \{x \in X_1 \mid f_i(x) \leq 0, i \in I_1, \quad f_i(x) = 0, i \in J \cup I_2\}.$$

We claim that if a row vector $(\lambda_1, \lambda_2, \gamma) \in \mathbb{R}^{|I_1|} \times \mathbb{R}^{|I_2|} \times \mathbb{R}^{|J|}$ satisfies $\lambda_1 \geq 0$ and

$$\lambda_1 \nabla f_{I_1}(\bar{x}) + \lambda_2 \nabla f_{I_2}(\bar{x}) + \gamma \nabla f_J(\bar{x}) = 0, \tag{17}$$

then $\lambda_1 = 0$.

Below we prove this claim by contradiction. Suppose that a triple $(\lambda_1, \lambda_2, \gamma)$ satisfies $\lambda_1 \geq 0$ and (17), with $\lambda_1 \neq 0$. Let i be a given element in I_2. By the way we defined I_2, there exist row vectors $\lambda_2' \geq 0$ and γ' satisfying

$$\lambda_2' \nabla f_{I_2}(\bar{x}) + \gamma' \nabla f_J(\bar{x}) = 0, \tag{18}$$

with the component of λ_2' corresponding to the given i being positive. Hence, by multiplying (18) by a positive large number and then adding it to (17), we obtain another triple $(\lambda_1, \hat{\lambda}_2, \hat{\gamma})$ satisfying (17), with $\lambda_1 \geq 0$ and the component of $\hat{\lambda}_2$ corresponding to the given i being positive. In this process λ_1 is unchanged. Repeating this process for each $i \in I_2$, we finally obtain a triple $(\lambda_1, \tilde{\lambda}_2, \tilde{\gamma})$ satisfying (17), with $\lambda_1 \geq 0$ and $\tilde{\lambda}_2 > 0$. Now define a subset I_3 of I_1 by $I_3 = \{i \in I_1 \mid (\lambda_1)_i > 0\}$. We assumed that $\lambda_1 \neq 0$, so $I_3 \neq \emptyset$. The set $I_3 \cup I_2$ is an element of the collection $\mathcal{K}$, which contradicts with the way we defined I_1 and I_2. This proves the claim.

Next, let K_1 be a maximal subset of $J \cup I_2$ such that $\nabla f_{K_1}(\bar{x})$ is of full row rank. Let j be an element of $J \cup I_2$ that is not in K_1. By Lemma 1, there exist neighborhoods X of $\bar{x}$ and Y of $f_{K_1}(\bar{x})$, and a continuously differentiable function G from Y to $\mathbb{R}$, such that $f_j(x) = G(f_{K_1}(x))$ for each $x \in X$. We have $G(0) = 0$ because $f_j(\bar{x}) = 0$ and $f_{K_1}(\bar{x}) = 0$. Consequently, each $x \in X$ with $f_{K_1}(x) = 0$ satisfies $f_j(x) = 0$. Hence, by making X_1 smaller if necessary, we have

$$S \cap X_1 = \{x \in X_1 \mid f_i(x) \leq 0, i \in I_1, \quad f_i(x) = 0, i \in K_1\}. \tag{19}$$

Finally, define $I' = I_1$, $I^* = I_2 \cap K_1$ and $J^* = J \cap K_1$. It follows from (19) that the set S^* defined in (16) locally coincides with S. It remains to prove that the MFCQ holds at $\bar{x} \in S^*$.

The way we defined I_1 and K_1 implies that (i) the matrix $\nabla f_{K_1}(\bar{x})$ is of full row rank, (ii) any row vector $(\lambda, \gamma) \in \mathbb{R}^{|I_1|} \times \mathbb{R}^{|K_1|}$ satisfying $\lambda \geq 0$ and $\lambda \nabla f_{I_1}(\bar{x}) + \gamma \nabla f_{K_1}(\bar{x}) = 0$ satisfies $\lambda = 0$. This means that the pair

$$(\{\nabla f_i(\bar{x}), i \in I_1\}, \{\nabla f_i(\bar{x}), i \in K_1\})$$

is positive-linearly independent in the sense of [22,25,33], which holds if and only if the MFCQ holds at $\bar{x} \in S^*$, see [27, Theorem 3]. □

Theorem 1 says that the CRCQ implies that the MFCQ holds in an alternative expression for S obtained by changing some inequality constraints to equations and then eliminating some constraints. The converse direction, from the MFCQ to the CRCQ, does not hold in general. For example, if S is the subset of $\mathbb{R}^2$ defined by two constraints $f_1(x, y) = y - x^3 \leq 0$ and $f_2(x, y) = y - x^5 \leq 0$, then the MFCQ holds at the origin of $\mathbb{R}^2$ but the CRCQ does not hold there, and one cannot find an alternative expression for S in which the CRCQ holds by changing inequalities to equations or constraint elimination. Also, Theorem 1 does not extend to situations in which the parameter u exists. For one thing, if the MFCQ holds at $(\bar{x}, \bar{u})$ then $S(u) \neq \emptyset$ for each u sufficiently close to $\bar{u}$, see, e.g., [27, Theorem 1]. Such property of local solvability may not hold under the CRCQ. For example, if for each $u \in \mathbb{R}$ the set $S(u)$ is the subset of $\mathbb{R}^2$ defined by two constraints $f_1(x, u) = x_1 - u \leq 0$ and $f_2(x, u) = -x_1 - u \leq 0$, then the CRCQ holds trivially at $(\bar{x}, \bar{u}) = (0, 0)$ and $S(u) = \emptyset$ for each $u < 0$.

4 Prox-regularity

In this section, we return to parametric sets $S(u)$ defined in (1). The main result of this section is that if the CRCQ holds at $(\bar{x}, \bar{u})$, then a property called *parametric prox-regularity* also holds there. Later in Section 6 we will use this result to prove properties of the Euclidean projector onto $S(u)$. For more details on prox-regularity and its use in variational analysis, see [16,23,24,35].

We start with a few definitions.

Definition 3 Let C be a subset of $\mathbb{R}^n$. For each point x of C, the *regular normal cone* of C at x, denoted by $\hat{N}_C(x)$, is the cone that consists of all vectors v satisfying

$$\langle v, x' - x \rangle \leq o(\|x' - x\|) \text{ for } x' \in C.$$

The *normal cone* of C at x, denoted by $N_C(x)$, is the cone that consists of all vectors v having the property that there is a sequence $\{x_k\}$ of points of C converging to x, and a sequence $\{v_k\}$ converging to v, in which for each k the vector v_k belongs to $\hat{N}_C(x_k)$. The *tangent cone* of C at x, denoted by $T_C(x)$, is the cone that consists of all vectors w having the property that there is a sequence $\{x_k\}$ of points of C converging to x, and a sequence of scalars τ_k converging to 0, with $[x_k - x]/\tau_k$ converging to w. For points x not belonging to C, all these cones are by convention the empty set. Finally, the set C is *Clarke regular* at $x \in C$ if C is locally closed at x and $N_C(x) = \hat{N}_C(x)$.

For details on the above, see [35]. Note that $N_C(x)$, $\hat{N}_C(x)$ and $T_C(x)$ are all closed cones by their definitions, and that $\hat{N}_C(x)$ is also convex and is indeed the polar of $T_C(x)$.

The following definition on parametric prox-regularity is from [16, Definition 2.1].

Definition 4 Let g be a lower semicontinuous extended real-valued function on $\mathbb{R}^n \times \mathbb{R}^m$, and let $(\bar{x}, \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^m$ with $\bar{v} \in \partial_x g(\bar{x}, \bar{u})$. The function g is *prox-regular in x at $\bar{x}$ for $\bar{v}$ with compatible parametrization by u at $\bar{u}$* if there exist neighborhoods U, V and X of $\bar{u}$, $\bar{v}$ and $\bar{x}$ respectively, with $\epsilon > 0$ and $\rho \geq 0$, such that

$$g(x', u) \geq g(x, u) + \langle v, x' - x \rangle - (\rho/2)\|x' - x\|^2 \tag{20}$$

whenever $x' \in X$, $(x, u, v) \in X \times U \times V$, $v \in \partial_x g(x, u)$ and $g(x, u) \leq g(\bar{x}, \bar{u}) + \epsilon$.

For our application we want the function $g(x, u)$ to be the indicator $\delta_{S(u)}(x)$ of $S(u)$ evaluated at x. The lower semicontinuity of g is equivalent to the closedness of its epigraph, which in this case is

$$\operatorname{epi} g = \operatorname{gph} S \times \mathbb{R}_+.$$

Hence, g is lower semicontinuous if and only if $\operatorname{gph} S$ is closed. The definition of S in (1) and the hypothesis that f is locally continuously differentiable imply that $\operatorname{gph} S$ is locally closed, and this suffices because all the analysis in this paper is local. As for the statement in (20), in the present case it becomes the assertion that

$$\langle v, x' - x \rangle - (\rho/2)\|x' - x\|^2 \leq 0 \tag{21}$$

whenever $x' \in X \cap S(u)$, $(x, u, v) \in X \times U \times V$, $x \in S(u)$ and $v \in N_{S(u)}(x)$.

A prime source of examples for parametric prox-regular functions are so-called *strongly amenable* functions; see [16, Proposition 2.2]. As for the case $g(x, u) = \delta_{S(u)}(x)$

with $S(u)$ defined by (1), the strong amenability property defined in that proposition holds at $(\bar{x}, \bar{u})$ if and only if the MFCQ holds there. Because the CRCQ does not imply the MFCQ in the presence of u, by assuming the CRCQ to hold at $(\bar{x}, \bar{u})$ we cannot guarantee that the function g is strongly amenable. However, as we will show in Theorem 2, the CRCQ does imply that g is parametric prox-regular. This result supplements [16, Proposition 2.2].

The following lemma says that under the CRCQ the tangent cone $T_{S(\bar{u})}(\bar{x})$ equals the cone defined by linearized constraints, that is, the CRCQ implies Abadie's CQ. This was proved in [11, Proposition 2.3] based on the constant rank theorem. We include here for the sake of completeness a simple proof based on Lemma 1.

Lemma 2 *Assume that the CRCQ holds at $(\bar{x}, \bar{u})$. Then*

$$T_{S(\bar{u})}(\bar{x}) = \{v \in \mathbb{R}^n \mid \langle \nabla_x f_i(\bar{x}, \bar{u}), v\rangle \le 0, i \in I(\bar{x}, \bar{u}) \cap I, \\ \langle \nabla_x f_j(\bar{x}, \bar{u}), v\rangle = 0, j \in J\}.$$

Proof It suffices to prove that the set on the right is included in $T_{S(\bar{u})}(\bar{x})$. Suppose that v belongs to the set on the right. Define

$$K = \{i \in I(\bar{x}, \bar{u}) : \langle \nabla_x f_i(\bar{x}, \bar{u}), v\rangle = 0\},$$

and let K_1 be a maximal subset of K such that $\{\nabla_x f_i(\bar{x}, \bar{u}) : i \in K_1\}$ is linearly independent. Write $K_2 = K \setminus K_1$. By Lemma 1 there exist neighborhoods X of $\bar{x}$ in $\bar{X}$ and Y of $f_{K_1}(\bar{x}, \bar{u})$ in $\mathbb{R}^{|K_1|}$ and a continuously differentiable function G from Y to $\mathbb{R}^{|K_2|}$ such that $f_{K_1}(x, \bar{u}) \in Y$ for each $x \in X$, with

$$f_{K_2}(x, \bar{u}) = G(f_{K_1}(x, \bar{u})).$$

Because $\{\nabla_x f_i(\bar{x}, \bar{u}) : i \in K_1\}$ is linearly independent, an application of the implicit function theorem will enable us to find a smooth arc $t \to x(t)$ from an interval $[0, \bar{t}]$ to X such that (i) $x(0) = \bar{x}$, (ii) $x'(0) = v$, (iii) $f_{K_1}(x(t), \bar{u}) = 0$. To see this, we may, for example, partition $x = (x_1, x_2)$ such that $\nabla_{x_1} f_{K_1}(\bar{x})$ is nonsingular, and partition $v = (v_1, v_2)$ accordingly. Then choose $x_2(t) = \bar{x}_2 + v_2 t$, and let $x_1(t)$ be the locally unique solution of $f_{K_1}(x_1, x_2(t)) = 0$.

The arc $x(t)$ satisfies $f_K(x(t), \bar{u}) = 0$ for each $t \in [0, \bar{t}]$, because

$$f_{K_2}(x(t), \bar{u}) = G(f_{K_1}(x(t), \bar{u})) = G(0) = 0.$$

By reducing $\bar{t}$ further if necessary, we have $f_i(x(t), \bar{u}) < 0$ for each $i \in I \setminus K$ and $t \in [0, \bar{t}]$. This proves that $x(t)$ is an arc in $S(\bar{u})$ with derivative v. □

In the corollary below, $\operatorname{pos}\{a_1, \ldots, a_k\}$ for a finite set $\{a_1, \ldots, a_k\}$ is defined as

$$\operatorname{pos}\{a_1, \ldots, a_k\} = \{0\} \cup \{\sum_{i=1}^{k} \tau_i a_i \mid \tau_i \in \mathbb{R}_+\},$$

and $\operatorname{span}\{a_1, \ldots, a_k\}$ is defined similarly with $\tau_i \in \mathbb{R}_+$ replaced by $\tau_i \in \mathbb{R}$. These definitions ensure that $\operatorname{pos} \emptyset = \operatorname{span} \emptyset = \{0\}$.

Corollary 2 *Assume that the CRCQ holds at $(\bar{x}, \bar{u})$. Then*

$$N_{S(\bar{u})}(\bar{x}) = \hat{N}_{S(\bar{u})}(\bar{x}) \\ = \operatorname{pos}\{\nabla_x f_i(\bar{x}, \bar{u}),\ i \in I(\bar{x}, \bar{u}) \cap I\} + \operatorname{span}\{\nabla_x f_j(\bar{x}, \bar{u}),\ j \in J\}.$$

Proof The second equality follows from Lemma 2 and the fact that $\hat{N}_{S(\bar{u})}(\bar{x})$ is the polar of $T_{S(\bar{u})}(\bar{x})$. The first equality holds because there exists a neighborhood X of $\bar{x}$ in $\bar{X}$ such that the CRCQ holds at each $x \in X \cap S(\bar{u})$ and that $I(x, \bar{u})$ is a subset of $I(\bar{x}, \bar{u})$ for each $x \in X$. □

For the rest of this paper, we use $H(x, u, v)$ to denote the set of Lagrange multipliers for a triple (x, u, v) satisfying $x \in S(u)$ and $v \in N_{S(u)}(x)$, defined as

$$H(x,u,v) = \{w \in \mathbb{R}_+^{|I|} \times \mathbb{R}^{|J|} : \quad v = \sum_{i \in I \cup J} \nabla_x f_i(x,u) w_i, \\ w_i = 0 \text{ for each } i \in I \setminus I(x,u)\}. \tag{22}$$

We use $\mathrm{supp}(w)$ to denote the set of indices $i \in I \cup J$ with $w_i \neq 0$.

Clearly, if the CRCQ holds at $(\bar{x}, \bar{u})$ then it follows from Corollary 2 that $H(\bar{x}, \bar{u}, \bar{v}) \neq \emptyset$ for each $\bar{v} \in N_{S(\bar{u})}(\bar{x})$. Moreover, as implied by the following lemma, the projection from the origin to $H(x, u, v)$ is uniformly bounded for all qualified triples (x, u, v) sufficiently close to $(\bar{x}, \bar{u}, \bar{v})$. This is similar to part of [12, Lemma 5], which says that the multiplier map associated with the KKT system of a nonlinear program has a *non-trivial closed and locally bounded selection* if either the MFCQ or the CRCQ holds. It is also closely related to [7, Lemma 3.2.8] which says that both the CRCQ and the MFCQ imply the *sequentially bounded constraint qualification*. We provide a proof for this lemma for completeness.

Lemma 3 *Assume that the CRCQ holds at $(\bar{x}, \bar{u})$. Let $\bar{v} \in N_{S(\bar{u})}(\bar{x})$. Then there exist a real number $r_0 > 0$ and neighborhoods X of $\bar{x}$ in $\bar{X}$, U of $\bar{u}$ in $\bar{U}$ and V of $\bar{v}$ in $\mathbb{R}^n$, such that for each $(x, u, v) \in X \times U \times V$ satisfying $x \in S(u)$ and $v \in N_{S(u)}(x)$, there exists $w \in H(x, u, v) \cap r_0 \mathbb{B}$ with*

$$\{\nabla_x f_i(x,u), i \in \mathrm{supp}(w)\}$$

being linearly independent.

Proof Select open bounded neighborhoods X of $\bar{x}$ in $\bar{X}$ and U of $\bar{u}$ in $\bar{U}$ such that the following hold for each $(x, u) \in X \times U$ with $x \in S(u)$:

(a) $I(x, u) \subset I(\bar{x}, \bar{u})$;
(b) For each $K \subset I(\bar{x}, \bar{u})$, the rank of $\{\nabla_x f_i(x, u), i \in K\}$ equals that of $\{\nabla_x f_i(\bar{x}, \bar{u}), i \in K\}$.

These imply that the CRCQ holds at each $(u, x) \in \mathrm{gph}\, S \cap (U \times X)$. Next, let V be a bounded neighborhood of $\bar{v}$. Consider a triple $(x, u, v) \in X \times U \times V$ satisfying $x \in S(u)$ and $v \in N_{S(u)}(x)$; we have $H(x, u, v) \neq \emptyset$. Moreover, by Carathéodory's theorem ([34, Corollary 17.1.2]) there exists $w \in H(x, u, v)$ such that

$$\{\nabla_x f_i(x,u), i \in \mathrm{supp}(w)\}$$

is linearly independent. Let $K = \mathrm{supp}(w)$; then

$$v = \sum_{i \in K} \nabla_x f_i(x,u) w_i.$$

Condition (b) above implies that $\{\nabla_x f_i(\bar{x}, \bar{u}), i \in K\}$ is linearly independent. By shrinking X, U and V further if necessary, we can find a real number $r_0 > 0$ so that $|w| < r_0$ for all w obtained in such a way. □

The following corollary applies Lemma 3 to the special case in which $\bar{v} = 0$.

Corollary 3 *Assume that the CRCQ holds at $(\bar{x}, \bar{u})$. For each real number $r > 0$, there exist neighborhoods X of $\bar{x}$ in $\bar{X}$, U of $\bar{u}$ in $\bar{U}$ and V of 0 in $\mathbb{R}^n$, such that for each $(x, u, v) \in X \times U \times V$ satisfying $x \in S(u)$ and $v \in N_{S(u)}(x)$, there exists $w \in H(x, u, v) \cap r\,\mathbb{B}$ with*

$$\{\nabla_x f_i(x, u), i \in \mathrm{supp}(w)\}$$

being linearly independent.

Proof Let $r > 0$ be given. Let $\bar{v} = 0$, and determine a positive scalar r_0 and neighborhoods X, U and V as in Lemma 3. If $r \geq r_0$ then these neighborhoods already satisfy the conclusion of the present corollary. Suppose that $r < r_0$. Because the multifunction $H(x, u, v)$ is positively homogeneous with respect to v when x and u are fixed, the conclusion of the present corollary holds for neighborhoods X, U and $\frac{r}{r_0} V$. □

The next theorem is the main result of this section. Its proof is similar to part of the proof of [16, Proposition 2.2].

Theorem 2 *Assume that the CRCQ holds at $(\bar{x}, \bar{u})$, and that f_i for each $i \in I \cup J$ is a C^2 function on $\bar{X} \times \bar{U}$. Let $\bar{v} \in N_{S(\bar{u})}(\bar{x})$. Then there exist neighborhoods X of $\bar{x}$ in $\bar{X}$, U of $\bar{u}$ in $\bar{U}$ and V of $\bar{v}$ in $\mathbb{R}^n$, and a real number $\rho > 0$, such that (21) holds whenever $x' \in X \cap S(u)$, $(x, u, v) \in X \times U \times V$, $x \in S(u)$ and $v \in N_{S(u)}(x)$.*

Proof Let neighborhoods X of $\bar{x}$ in $\bar{X}$, U of $\bar{u}$ in $\bar{U}$ and V of $\bar{v}$ in $\mathbb{R}^n$ and a real number $r_0 > 0$ have the property of Lemma 3. By shrinking X and U if necessary, we may assume that they are compact and convex. Define a function $h : \bar{X} \times \bar{U} \times \mathbb{R}^{|I|+|J|} \to \mathbb{R}$ by

$$h(x, u, w) = \langle w, f(x, u) \rangle.$$

Because f is assumed to be a C^2 function, so is h. An application of Taylor's theorem implies the existence of a scalar $\rho > 0$ such that

$$\begin{aligned} \langle w, f(x', u) - f(x, u) \rangle &= h(x', u, w) - h(x, u, w) \\ &\geq \langle \nabla_x f(x, u)^* w, x' - x \rangle - \frac{\rho}{2} \|x' - x\|^2 \end{aligned} \tag{23}$$

for each $x', x \in X$, $u \in U$ and $w \in r_0\,\mathbb{B}$.

Next, consider a triple $(x, u, v) \in X \times U \times V$ satisfying $x \in S(u)$ and $v \in N_{S(u)}(x)$, and let $x' \in X \cap S(u)$. Find $w \in H(x, u, v) \cap r_0\,\mathbb{B}$ whose existence is guaranteed by Lemma 3. We have $v = \nabla_x f(x, u)^* w$ and

$$\langle w, f(x', u) - f(x, u) \rangle = \langle w, f(x', u) \rangle \leq 0.$$

A combination of this and (23) proves the theorem. □

5 Continuity

The main result of this section is Theorem 3, just below, which says that if the CRCQ holds at $(\bar{x}, \bar{u})$ then the multifunction S defined in (1) satisfies a continuity property (24) for each $u, u' \in U$ with $S(u) \cap X$ and $S(u') \cap X$ nonempty, where X and U are some neighborhoods of $\bar{x}$ and $\bar{u}$ respectively. This continuity property is analogous to the pseudo-Lipschitz continuity defined in [2], called the Aubin property in [35], but it is a different property, because the Aubin property would require (24) to hold for each $u, u' \in U$. Clearly, the CRCQ alone is not sufficient to imply the Aubin property, because it allows $S(u) = \emptyset$ for u arbitrarily close to $\bar{u}$. On the other hand, it is well known that the MFCQ is sufficient for S to have the Aubin property, and is also necessary under the *ample parametrization* condition, see, e.g., [6, Example 4D.3] or [35, Example 9.44].

Theorem 3 is an extension of [11, Proposition 2.5], which says that (24) holds for $u = 0$ and each $u' \in U$. The latter property is analogous to the calmness concept in [35].

Theorem 3 *Assume that the CRCQ holds at $(\bar{x}, \bar{u})$. Then there exist neighborhoods X of $\bar{x}$ in $\bar{X}$ and U of $\bar{u}$ in $\bar{U}$ and a real number $\kappa > 0$ such that*

$$S(u') \cap X \subset S(u) + \kappa \|u - u'\| \, \mathbb{B} \tag{24}$$

for each $u, u' \in U$ satisfying $S(u) \cap X \neq \emptyset$ and $S(u') \cap X \neq \emptyset$.

Before proceeding to prove this theorem, we make some preliminary arrangements and prove some technical results.

First, note that we may assume without loss of generality that the index set J in the definition of $S(u)$ is empty, because we can rewrite each equality constraint as two opposite inequality constraints without affecting the CRCQ assumption. We may also assume that $I(\bar{x}, \bar{u}) = I$, because Theorem 3 is about local behavior of S around $(\bar{x}, \bar{u})$, and the inactive constraints at $(\bar{x}, \bar{u})$ will remain inactive for all (x, u) in some neighborhood of $(\bar{x}, \bar{u})$. Finally, for convenience we temporarily use $\|\cdot\|$ to refer to the 1-norm in the proof (but not the statement) of the theorem.

Let k be the rank of $\nabla_x f_I(\bar{x}, \bar{u})$. If $k = 0$, then it follows from the CRCQ assumption that there exist neighborhoods X of x^0 in $\bar{X}$ and U of u^0 in $\bar{U}$, with X being convex, such that $\nabla_x f_I(x, u) \equiv 0$ on $X \times U$. Consequently, for each fixed $u \in U$ the function $f_I(x, u)$ is a constant over X, and $S(u)$ is either X or $\emptyset$. Hence, the conclusion of Theorem 3 trivially holds in this case. Suppose for the rest of this section that $k \geq 1$.

Let $\mathcal{K}$ be the family of sets $K \subset I$ such that $|K| = k$ and $\nabla_x f_K(\bar{x}, \bar{u})$ is of full row rank. Let A be an $(n-k) \times n$ matrix of full row rank the row space of which complements that of $\nabla_x f_I(\bar{x}, \bar{u})$, and let a_i be the ith row of A. As a result, for each $K \in \mathcal{K}$, the matrix

$$\begin{bmatrix} \nabla_x f_K(\bar{x}, \bar{u}) \\ A \end{bmatrix}$$

is nonsingular.

We mention that it is possible to assume without loss of generality that $k = n$, as in [11], to avoid introducing the matrix A, by restricting the variable x to a certain manifold. However, we choose to allow $k < n$, as keeping A in the proofs does not use much space and makes it easy for the readers to follow.

Now, choose compact neighborhoods X_0 of $\bar{x}$ in $\bar{X}$, U_0 of $\bar{u}$ in $\bar{U}$, Y_0 of 0 in $\mathbb{R}^k$ and Z_0 of 0 in $\mathbb{R}^{n-k}$, with Y_0 and Z_0 being boxes, such that the following conditions hold:

(a) For each $i \in I$ the function $f_i(x,u)$ is Lipschitz continuous on $X_0 \times U_0$ with a constant θ independent of i. That is

$$\|f_i(x,u) - f_i(x',u')\| \leq \theta(\|x - x'\| + \|u - u'\|) \tag{25}$$

for each (x,u) and (x',u') in $X_0 \times U_0$ and each $i \in I$.

(b) $I(x,u) \subset I$ for each $(x,u) \in X_0 \times U_0$.

(c) For each $K \subset I$ and each $J \subset \{1, \cdots, n-k\}$, the family $\{\nabla_x f_i(x,u), i \in K\} \cup \{a_j, j \in J\}$ is of constant rank on $X_0 \times U_0$.

(d) For each $K \subset I$ and $j \in I \setminus K$ such that the family $\{\nabla_x f_i(\bar{x},\bar{u}), i \in K\}$ is linearly independent and $\nabla_x f_j(\bar{x},\bar{u})$ is a linear combination of elements in that family, the unique solution $\lambda(x,u) \in \mathbb{R}^{|K|}$ determined by the equation

$$\nabla_x f_j(x,u) = \sum_{i \in K} \lambda_i(x,u) \nabla_x f_i(x,u)$$

has constant signs as (x,u) varies within $X_0 \times U_0$. That is, $\operatorname{sgn} \lambda_i(x,u) = \operatorname{sgn} \lambda_i(\bar{x},\bar{u})$ for each $i \in K$ and $(x,u) \in X_0 \times U_0$. Moreover, there exist a convex neighborhood V of $f_K(\bar{x},\bar{u})$ in $\mathbb{R}^{|K|}$ and a continuously differentiable function $g : V \times U_0 \to \mathbb{R}$, such that $f_K(x,u) \in V$ and $f_j(x,u) = g(f_K(x,u),u)$ for each $(x,u) \in X_0 \times U_0$, and that

$$\operatorname{sgn} \frac{\partial}{\partial v_i} g(v,u) = \operatorname{sgn} \lambda_i(\bar{x},\bar{u})$$

for each $i \in K$ and each $(v,u) \in V \times U_0$.

(e) For each $K \in \mathcal{K}$ the system of equations

$$\begin{aligned} f_K(x,u) - y &= 0, \\ A(x - \bar{x}) - z &= 0 \end{aligned} \tag{26}$$

holds for some $(x,u,y,z) \in X_0 \times U_0 \times Y_0 \times Z_0$ if and only if $x = h_K(u,y,z)$, where $h_K : U_0 \times Y_0 \times Z_0 \to \mathbb{R}^n$ is a continuously differentiable function with $h_K(u,y,z) \in \operatorname{int} X_0$ for each $(u,y,z) \in U_0 \times Y_0 \times Z_0$. Moreover, h_K is Lipschitz continuous on $U_0 \times Y_0 \times Z_0$ with a constant M independent of K. That is,

$$\|h_K(u,y,z) - h_K(u',y',z')\| \leq M(\|u - u'\| + \|y - y'\| + \|z - z'\|) \tag{27}$$

for each (u,y,z) and (u',y',z') in $U_0 \times Y_0 \times Z_0$ and each $K \in \mathcal{K}$.

For brevity we will in the rest of this section refer to the conditions above as conditions (a) to (e). The following are a few comments about how to choose neighborhoods X_0, U_0, Y_0 and Z_0 to satisfy these conditions. It follows from the continuous differentiability of f_i, the CRCQ assumption and the way we chose the matrix A that we can guarantee (a), (b), (c) and the constant sign statement about $\lambda(x,u)$ in (d) to hold by choosing X_0 and U_0 to be sufficiently small. The second part of (d) follows from an application of Corollary 1 for each pair of qualified K and j. The function g and the neighborhood V in (d) depend on the specific choices of K and j; for notational simplicity we did not use (K,j) as subscripts for g and V. Finally, we obtain (e) by applying the standard implicit function theorem, because the Jacobian matrix of the left hand side of (26) with respect to x at $(\bar{x},\bar{u},0,0)$, given by

$$\begin{bmatrix} \nabla_x f_K(\bar{x},\bar{u}) \\ A \end{bmatrix},$$

is nonsingular for each $K \in \mathcal{K}$.

Next, choose neighborhoods $X_k \subset X_{k-1} \subset \cdots \subset X_1$ of $\bar{x}$ in X_0, $U_k^a \subset U_{k-1}^b \subset U_{k-1}^a \subset \cdots U_1^b \subset U_1^a$ of $\bar{u}$ in U_0, $Y_{k-1} \subset \cdots \subset Y_1$ of 0 in Y_0, $Z_{k-1} \subset \cdots \subset Z_1$ of 0 in Z_0 in the following inductive way.

1. Choose neighborhoods X_1 of $\bar{x}$ in X_0 and U_1^a of $\bar{u}$ in U_0 such that $f_K(x,u) \in Y_0$ and $A(x-\bar{x}) \in Z_0$ for each $(x,u) \in X_1 \times U_1$ and each $K \in \mathcal{K}$. Set $j := 1$.
2. Choose neighborhoods Y_j of $\bar{y}$ in Y_{j-1}, Z_j of $\bar{z}$ in Z_{j-1}, U_j^b of $\bar{u}$ in U_j^a such that $h_K(u,y,z) \in X_j$ for each $(u,y,z) \in U_j^b \times Y_j \times Z_j$; let Y_j and Z_j be boxes. Then choose neighborhoods X_{j+1} of $\bar{x}$ in X_j and U_{j+1}^a of $\bar{u}$ in U_j^b such that $f_K(x,u) \in Y_j$ and $A(x-\bar{x}) \in Z_j$ for each $(x,u) \in X_{j+1} \times U_{j+1}^a$ and each $K \in \mathcal{K}$. Set $j := j+1$.
3. Repeat step 2 until $j = k$.

We are now ready to prove the following lemma. Roughly, this lemma says that if u is close to $\bar{u}$ and $S(u)$ contains some point close to $\bar{x}$, then $S(u)$ must contain a point of the form $h_K(u,0,\hat{z})$ for some $k \in \mathcal{K}$. This result is closely related to [11, Lemma 2.1]. There is, in our view, an unclear point in the proof of the cited lemma. The way that proof defined the point $\bar{y}_k$ does not imply that $\bar{y}_k$ belongs to the set W defined there; consequently, the arc $x_k(t)$ in that proof might not be well defined. To explain this in the setting of the present paper, note that in applying the implicit function theorem to (26) we cannot guarantee simultaneously that $f_K(x,u) \in Y_0$ and $A(x-\bar{x}) \in Z_0$ for each $(x,u) \in X_0 \times U_0$ and that $h_K(u,y,z) \in X_0$ for each $(u,y,z) \in U_0 \times Y_0 \times Z_0$. The proof of the cited lemma, however, implicitly assumed that this could be done. To fix this deficiency in that proof, we need to define sequences of neighborhoods in the above three steps.

Lemma 4 *Let $u \in U_k^a$ with $S(u) \cap X_k \neq \emptyset$. Then there exist $K \in \mathcal{K}$ and $\hat{z} \in Z_{k-1}$ such that $h_K(u,0,\hat{z}) \in S(u)$.*

Proof Let $x^k \in S(u) \cap X_k$. The main idea of this proof is to start with x^k to find a point in $S(u) \cap X_{k-1}$, and then a point in $S(u) \cap X_{k-2}$, and so on. In at most k steps we will find a point in $S(u) \cap X_0$ of the desired form.

Let $I(x^k,u)$ denote the set of indices $i \in I$ such that $f_i(x^k,u) = 0$. Let L^k be a maximal subset of $I(x^k,u)$ with $\{\nabla_x f_i(x^k,u), i \in L^k\}$ being linearly independent, and let $\hat{z} = A(x^k - \bar{x})$. It follows from the definition of X_k that $\hat{z} \in Z_{k-1}$. If $L^k \in \mathcal{K}$, then we would have $x^k = h_{L^k}(u,0,\hat{z})$ and could stop with $K := L^k$.

Suppose that $L^k \notin \mathcal{K}$. Then there exists some $K^k \in \mathcal{K}$ with $L^k \subset K^k$. If $h_{K^k}(u,0,\hat{z}) \in S(u)$, then we could stop with $K = K^k$. Suppose that $h_{K^k}(u,0,\hat{z}) \notin S(u)$. Let $y^k = f_{K^k}(x^k,u)$; then the definition of X_k implies that $y^k \in Y_{k-1}$. For each t in the interval $[0,1]$ define

$$x(t) = h_{K^k}(u, ty^k, \hat{z}),$$

which is well defined because $ty^k \in Y_{k-1}$ for each t. Moreover, $x(t) \in X_{k-1}$ for each t, with $x(1) = x^k$ and $x(0) = h_{K^k}(u,0,\hat{z})$. Note that $x(1) \in S(u)$ and $x(0) \notin S(u)$. Define

$$\bar{t} = \inf\{t \in [0,1] \mid x(s) \in S(u) \text{ for each } s \geq t\}.$$

We claim that $0 < \bar{t} < 1$. To prove this claim, first let $i \in K^k$. For each $t \in [0,1]$ we have $f_i(x(t),u) = ty_i^k$ by the definition of $x(t)$. In particular, $f_i(x(t),u) = 0$ for $i \in L^k$, and $f_i(x(t),u) < 0$ for $i \in K^k \setminus L^k$. Second, let $i \in I(x^k,u) \setminus L_k$. The definition of L_k implies that $\nabla_x f_i(x^k,u)$ is a linear combination of elements of $\{\nabla_x f_j(x^k,u), j \in L_k\}$.

By condition (d), the value of $f_i(x,u)$ is a function of $f_{L_k}(x,u)$ and u on $X_0 \times U_0$. It then follows from $f_{L_k}(x(t),u) = 0 = f_{L_k}(x^k,u)$ that $f_i(x(t),u) = f_i(x^k,u) = 0$ for each $t \in [0,1]$. Next, let $i \in I \setminus I(x^k,u)$. We have $f_i(x^k,u) < 0$, so $f_i(x(t),u) < 0$ holds for t sufficiently close to 1. Taking all these into account, we conclude that $x(t) \in S(u)$ for t sufficiently close to 1. This shows that $\bar{t} < 1$. Because $S(u)$ is closed, we have $x(\bar{t}) \in S(u)$ and therefore $\bar{t} > 0$. This proves the claim that $0 < \bar{t} < 1$.

Let $x^{k-1} = x(\bar{t})$; then $x^{k-1} \in X_{k-1}$. Let L^{k-1} be a maximal subset of $I(x^{k-1},u)$ with $\{\nabla_x f_i(x^{k-1},u), i \in L^{k-1}\}$ being linearly independent.

We claim that $|L^{k-1}| > |L^k|$. To prove this claim, first note that $f_{L^k}(x^{k-1},u) = 0$, so $L^k \subset I(x^{k-1},u)$. Second, the definition of $\bar{t}$ implies that there exists a sequence $t_j \uparrow \bar{t}$ with $x(t_j) \notin S(u)$. By passing to a subsequence if necessary, there exists $i_0 \in I$ such that $f_{i_0}(x(t_j),u) > 0$ for each t_j. This implies that $f_{i_0}(x^{k-1},u) = 0$, so $i_0 \in I(x^{k-1},u)$. Note that the family $\{\nabla_x f_i(x^k,u), i \in L_k \cup \{i_0\}\}$ must be linearly independent: if not, then $\nabla_x f_{i_0}(x^k,u)$ would be a linear combination of $\{\nabla_x f_i(x^k,u), i \in L_k\}$, and we would have $f_{i_0}(x(t),u) = 0$ for each $t \in [0,1]$. Hence, $L^k \cup \{i_0\}$ is a subset of $I(x^{k-1},u)$ with $\{\nabla_x f_i(x^{k-1},u), i \in L^k \cup \{i_0\}\}$ being linearly independent. This proves the claim that $|L^{k-1}| > |L^k|$.

In summary, starting from a point $x^k \in S(u) \cap X_k$, we have found a point $x^{k-1} \in S(u) \cap X_{k-1}$, with $A(x^{k-1} - \bar{x}) = A(x^k - \bar{x}) = \hat{z}$ and $|L^{k-1}| > |L^k|$. If $L^{k-1} \in \mathcal{K}$ then we stop with $K := L^{k-1}$, otherwise we repeat the previous procedure to find a point $x^{k-2} \in S(u) \cap X_{k-2}$ with $A(x^{k-2} - \bar{x}) = \hat{z}$ and $|L^{k-2}| > |L^{k-1}|$. Conduct this procedure iteratively, and note that the cardinality of L^d strictly increases as d decreases. On the other hand, $|L^d|$ cannot exceed k by its definition. Accordingly, we must have $|L^d| = k$ for some $0 \le d \le k$; by that time, we have found $x^d \in S(u) \cap X_d$, with $A(x^d - \bar{x}) = \hat{z}$ and $L^d \in \mathcal{K}$. We then stop with $K := L^d$ and $x^d = h_K(u,0,\hat{z})$. □

One more remark about Lemma 4. The point $\hat{z}$ in that lemma was defined by $\hat{z} = A(x^k - \bar{x})$. Indeed, the choice of $\hat{z}$ is not important in the sense that if $h_K(u,0,\hat{z}) \in S(u)$ then $h_K(u,0,z) \in S(u)$ for each $z \in Z_0$. To see this, note that the definition of $\mathcal{K}$ implies that $\nabla_x f_i(\bar{x},\bar{u})$ for each $i \in I \setminus K$ is a linear combination of elements of $\{\nabla_x f_j(\bar{x},\bar{u}), j \in K\}$, so by condition (d) the value of $f_i(\cdot,u)$ is a function of $f_K(\cdot,u)$ and u. It then follows from the fact

$$f_K(h_K(u,0,z),u) = f_K(h_K(u,0,\hat{z}),u) = 0$$

that

$$f_i(h_K(u,0,z),u)) = f_i(h_K(u,0,\hat{z}),u) \le 0$$

for each $i \in I \setminus K$, so $h_K(u,0,z) \in S(u)$. This proves the following

Corollary 4 *Let $u \in U_k^a$ with $S(u) \cap X_k \neq \emptyset$, and let $z \in Z_0$. Then there exists $K \in \mathcal{K}$ such that $h_K(u,0,z) \in S(u)$.*

Below we prove Theorem 3.

Proof (of Theorem 3) Define

$$\kappa = M + n\theta M^2 + n\theta M,$$

where θ and M are defined in (25) and (27) respectively. Let $u, u' \in U_k^a$ with $S(u) \cap X_k \neq \emptyset$ and $S(u') \cap X_k \neq \emptyset$, and let $x' \in S(u') \cap X_k$. We will prove by induction that there exists $x \in S(u)$ such that

$$\|x - x'\| \le \kappa \|u - u'\|. \tag{28}$$

We start by setting up the initialization stage for the induction procedure. Let $z' = A(x' - \bar{x})$; then $z' \in Z_{k-1}$. Apply Corollary 4 to find $K \in \mathcal{K}$ such that $h_K(u, 0, z') \in S(u)$. Define the following set of points

$$P = \left\{ x \in S(u) \cap X_0 \,\middle|\, \begin{array}{l} f_i(x', u') \leq f_i(x, u) \leq 0 \text{ for each } i \in K, \\ A(x - \bar{x}) = z' \end{array} \right\}. \tag{29}$$

The set P as defined is nonempty, because it contains at least $h_K(u, 0, z')$. Choose x^0 to be a point in P such that the set

$$L = \{i \in K \mid f_i(x^0, u) = f_i(x', u')\}$$

has the maximal cardinality. Let $r := |K \setminus L|$, and write $K \setminus L$ as

$$K \setminus L = \{j(1), \cdots, j(r)\}. \tag{30}$$

Finally, for each $d = 0, 1, \cdots, r$ define a vector $\xi^d \in \mathbb{R}^r$ by

$$\xi^d_j = \begin{cases} \min\{0, f_j(x', u') + \theta(M+1)\|u - u'\|\}, & j = j(1), \cdots, j(d), \\ 0, & j = j(d+1), \cdots, j(r). \end{cases}$$

This definition implies that

$$\xi^r \leq \xi^{r-1} \leq \cdots \leq \xi^0 = 0 \in \mathbb{R}^r.$$

In addition, the definitions of x^0 and L imply that

$$\begin{array}{rl} f_i(x', u') \leq f_i(x^0, u) \leq \xi^0_i, & i \in K \setminus L, \\ f_i(x^0, u) = f_i(x', u'), & i \in L. \end{array} \tag{31}$$

So far we have set up the initialization stage for the induction. For the induction step, assume that for some $d = 0, \cdots, r-1$ there exists a point $x^d \in P$ such that

$$\begin{array}{rl} f_i(x', u') \leq f_i(x^d, u) \leq \xi^d_i, & i \in K \setminus L, \\ f_i(x^d, u) = f_i(x', u'), & i \in L. \end{array} \tag{32}$$

The main step in our proof will be to show, based on the inductive hypothesis, that there exists a point $x^{d+1} \in P$ satisfying

$$\begin{array}{rl} f_i(x', u') \leq f_i(x^{d+1}, u) \leq \xi^{d+1}_i, & i \in K \setminus L, \\ f_i(x^{d+1}, u) = f_i(x', u'), & i \in L. \end{array} \tag{33}$$

Once we have established this, we may proceed the induction by increasing d from 0 to $r-1$ one by one, because it follows from (31) that x^0 satisfies (32) for $d = 0$. By the end, we will find a point $x^r \in P$ such that

$$\begin{array}{rl} f_i(x', u') \leq f_i(x^r, u) \leq \xi^r_i, & i \in K \setminus L, \\ f_i(x^r, u) = f_i(x', u'), & i \in L, \end{array} \tag{34}$$

which implies that

$$\|f_i(x^r, u) - f_i(x', u')\| \leq \theta(M+1)\|u - u'\|$$

for each $i \in K$. Because $x^r = h_K(u, f_K(x^r, u), z')$ and $x' = h_K(u', f_K(x', u'), z')$, we have

$$\begin{aligned}\|x^r - x'\| &= \|h_K(u, f_K(x^r, u), z') - h_K(u', f_K(x', u'), z')\| \\ &\le M(\|u - u'\| + k\theta(M+1)\|u - u'\|) \\ &\le (M + n\theta M^2 + n\theta M)\|u - u'\|,\end{aligned}$$

where the first inequality follows from (27) and the second holds because $k \le n$. This will show that x^r satisfies (28) and thereby complete the proof.

In the rest of this proof, we assume the existence of a point $x^d \in P$ satisfying (32) for some $d = 0, \cdots, r-1$, and prove the existence of $x^{d+1} \in P$ satisfying (33).

To find such x^{d+1}, consider the following optimization problem

$$\begin{aligned}\min_{x \in X_0} \quad & f_{j(d+1)}(x, u) \\ \text{s.t. } f_i(x', u') \le & f_i(x, u) \le \xi_i^d, && i \in K \setminus L, \\ & f_i(x, u) = f_i(x', u'), && i \in L, \\ & f_i(x, u) \le 0, && i \in I \setminus K, \\ & A(x - \bar{x}) = z'.\end{aligned} \tag{35}$$

This problem is feasible, because by the induction hypothesis x^d is a feasible solution. Moreover, the feasible region is a compact set in $\operatorname{int} X_0$. To see this, first note that $\xi^d \le 0$ and that $f_K(x', u') \in Y_{k-1}$ because $(x', u') \in X_k \times U_k$. By definition Y_{k-1} is a box neighborhood of 0, so it contains any $y \in \mathbb{R}^k$ satisfying $f_K(x', u') \le y \le 0$. Hence, each feasible solution x of (35) satisfies

$$\begin{aligned}f_K(x, u) &= y \\ A(x - \bar{x}) &= z'\end{aligned}$$

for some $y \in Y_{k-1}$. Because $x \in X_0$, $u \in U_k$ and $z' \in Z_{k-1}$, we have $x = h_K(u, y, z')$, which belongs to $\operatorname{int} X_0$ by condition (e). This proves that the feasible region of (35) is a compact set in $\operatorname{int} X_0$. Accordingly, there exists an optimal solution to (35); denote it by x^{d+1}. Because $x^{d+1} \in \operatorname{int} X_0$ and the inequality and equality constraints in (35) satisfy the CRCQ assumption, the following first order necessary conditions hold:

$$\begin{aligned}\nabla_x f_{j(d+1)}(x^{d+1}, u) &= \sum_{i \in J_1 \cup J_2 \cup L \cup J} \lambda_i \nabla_x f_i(x^{d+1}, u) + \sum_{i=1}^{n-k} \mu_i a_i, \\ \lambda_i > 0, \quad & f_i(x^{d+1}, u) = f_i(x', u'), && i \in J_1 \subset K \setminus L, \\ \lambda_i < 0, \quad & f_i(x^{d+1}, u) = \xi_i^d, && i \in J_2 \subset K \setminus L, \\ \lambda_i \text{ free}, \quad & f_i(x^{d+1}, u) = f_i(x', u'), && i \in L, \\ \lambda_i < 0, \quad & f_i(x^{d+1}, u) = 0, && i \in J \subset I \setminus K, \\ \mu \text{ free}, \quad & A(x - \bar{x}) = z',\end{aligned} \tag{36}$$

where λ and μ are multipliers, J_1 and J_2 are subsets of $K \setminus L$ with positive and negative multipliers respectively, and J is the subset of $I \setminus K$ with negative multipliers. For brevity we omitted some feasibility constraints with zero multipliers. By an argument similar to the proof of Carathéodory's theorem, it is easy to show that we can choose

the multipliers so that the family $\{\nabla_x f_i(x^{d+1}, u), i \in J_1 \cup J_2 \cup L \cup J\}$ is linearly independent.

Being a feasible solution to (35), x^{d+1} belongs to P. The way we defined L then implies that $J_1 = \emptyset$, because otherwise the set

$$\{i \in K \mid f_i(x^{d+1}, u) = f_i(x', u')\},$$

which contains $J_1 \cup L$, would have larger cardinality than L, contradicting the maximality of L. On the other hand, it follows from condition (c) and the definition of matrix A that the spaces spanned by $\{a_i, i = 1, \cdots, n-k\}$ and $\{\nabla_x f_i(x^{d+1}, u), i \in I\}$ are independent. The first equation in (36) therefore implies that $\mu_i = 0$ for each $i = 1, \cdots, n-k$. Consequently, that equation simplifies to

$$\nabla_x f_{j(d+1)}(x^{d+1}, u) = \sum_{i \in J_2 \cup L \cup J} \lambda_i \nabla_x f_i(x^{d+1}, u). \tag{37}$$

Note that we may assume without loss of generality that $j(d+1) \notin J_2$, because if $j(d+1)$ would belong to J_2 then we could move the term $\lambda_{j(d+1)} \nabla_x f_{j(d+1)}(x^{d+1}, u)$ in the right hand side of (37) to its left hand side, and then divide both sides by the positive number $1 - \lambda_{j(d+1)}$. In doing so we would obtain a new set of multipliers that satisfies (36), with $j(d+1) \notin J_2$.

The linear independence of $\{\nabla_x f_i(x^{d+1}, u), i \in J_2 \cup L \cup J\}$, equation (37) and condition (c) together imply that there exists a unique $\bar{\lambda} \in \mathbb{R}^{|J_2|+|L|+|J|}$ such that

$$\nabla_x f_{j(d+1)}(\bar{x}, \bar{u}) = \sum_{i \in J_2 \cup L \cup J} \bar{\lambda}_i \nabla_x f_i(\bar{x}, \bar{u}). \tag{38}$$

Moreover, the first part of condition (d) implies that

$$\operatorname{sgn} \bar{\lambda}_i = \operatorname{sgn} \lambda_i \tag{39}$$

for each $i \in J_2 \cup L \cup J$. The second part of condition (d) implies that there exist a convex neighborhood V of $f_{J_2 \cup L \cup J}(\bar{x}, \bar{u})$ and a continuously differentiable function $g : U_0 \times V \to \mathbb{R}$ such that

$$f_{J_2 \cup L \cup J}(x, u) \in V \quad \text{and} \quad f_{j(d+1)}(x, u) = g(f_{J_2 \cup L \cup J}(x, u), u) \tag{40}$$

whenever $(x, u) \in X_0 \times U_0$, with

$$\operatorname{sgn} \frac{\partial}{\partial v_i} g(v, u) = \operatorname{sgn} \bar{\lambda}_i \tag{41}$$

for each $i \in J_2 \cup L \cup J$ and each $(v, u) \in V \times U_0$. Combining (39) and (41), we see that

$$\operatorname{sgn} \frac{\partial}{\partial v_i} g(v, u) = \operatorname{sgn} \lambda_i = -1 \tag{42}$$

for each $i \in J_2 \cup J$ and each $(v, u) \in V \times U_0$.

Now, choose $\hat{K} \in \mathcal{K}$ such that $J_2 \cup L \cup J \subset \hat{K}$. Let $\beta = f_{\hat{K}}(x', u')$, and $\hat{x} = h_{\hat{K}}(u, \beta, z')$. Then $\hat{x} \in X_{k-1} \subset X_0$, because $\beta \in Y_{k-1}$ and $z' \in Z_{k-1}$. It is possible that $\hat{x} \notin S(u)$. We have

$$\begin{aligned} f_i(\hat{x}, u) = \beta_i = f_i(x', u') &\le \xi_i^d = f_i(x^{d+1}, u), \quad i \in J_2, \\ f_i(\hat{x}, u) = \beta_i = f_i(x', u') &= f_i(x^{d+1}, u), \qquad\quad i \in L, \\ f_i(\hat{x}, u) = \beta_i = f_i(x', u') &\le 0 = f_i(x^{d+1}, u), \quad i \in J, \end{aligned} \tag{43}$$

where the equalities $f_i(\hat{x}, u) = \beta_i$ and $\beta_i = f_i(x', u')$ follow from the definitions of $\hat{x}$ and β, the inequality $f_i(x', u') \leq \xi_i^d$ holds for $i \in J_2$ by the definition of ξ_i^d, and $f_i(x', u') \leq 0$ for $i \in J$ because $x' \in S(u')$.

By (40) we have

$$f_{j(d+1)}(\hat{x}, u) = g(f_{J_2 \cup L \cup J}(\hat{x}, u), u) \quad \text{and} \quad f_{j(d+1)}(x^{d+1}, u) = g(f_{J_2 \cup L \cup J}(x^{d+1}, u), u).$$

It then follows from (42), (43) and an application of the mean value theorem that

$$f_{j(d+1)}(\hat{x}, u) \geq f_{j(d+1)}(x^{d+1}, u). \tag{44}$$

Recall that $j(d+1) \in K \setminus L$ and that x^{d+1} is feasible to (35); so we have

$$f_{j(d+1)}(x^{d+1}, u) \geq f_{j(d+1)}(x', u'). \tag{45}$$

Combining this with (44), we have

$$\|f_{j(d+1)}(x^{d+1}, u) - f_{j(d+1)}(x', u')\| \leq \|f_{j(d+1)}(\hat{x}, u) - f_{j(d+1)}(x', u')\|. \tag{46}$$

On the other hand, the definition of β implies that $x' = h_{\hat{K}}(u', \beta, z')$, so

$$\|\hat{x} - x'\| = \|h_{\hat{K}}(u, \beta, z') - h_{\hat{K}}(u', \beta, z')\| \leq M\|u' - u\|,$$

where the inequality follows from (27). In view of (25), we have

$$\|f_{j(d+1)}(\hat{x}, u) - f_{j(d+1)}(x', u')\| \leq \theta(\|\hat{x} - x'\| + \|u - u'\|) \leq \theta(M+1)\|u - u'\|.$$

Combining the inequality above with (46), we obtain

$$\|f_{j(d+1)}(x^{d+1}, u) - f_{j(d+1)}(x', u')\| \leq \theta(M+1)\|u - u'\|.$$

This and (45) together imply

$$f_{j(d+1)}(x', u') \leq f_{j(d+1)}(x^{d+1}, u) \leq f_{j(d+1)}(x', u') + \theta(M+1)\|u - u'\|.$$

Again, because $j(d+1) \in K \setminus L$ and x^{d+1} is feasible to (35), we have

$$f_{j(d+1)}(x^{d+1}, u) \leq \xi_{j(d+1)}^d = 0,$$

where $\xi_{j(d+1)}^d = 0$ by the definition of ξ^d. Therefore, we have

$$\begin{aligned} f_{j(d+1)}(x', u') &\leq f_{j(d+1)}(x^{d+1}, u) \\ &\leq \min\{0, f_{j(d+1)}(x', u') + \theta(M+1)\|u - u'\|\} = \xi_{j(d+1)}^{d+1}, \end{aligned}$$

so x^{d+1} satisfies the inequality in (33) for $i = j(d+1)$. By definition, $\xi_i^d = \xi_i^{d+1}$ for each $i \neq j(d+1)$. Therefore, as a feasible solution to (35), x^{d+1} satisfies all the other constraints in (33) as well. We already noted that $x^{d+1} \in P$, and this completes the proof. □

6 The Euclidean projector

This section combines the prox-regularity in Section 4 and the continuity in Section 5 to show that under the CRCQ the Euclidean projection of a point $z \in \mathbb{R}^n$ onto $S(u)$, denoted by $\Pi_{S(u)}(z)$, is locally a single-valued PC^1 function around $(\bar{u}, \bar{x})$ in the sense of Definition 5. For details on PC^1 (or more general PC^r) functions, see [3,36]. Here we allow the domain of a PC^1 function to be any subset of the Euclidean space, while in its original definitions in [3,36] its domain has to be an open set. We make this slight extension to cover the situation concerned in the present paper where $S(u)$ can be empty for u arbitrarily close to $\bar{u}$.

Definition 5 If D is a subset of $\mathbb{R}^k$, then a continuous function f from D to $\mathbb{R}^n$ is a PC^1 function on D if for each $x \in D$ there exist an open neighborhood N of x in $\mathbb{R}^k$ and a finite collection of C^1 functions $f_j : N \to \mathbb{R}^n, j = 1, \cdots, l$ such that the inclusion $f(x') \in \{f_1(x'), \cdots, f_l(x')\}$ holds for each $x' \in N \cap D$.

The following lemma applies results in [32] to show that the Euclidean projector onto $S(u)$ is locally a single-valued continuous function.

Lemma 5 *Assume that the CRCQ holds at $(\bar{x}, \bar{u})$, and that f_i for each $i \in I \cup J$ is a C^2 function on $\bar{X} \times \bar{U}$. Let β be a real number such that $\beta > 1$. Then there exist neighborhoods X_0 of $\bar{x}$ in $\bar{X}$, U_0 of $\bar{u}$ in $\bar{U}$ and Z_0 of the point $\bar{z} := \bar{x}$ in $\mathbb{R}^n$, such that the following properties hold with a set U_0' defined by*

$$U_0' = \{u \in U_0 \mid S(u) \cap X_0 \neq \emptyset\}. \tag{47}$$

(a) The localization to $(U_0' \times Z_0) \times X_0$ of the multifunction taking $(u, z) \in \bar{U} \times \mathbb{R}^n$ to $(I + N_{S(u)})^{-1}(z) \subset \mathbb{R}^n$ is a single-valued function π that coincides with the localization to $(U_0' \times Z_0) \times X_0$ of the multifunction taking $(u, z) \in \bar{U} \times \mathbb{R}^n$ to the projection $\Pi_{S(u)}(z)$.

(b) There exist real numbers $M_1 > 0$ and $M_2 > 0$ such that

$$\|\pi(u, z) - \pi(u', z')\| \leq \beta\|z - z'\| + M_1\|u - u'\| + M_2\|u - u'\|^{\frac{1}{2}} \tag{48}$$

for each (u, z) and (u', z') in $U_0' \times Z_0$.

Proof Let $\bar{v} = 0$. The CRCQ and C^2 assumptions imply by Theorem 2 that the indicator of $S(u)$ is prox-regular in x at $\bar{x}$ for $\bar{v}$ with compatible parametrization by u at $\bar{u}$, that is, there exist neighborhoods X' of $\bar{x}$ in $\bar{X}$, U_1 of $\bar{u}$ in $\bar{U}$, V' of $\bar{v}$ in $\mathbb{R}^n$, and a scalar $\rho > 0$, such that (21) holds whenever $x' \in X' \cap S(u)$, $(x, u, v) \in X' \times U_1 \times V'$, $x \in S(u)$ and $v \in N_{S(u)}(x)$. The CRCQ assumption further implies by Theorem 3 that, by shrinking X' and U_1 if necessary, we may find a scalar $\kappa > 0$ such that

$$S(u') \cap X' \subset S(u) + \kappa\|u - u'\|\,\mathbb{B} \tag{49}$$

holds for each $u, u' \in U_1$ satisfying $S(u) \cap X' \neq \emptyset$ and $S(u') \cap X' \neq \emptyset$.

We can then follow the proof of [32, Theorem 2] to obtain neighborhoods X_0, U_0 and Z_0 satisfying condition (a) of the present lemma, with

$$\|\pi(u', z) - \pi(u', z')\| \leq \beta\|z - z'\| \tag{50}$$

for each $u' \in U_0'$ and each $z, z' \in Z_0$. The only change we need to make to that proof is that in choosing the neighborhood U_0 we use (49) instead of the inner semicontinuity assumption in that theorem.

To prove (b), we first proceed as in the proof of [32, Theorem 3] to show that for each $z \in Z_0$ and each $u, u' \in U_0'$ one has

$$\begin{aligned} \|\pi(u', z) - \pi(u, z)\| \leq \delta(\beta/2) \\ + \delta^{\frac{1}{2}} \{2\beta\|z - \pi(u, z)\| + \delta[\beta^2/4 + \beta - 1]\}^{\frac{1}{2}} \end{aligned} \tag{51}$$

where $\delta = \kappa(u' - u)$. In doing so we need to make the following changes to the proof of that theorem.

1. In the proof of [32, Theorem 3], choose points y and y' from $S(u)$ and $S(u')$ respectively so that $r = y' - x$ and $r' = y - x'$ satisfy $\|r\| \leq \delta$ and $\|r'\| \leq \delta$.
2. By making X_0, U_0 and Z_0 smaller if necessary, arrange that

$$X_0 + \kappa\|u - u'\|\,\mathbb{B} \subset X'$$

for each $u, u' \in U_0$. This ensures that the points y and y' defined above belong to X'.

Once we have established (51), apply (50) and (51) with $u = \bar{u}$ and $z = \bar{z}$ to obtain

$$\begin{aligned} \|\pi(u', z') - \pi(\bar{u}, \bar{z})\| &\leq \|\pi(u', \bar{z}) - \pi(u', z')\| + \|\pi(u', \bar{z}) - \pi(\bar{u}, \bar{z})\| \\ &\leq \beta\|z' - \bar{z}\| + \delta(\beta/2) + \delta^{\frac{1}{2}} \{2\beta\|\bar{z} - \pi(\bar{u}, \bar{z})\| + \delta[\beta^2/4 + \beta - 1]\}^{\frac{1}{2}} \end{aligned}$$

for each $(u', z') \in U_0' \times Z_0$. In particular, by choosing U_0 and Z_0 to be bounded, the norm $\|\pi(u, z)\|$ is bounded on $U_0' \times Z_0$. The existence of M_1 and M_2 then follows from combining (50) and (51). □

For the rest of this section, let $\mathcal{B}$ denote the family of sets $K \subset I(\bar{x}, \bar{u})$ such that $\{\nabla_x f_i(\bar{x}, \bar{u}), i \in K\}$ is linearly independent, and assume that f_i for each $i \in I \cup J$ is C^2 on $\bar{X} \times \bar{U}$. For each $K \in \mathcal{B}$, define the following system of equations where $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^{|I|+|J|}$ are variables, and $u \in \mathbb{R}^m$ and $z \in \mathbb{R}^n$ are parameters:

$$\begin{aligned} & x - z + \sum_{i \in K} \nabla_x f_i(x, u) w_i = 0, \\ & f_i(x, u) = 0 \quad \text{for each } i \in K, \\ & w_i = 0 \qquad\quad \text{for each } i \in (I \cup J) \setminus K. \end{aligned} \tag{52}$$

This is the first-order necessary conditions of the nonlinear program of minimizing $\frac{1}{2}\|z - x\|^2$ subject to the equality constraints $f_i(x, u) = 0$ for $i \in K$. Let $\bar{z} := \bar{x}$, and note that $f_i(\bar{x}, \bar{u}) = 0$ for each $i \in K$ because $K \subset I(\bar{x}, \bar{u})$. Hence, $(x, w) = (\bar{x}, 0)$ solves (52) under parameter $(\bar{u}, \bar{z})$. Moreover, the Jacobian matrix of the function on the left hand side of (52) with respect to (x, w) at $(\bar{x}, 0, \bar{u}, \bar{z})$ is

$$\begin{bmatrix} I & \nabla_x f_K(\bar{x}, \bar{u})^T & 0 \\ \nabla_x f_K(\bar{x}, \bar{u}) & 0 & 0 \\ 0 & 0 & I \end{bmatrix},$$

which is nonsingular, so it follows from the classical implicit function theorem that there exist a neighborhood $X_K \times W_K \times U_K \times Z_K$ of $(\bar{x}, 0, \bar{u}, \bar{z})$ in $\mathbb{R}^n \times \mathbb{R}^{|I|+|J|} \times \mathbb{R}^m \times \mathbb{R}^n$

and C^1 functions $x_K : U_K \times Z_K \to X_K$ and $w : U_K \times Z_K \to W_K$ such that for each $(u, z) \in U_K \times Z_K$, $(x_K(u, z), w_K(u, z))$ is the unique solution in $X_K \times W_K$ that satisfies (52).

In the next proposition, Proposition 2, we show that for (u, z) close to $(\bar{u}, \bar{z})$ the function $\pi(u, z)$ defined in Lemma 5 takes the value of $x_K(u, z)$ for some $K \in \mathcal{B}$. The proof of Proposition 2 resembles those of [26, Proposition 7] and [19, Lemma 4.2.17]. The result here is different from those earlier results in that it does not assume the MFCQ.

Proposition 2 *Assume the notation and hypotheses of Lemma 5. Determine neighborhoods X_0 of $\bar{x}$ in $\bar{X}$, U_0 of $\bar{u}$ in $\bar{U}$, Z_0 of the point $\bar{z} = \bar{x}$ in $\mathbb{R}^n$, the set U_0', and the function $\pi : U_0' \times Z_0 \to X_0$ as in that lemma. Then there exist neighborhoods U_1 of $\bar{u}$ in U_0 and Z_1 of $\bar{z}$ in Z_0, such that for each $(u, z) \in (U_1 \cap U_0') \times Z_1$ there exists $K \in \mathcal{B}$ satisfying $\pi(u, z) = x_K(u, z)$ and $w_K(u, z) \in H(\pi(u, z), u, z - \pi(u, z))$.*

Proof Choose a neighborhood $X_1 \times W_1 \times U_1 \times Z_1$ of $(\bar{x}, 0, \bar{u}, \bar{z})$ in $\mathbb{R}^n \times \mathbb{R}^{|I|+|J|} \times \mathbb{R}^m \times \mathbb{R}^n$, such that

(a) $X_1 \times U_1 \times Z_1 \subset X_0 \times U_0 \times Z_0$ and $X_1 \times W_1 \times U_1 \times Z_1 \subset X_K \times W_K \times U_K \times Z_K$ for each $K \in \mathcal{B}$,
(b) $I(x, u) \subset I(\bar{x}, \bar{u})$ for each $(x, u) \in X_1 \times U_1$,
(c) for each $K \subset I(\bar{x}, \bar{u})$ the family $\{\nabla_x f_i(x, u), i \in K\}$ is of constant rank over $X_1 \times U_1$,
(d) $\pi(u, z) \in X_1$ for each $(u, z) \in (U_1 \cap U_0') \times Z_1$,
(e) for each $(x, u, z) \in X_1 \times U_1 \times Z_1$ satisfying $x \in S(u)$ and $z - x \in N_{S(u)}(x)$, there exists $w \in H(x, u, z - x) \cap W_1$ with

$$\{\nabla_x f_i(x, u), i \in \operatorname{supp}(w)\}$$

being linearly independent.

Among the above, (a), (b) and (c) are standard, and (d) and (e) can be established by Lemma 5 and Corollary 3 respectively.

Next, let $(u, z) \in (U_1 \cap U_0') \times Z_1$, and let $x = \pi(u, z)$. By Lemma 5 we have

$$z \in x + N_{S(u)}(x).$$

It follows from (d) and (e) that there exists $w \in H(x, u, z-x) \cap W_1$ with $\{\nabla_x f_i(x, u), i \in \operatorname{supp}(w)\}$ being linearly independent. Let $K = \operatorname{supp}(w)$. The fact that $w \in H(x, u, z - x)$ implies $K \subset I(x, u)$, so $K \subset I(\bar{x}, \bar{u})$ by (b). It then follows from (c) that $\{\nabla_x f_i(\bar{x}, \bar{u}), i \in K\}$ is linearly independent. Hence, K is an element of $\mathcal{B}$. The fact that $w \in H(x, u, z-x)$ also implies that (x, w, u, z) satisfies (52) for this particular choice of K. Accordingly, (x, w) is the unique solution for (52) in $X_1 \times W_1$ under parameter (u, z). It follows that $x = x_K(u, z)$ and $w = w_K(u, z)$. □

It is known from [36, Proposition 4.1.2] that a PC^1 function over a convex domain is locally Lipschitz continuous. As far as we know, no results exist on whether a PC^1 function over an arbitrary domain is Lipschitz continuous, so the result here does not lead to Lipschitz continuity of the Euclidean projector. On the other hand, under the MFCQ the Euclidean projector onto $S(u)$ is a continuous function of (u, z) in a neighborhood of $(\bar{u}, \bar{x})$ in $\mathbb{R}^{m+n}$ by a fundamental result in [13], but is not necessarily Lipschitz continuous, as shown by an example in [29]. The latter example shows that the MFCQ does not imply the PC^1 property of the projector, because for this case the domain of the projector can be easily arranged to be convex.

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