Stability of User-Equilibrium Route Flow Solutions for the Traffic Assignment Problem

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Abstract

This paper studies stability of user equilibrium (UE) route flow solutions with respect to inputs to a traffic assignment problem, namely the travel demand and parameters in the link cost function. It shows, under certain continuity and strict monotonicity assumptions on the link cost function, that the UE link flow is a continuous function of the inputs, that the set of UE route flows is a continuous multifunction of the inputs, and that the UE route flow selected to maximize an objective function with certain properties is a continuous function of the inputs. The maximum entropy UE route flow is an example of the last. On the other hand, a UE route flow arbitrarily generated in a standard traffic assignment procedure may not bear such continuity property, as demonstrated by an example in this paper.

Keywords: traffic assignment; stability; maximum entropy; user equilibrium

1 Introduction

A common technique used to predict travelers' route choices is by solving a user-equilibrium traffic assignment problem (UE-TAP) (Beckmann, McGuire & Winsten 1956). Such a problem can be formulated in the space of link flows or route flows. It is well known that the link flow formulation has a unique solution under mild conditions (Smith 1979). We call the unique link flow solution the *UE link flow*. If we regard the UE link flow as a function of the input data of the UE-TAP, this function is continuous under mild conditions; see, e.g., (Hall 1978), (Dafermos & Nagurney 1984).

The route flow formulation of the UE-TAP usually has multiple solutions. Each of its solutions is called a *UE route flow*. Indeed, a nonnegative route flow is a UE route flow if and only if it meets the travel demand and produces the UE link flow. Hence, once the UE link flow is known, the set

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of UE route flows is a polyhedral convex set defined by the nonnegativity constraints and flow-consistency constraints. This set usually contains more than one point; an intuitive explanation for such non-uniqueness is that the number of UE routes in a network is generally larger than the sum of the numbers of links and of OD pairs. If the set of UE route flows is considered as a *multifunction* of the input data of the UE-TAP, this multifunction is also continuous (see Section 3.3 for more details).

Despite the nonuniqueness of UE route flows, they are used in many practical applications. Decomposing total link flows by origin-destination (O-D) pairs, known as *select link analysis* in practice, can be used to examine the spatial impacts of a transportation project, such as road expansion. Such a decomposition is usually accomplished by aggregating UE route flows by O-D pairs and therefore is not unique itself. Determining class-specific link flows in a multi-class traffic assignment is another example, which is subject to the same nonuniqueness issue. Since considering the entire set of UE route flows in these analysis is infeasible, practitioners commonly select one from the set as a representative. Nonetheless, this selection process, which motivates the current paper, has not received much attention from researchers, let alone practitioners [1]. The state-of-the-practice method is indeed *arbitrary*, implicitly assuming that any UE route flow in the set is no better (or worse) than others. Accordingly, the UE route flow generated from a traffic assignment algorithm, sometimes as its by-product, is most often used.

It is apparent that additional criteria have to be employed in order to "rank" all UE route flows that are otherwise seemingly of equal merit. One option is to use the entropy measure, which leads to the so-called maximum entropy user equilibrium (MEUE) problem (Rossi, McNeil & Hendrickson 1989, Oppenheim 1995, Bell & Iida 1997). That problem looks for, among all UE route flows, the one that maximizes the entropy. It has a unique solution, which we shall call the MEUE route flow. Besides being unique, the MEUE route flow has an interesting behavioral interpretation, namely travelers should distribute in the same proportion on each of any two alternative route segments regardless of their origin or their destination (Bar-Gera & Boyce 1999, Bar-Gera 2009). The assumption of *proportionality* enables highly efficient solution procedures that concurrently solve traffic assignment and the selection of the MEUE route flow, as recently discovered by (Bar-Gera 2009) [2]. It remains an open question, however, whether or not the proportionality assumption accords to real travel behavior. This paper does not attempt to address that question. Instead, we argue that the MEUE route flow is *stable* and therefore *more useful* than an arbitrarily selected UE route flow, regardless of the realism of behavior implied by proportionality.

[1] The issue of non-uniqueness is well-known at least to researchers. In fact, software vendors (e.g. INR 1998) has warned practitioners of the issue and discouraged the applications of the tools relying on non-unique route flows. Nevertheless, little attention has been given to the resolution of the problem, which has been more or less regarded as something that one has to live with.

[2] It is worth noting (Bar-Gera 2006) that, although paired proportionality condition only produces an approximate MEUE solution on general networks, the difference between the two conditions are practically insignificant.

Note that stability has different meanings in different contexts. In (Nie 2009), for instance, the stability is characterized by whether or not a perturbed equilibrium (e.g., when a tiny number of traveler are assumed to switch from one path to the other) will be driven back to the original one. In this paper, we discuss solution stability from a sensitivity analysis point of view. We say that a solution is stable, if small perturbations of model inputs lead to small changes in the solution. Below, we give a more precise definition of stability in the context of MEUE problems. The inputs for a MEUE problem include the travel demand, denoted by d, and the link cost function. Let us introduce a parameter u into the link cost function, and treat both u and d as parameters subject to perturbations. The output for such a problem is the MEUE route flow. Let Ω denote the set of all pairs (u, d) under which there exist at least one UE route flow. Suppose that in the unperturbed MEUE problem the parameters (u, d) take the values of (u^0, d^0) and there is a MEUE route flow q^0. We say that q^0 is stable, if for every neighborhood Q of q^0 there exists a neighborhood N of (u^0, d^0) in Ω, such that whenever (u, d) belongs to N the perturbed MEUE problem has a solution in Q. Since the MEUE problem is known to have a unique solution, denoted by $q(u, d)$, the above definition of stability says that q^0 is stable if and only if the function q is continuous at (u^0, d^0) with respect to Ω.

The objectives of the paper are to 1) derive continuity of the UE link flow and route flows under strict monotonicity of the link cost function; 2) propose a general resolution to the non-unique issue; 3) demonstrate the stability of maximum entropy model with respect to inputs. To these ends, we give conditions that lead to a continuous selection of UE route flows and show that the maximum entropy model meets those conditions. Consequently, the MEUE route flow provides a continuous selection from the set of all UE route flows. On the other hand, a UE route flow arbitrarily generated in a standard traffic assignment procedure may not bear such continuity. Stability is of both theoretical and practical significance. It is important to *scenario comparison*, which uses UE flows to analyze and compare different scenarios of infrastructure investment and management policies. Often such analysis involves policies that are expected to make gradual instead of radical changes. For instance, adding one lane to a 3 mile freeway section may modestly change the composition of its users (in terms of their origin and destination). However, a dramatic change of this composition could well result from an arbitrary choice of UE route flows, which could in turn cast doubts on the usefulness of either the UE model itself, or software systems being used.

The next section provides a numerical example to highlight the instability associated with the arbitrary choice of UE route flows, which is followed by the main results presented in Section 3. Section 4 revisits the numerical example and concludes the paper.

2 An illustrative example

2.1 Justification

In this section, we show by an example that the UE route flow arbitrarily generated from a traffic assignment algorithm is not stable with respect to input perturbations. Specifically, a route-based algorithm known as gradient projection (GP) (Bertsekas 1976, Jayakrishnan, Tsai, Prashker & Rajadhyaksha 1994) is chosen for the demonstration. Route-based algorithms have a greater degree of arbitrariness in the choice of UE route flows than their link-based (such as the Frank-Wolfe algorithm (Frank & Wolfe 1956)) and bush-based (such as the origin-based algorithm (Bar-Gera 2002)) counterparts. See Bar-Gera, Nie & Boyce (2009) for a very recent study that has revealed this interesting difference, which is attributed to the fact that route-based algorithms operate on a small set of "used" routes obtained from column generation. In contrast, the number of routes stored in link-based and origin-based algorithms are much larger, albeit through different mechanisms [3]. Therefore, the results from this example probably represent an extremal of instability. However, any arbitrarily selected solution, regardless of how it is obtained, is equally questionable in principle.

The GP algorithm decomposes the TAP with respect to O-D pairs in the spirit of Gauss-Seidel scheme, and solves each decomposed subproblem using a projected Newton method (Bertsekas 1976). This subproblem solution process is combined with column generation, which iteratively generates routes only when they become required to optimally solve the problem (implicitly, this also implies that routes with zero-flows will be excluded from the set of "used paths"). The reader is referred to (Jayakrishnan et al. 1994) for more details about GP. To exclude the noise caused by insufficient convergence, high precision assignment solutions are obtained, corresponding to a *relative gap* [4] smaller than 10^{-12}. As a comparison, a relative gap of 10^{-6} is usually considered satisfactorily precise in practical applications.

2.2 Results

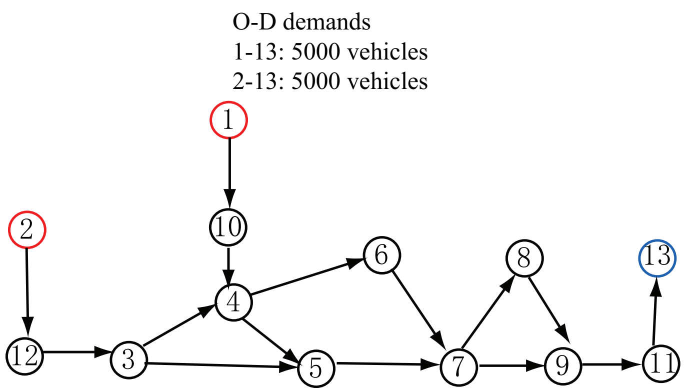
The example is concerned with the network shown in the Figure 1, which consists of 12 links whose properties are also shown in the figure. The travel demands between O-D pairs 1-13 and 2-13 are both 5,000. Links 2-12, 1-10 and 11-13 are centroid connectors whose free flow travel times are set to zero. The classical BPR function is used to model link performance, namely,

$$t(x) = t_0 \left[1 + 0.15(\frac{x}{C})^4\right]$$

where x and C are link volume and capacity, respectively; $t(\cdot)$ is link cost function, $t_0 = t(0)$ is free flow travel time.

[3] In link-based algorithms such as Frank-Wolfe, the routes generated from solving each linearized subproblem will always be carried over and have non-zero flows. In bush-based algorithms, (Bar-Gera & Boyce 1999) showed that the entropy is maximized separately for each origin.

[4] The relative gap measures the violation of UE conditions.



Link properties

From	To	Capacity (veh/hour)	Length (mile)	Free flow travel time (min)
1	10	4.99E+09	0	0
2	12	4.99E+09	0	0
3	4	4000	0.25	0.25
3	5	6000	0.5	1
4	5	4000	0.25	0.25
4	6	6000	0.5	1
5	6	4000	0.25	0.25
5	7	6000	0.5	1
6	7	4000	0.25	0.25
7	8	4000	0.25	0.25
7	9	6000	0.5	1
8	9	4000	0.25	0.25
9	11	8000	0.5	0.5
10	4	8000	0.25	0.25
11	13	4.99E+09	0	0
12	3	8000	0.25	0.25

Figure 1: Network topology and link properties

Table 1: Optimal route-flow solution obtained by the GP algorithm

Flow	Base	Scenario I	Scenario II	Scenario III	Route Topology
Route				OD 1-13	
1	1157.222	838.8729	226.9197	1339.149	1-10-4-6-7-8-9-11-15
2	625.0449	41.81449	780.0178	0	1-10-4-6-7-9-11-15
3	84.8177	1897.457	1465.647	907.5059	1-10-4-5-6-7-8-9-11-15
4	1387.399	1223.474	1047.532	1757.743	1-10-4-5-6-7-9-11-15
5	1728.526	70.12834	1479.884	1495.602	1-10-4-5-7-8-9-11-15
6	16.9907	928.2534	0	0	1-10-4-5-7-9-11-15
link 5-7	2665.65	2280.921	2615.138	2631.587	

To reveal the stability performance, four scenarios, including the base scenario and three perturbed scenarios, are examined.

Perturbed scenario I: The free flow speed on link 5-7 is reduced by 10% due to a minor construction project, which increases the free flow travel time on that link from 1 minute to 1.111 minutes;

Perturbed scenario II: The capacity on link 5-6 is expanded by 10% to 4400;

Perturbed scenario III: The O-D demand between 1-13 is increased by 10% and the demand for 2-13 is reduced by 10%.

Optimal route flow solutions from each scenario are given in Table 1 (for brevity O-D pair 2-13 is ignored). Total flows on link 5-7 are also reported. As expected, the four route flow patterns,

Table 2: Comparison of route flow and select link analysis: arbitrary UE solution

Performance Measures	Base flow	%*	Scenario I flow	%	Scenario II flow	%	Scenario III flow	%
$q_3^{1,13}$	84.8177	0	1897.457	2137.1	1465.647	1628.0	907.5059	969.9
$x_{5-7}^{1,13}$	1744.691	0	998.3817	-42.8	1479.884	-15.2	1495.602	-14.3

* - percent change

which are all highly converged UE solutions, are vastly different. It should be noted that all perturbations in inputs are relatively minor, and may simply arise from inaccuracy of data in practical applications.

We inspect these results more closely in the following, focusing on flows on route 3 ($q_3^{1,13}$) and the flows on link 5-7 that come from O-D pair 1-13 ($x_{5-7}^{1,13}$). The latter is known as select link analysis, which is often used in travel planning practice. Table 2 reports these results in the four scenarios. The instability of route flows with respect to input parameters is clear; for example, a 10% capacity increase on a single link in the network (Scenario II) has caused the flow on that route to increase more than 16 times. The results of select link analysis are also unstable. Take Scenario II as an example again. The select link analysis suggests that about 264.8 travelers (15.2%) from origin 1 will shift to link 5-6 after it is expanded. In the meanwhile, the total UE flow on link 5-7 was reduced from 2665.65 (base) to 2615.138 (Scenario II). To balance the flow, about 214.3 more travelers from origin 2 would have to move to link 5-7, a relatively less attractive alternative after the expansion. Consequently, the analysis suggests that the travelers from origin 2 actually yield the benefit of the capacity expansion to those from origin 1, which seems behaviorally unjustifiable.

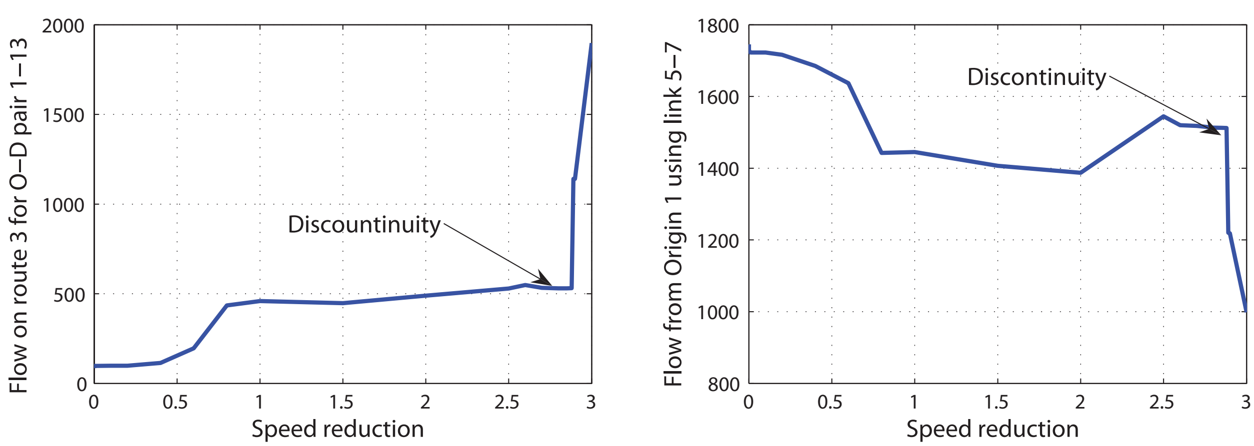


Figure 2: A stress test of stability: continuous speed reduction in Scenario I

For a further demonstration, we revisit Scenario I, examining a number of intermediate speed reductions ranging from 0.1 to 3 (mile per hour). The interval of speed reduction varies based on trial and error; namely higher resolution is employed when rapid changes in the output are detected. The results are reported in Figure 2. The plots identify a distinct discontinuous change in the route flow and select link analysis results when speed on link 5-6 is reduced from 27.12 (mph) to 27.11 (mph). If desired, one could further increase the resolution to find a more precise estimation of the discontinuity point. We did not attempt it since a numerical demonstration of this kind will never find the exact discontinuity point thanks to computers' incapability of storing "exact" real numbers.

3 Stability analysis

The stability analysis of this section starts with Theorem 1, which gives conditions for existence, uniqueness and continuity of the UE link flow. Following that, Theorem 2 proves continuity of the UE route flow as a multifunction, under the same conditions assumed in Theorem 1. Based on these results, Theorem 5 provides conditions that lead to a continuous selection of UE route flows, and Corollary 6 establishes continuity of the MEUE route flow as a single-valued function.

We use $\mathbb{R}^n$ and $\mathbb{R}^n_+$ to denote the n-dimensional Euclidean space and its nonnegative orthant respectively, $\langle \cdot, \cdot \rangle$ to denote the inner product of two column vectors, $\| \cdot \|$ to denote the Euclidean norm, and B to denote the unit closed ball in an Euclidean space. A multifunction F from a set $X \subset \mathbb{R}^n$ to $\mathbb{R}^m$ refers to an assignment for each $x \in X$ to a set $F(x) \subset \mathbb{R}^m$. The *graph* of F is a subset of $X \times \mathbb{R}^m$, defined to be $\operatorname{gph} F := \{(x, y) \mid x \in X, y \in F(x)\}$. The *domain* of F is a subset of X, defined to be $\operatorname{dom} F := \{x \in X \mid F(x) \neq \emptyset\}$.

3.1 Notations and UE-TAP formulations

A UE-TAP considers a network consisting of a set of nodes $\mathcal{N}$, a set of links $\mathcal{A}$, and a set of origin-destination (OD) pairs $\mathcal{W} \subset \mathcal{N}^2$. Each OD pair $w \in \mathcal{W}$ is connected by a set of routes $\mathcal{P}_w$, each member of which is a set of sequentially connected links. Let $\mathcal{P} = \cup_{w \in \mathcal{W}} \mathcal{P}_w$ denote the set of all routes, and let $\alpha = |\mathcal{A}|$, $\omega = |\mathcal{W}|$ and $\pi = |\mathcal{P}|$ denote the cardinalities of $\mathcal{A}$, $\mathcal{W}$ and $\mathcal{P}$ respectively. Let the matrix $(\Lambda = [\Lambda_{wp}]) \in \mathbb{R}^{\omega \times \pi}$ denote the OD-route incidence matrix in which $\Lambda_{wp} = 1$ if route $p \in \mathcal{P}_w$ and $\Lambda_{wp} = 0$ otherwise, and the matrix $(\Delta = [\Delta_{ap}]) \in \mathbb{R}^{\alpha \times \pi}$ denote the link-route incidence matrix; here $\Delta_{ap} = 1$ if link a is in route p and $\Delta_{ap} = 0$ otherwise.

We use column vectors $(d = [d_w]) \in \mathbb{R}^\omega$, $(q = [q_p]) \in \mathbb{R}^\pi$ and $(x = [x_a]) \in \mathbb{R}^\alpha$ to denote the travel demand (also called the OD flow), the route flow, and the link flow respectively. The travel demand d will be treated as a parameter to the UE-TAP, and q and x will be variables in the problem. We also consider a parameter u, taking values in a set $U \subset \mathbb{R}^m$, that affects the link cost function. We use $f : U \times \mathbb{R}^\alpha_+ \to \mathbb{R}^\alpha$ to denote the parametric link cost function; elements of the vector $f(u, x)$ give the cost on each link $a \in \mathcal{A}$ under the parameter u and the link flow x.

The link flow formulation of the UE-TAP can be written as a variational inequality (Smith 1979, Dafermos 1980). Let $S(d)$ denote the set of feasible link flows under demand d:

$$S(d) = \{x \in \mathbb{R}^\alpha \mid \text{there exists some } q \in \mathbb{R}^\pi_+, \Lambda\, q = d, \Delta\, q = x\}. \tag{1}$$

We can regard S as a multifunction from $\mathbb{R}^\omega$ to $\mathbb{R}^\alpha$. A link flow x is a UE link flow under (u, d) if and only if it satisfies the following condition:

$$x \in S(d) \text{ and } \langle f(u, x), x' - x\rangle \geq 0 \text{ for each } x' \in S(d). \tag{2}$$

The variational inequality (2) is equivalent to an optimization problem, if the Jacobian matrix of f with respect to x is symmetric and positive semidefinite.

The route flow formulation of the UE-TAP can also be written as a variational inequality; see, e.g., (Smith 1979, Patriksson 1994). We omit the details, but mention the fact that q is a UE route flow if and only if it belongs to $\mathbb{R}^\pi_+$ and satisfies $\Lambda\, q = d$ and $\Delta\, q = x$ for a UE link flow x.

3.2 Continuity of the UE link flow

The theorem below, Theorem 1, is mainly a recapitulation of the well-known result about existence, uniqueness and continuity of the UE link flow. One detail in this theorem is new, in that it obtains continuity of the UE link flow under a global *strict* monotonicity assumption on the cost function. We say that a function g from a set $X \subset \mathbb{R}^n$ to $\mathbb{R}^n$ is *strictly monotone* on X, if

$$\langle g(x) - g(x'), x - x'\rangle > 0$$

whenever $x, x' \in X$ and $x \neq x'$. We say that the function g is *strongly monotone* on X, if there exists a real number $\alpha > 0$ such that

$$\langle g(x) - g(x'), x - x'\rangle \geq \alpha \|x - x'\|^2$$

whenever $x, x' \in X$. Clearly, strict monotonicity is a weaker condition than strong monotonicity.

Most existing work on UE solution properties use some version of the strong monotonicity assumption to obtain continuity, see, e.g., (Dafermos & Nagurney 1984, Tobin & Friesz 1988, Qiu & Magnanti 1989, Nagurney 1993, Yen 1995, Outrata 1997, Cho, Smith & Friesz 2000, Patriksson & Rockafellar 2003, Lu 2008). Those works that focused on local sensitivity analysis typically use a strong monotonicity assumption over some subset of the link flow space. The papers (Smith 1979) and (Hall 1978) are based on strict monotonicity, but the former did not give a continuity result, and the latter considered only the case in which the cost function is separable and gave the continuity result only with respect to travel demands (not parameters in the cost function). On the other hand, many commonly used cost functions, such as the BPR travel time function, are strictly monotone but not strongly monotone around the origin. Thus, by using the strict monotonicity instead of strong monotonicity assumption, Theorem 1 allows us to obtain continuity of the UE link flow when some elements of the unperturbed UE link flow are zero.

The statement of Theorem 1 contains the notation $\operatorname{dom} S$, which by definition means the set of d such that $S(d)$ is nonempty. In other words, a travel demand d belongs to $\operatorname{dom} S$ if and only if it can be satisfied. For example, in a network in which each OD pair is connected by at least one route, $\operatorname{dom} S$ is exactly $\mathbb{R}^{\omega}_{+}$, the nonnegative orthant of the demand space. It will become clear by Section 3.4 that the set $U \times \operatorname{dom} S$ that appears in the statement is precisely the domain Ω of the MEUE route flow mentioned in Section 1.

Theorem 1. *Suppose that the cost function f is continuous on $U \times \mathbb{R}^{\alpha}_{+}$, and that for each $u \in U$ the function $f(u, \cdot)$ is strictly monotone on $\mathbb{R}^{\alpha}_{+}$, i.e.,*

$$\langle f(u, x) - f(u, x'), x - x' \rangle > 0$$

whenever $x, x' \in \mathbb{R}^{\alpha}_{+}$, $x \neq x'$. For each $d \in \operatorname{dom} S$ and $u \in U$, the variational inequality (2) *has a unique solution in $\mathbb{R}^{\alpha}$, denoted as $x(u, d)$. Moreover, the function $x(u, d)$ is continuous on $U \times \operatorname{dom} S$.*

Proof. The fact that (2) has a unique solution $x(u, d)$ in $\mathbb{R}^{\alpha}$ for each $(u, d) \in U \times \operatorname{dom} S$ is a result of (Smith 1979). It suffices to prove continuity of the function x.

The graph of S is a polyhedral convex set, because it is the projection of the polyhedral convex set

$$\{(d, q, x) \in \mathbb{R}^{\omega} \times \mathbb{R}^{\pi}_{+} \times \mathbb{R}^{\alpha} \mid \Lambda\, q = d, \Delta\, q = x\}$$

onto the (d, x) space. By (Rockafellar & Wets 1998, Example 9.35), S is Lipschitz continuous on $\operatorname{dom} S$ with some constant $\lambda > 0$, that is,

$$S(d) \subset S(d') + \lambda \|d - d'\| B \tag{3}$$

whenever $d, d' \in \operatorname{dom} S$.

In the following, let $\{(u^n, d^n)\}, n = 1, 2, 3, \ldots$, be a sequence in $U \times \operatorname{dom} S$ converging to $(u, d) \in U \times \operatorname{dom} S$, and let $x^n = x(u^n, d^n)$. We need to prove $\lim_{n\to\infty} x^n = x(u, d)$.

For each n and each $a \in \mathcal{A}$, we have $0 \leq x^n_a \leq \sum_{w \in \mathcal{W}} d^n_w$, so the sequence $\{x^n\}$ is bounded. Because $x^n \in S(d^n)$, the Lipschitz continuity of S ensures that all limit points of the sequence $\{x^n\}$ belong to $S(d)$. Let x be one such limit point; it suffices to prove that $x = x(u, d)$.

Let $\tilde{x} \in S(d)$. By the Lipschitz continuity of S, there exists a sequence $\{\tilde{x}^n\}$ converging to $\tilde{x}$, with $\tilde{x}^n \in S(d^n)$ for each n. The definition of x^n implies that

$$\langle f(u^n, x^n), \tilde{x}^n - x^n \rangle \geq 0$$

for each n. Passing to a subsequence and taking limits, we obtain

$$\langle f(u, x), \tilde{x} - x \rangle \geq 0.$$

Thus, x solves (2) under (u, d). This proves that $x = x(u, d)$. □

3.3 Continuity of the set of UE route flows

This section discusses a continuity property of the set of UE route flows with respect to (u,d). We need the following concepts about limits and continuity of a multifunction.

Let F be a multifunction from $X \subset \mathbb{R}^n$ to $\mathbb{R}^m$. The *outer limit* of F at $x \in X$, denoted by $\limsup_{x' \to x} F(x')$, is the set that consists of all vectors y having the property that there is a sequence $\{x^k\}$ of points of X converging to x, and a sequence $\{y^k\}$ converging to y, in which for each k the vector y^k belongs to $F(x^k)$. The *inner limit* of F at $x \in X$, denoted by $\liminf_{x' \to x} F(x')$, is the set that consists of all vectors y having the property that for each sequence $\{x^k\}$ of points of X converging to x, there is a sequence $\{y^k\}$ converging to y, in which for each k the vector y^k belongs to $F(x^k)$. We say that F is continuous at x, if

$$\limsup_{x' \to x} F(x') = \liminf_{x' \to x} F(x') = F(x).$$

We say that F is continuous on X if it is continuous at each $x \in X$.

Now, define a multifunction H from $\mathbb{R}^\omega \times \mathbb{R}^\alpha$ to $\mathbb{R}^\pi$ by

$$H(d,x) = \{q \in \mathbb{R}^\pi_+ \mid \Lambda\, q = d, \Delta\, q = x\}. \tag{4}$$

The set $H(d,x)$ for each pair (d,x) consists of nonnegative route flows that meet the travel demand d and produce the link flow x. Clearly, in order for $H(d,x)$ to be nonempty, both d and x have to be nonnegative, with $x \in S(d)$.

The graph of H is a polyhedral convex set, so by (Rockafellar & Wets 1998, Example 9.35) there is some nonnegative number θ such that H is Lipschitz continuous on $\operatorname{dom} H$ with constant θ, i.e.,

$$H(d,x) \subset H(d',x') + \theta \|(d,x) - (d',x')\| B \tag{5}$$

whenever (d,x) and (d',x') belong to $\operatorname{dom} H$. Note that Lipschitz continuity of a multifunction on a set implies its continuity on that set; thus, the multifunction H is continuous on $\operatorname{dom} H$.

Theorem 2. *Assume the hypotheses in Theorem 1, and determine the function $x : U \times \operatorname{dom} S \to \mathbb{R}^\alpha$ as in that theorem. For each $(u,d) \in U \times \operatorname{dom} S$, the set of UE route flows is a nonempty set given by*

$$Q(u,d) = H(d, x(u,d)). \tag{6}$$

Moreover, the multifunction Q is continuous on $U \times \operatorname{dom} S$.

Proof. The fact that $x(u,d) \in S(d)$ implies that the set $Q(u,d)$ defined in (6) is nonempty for each $(u,d) \in U \times \operatorname{dom} S$. We can then define a single-valued continuous function $G : U \times \operatorname{dom} S \to \operatorname{dom} H$ by $G(u,d) = (d, x(u,d))$. We have $Q(u,d) = H(G(u,d))$. Continuity of G as a single-valued function implies that any sequence $\{(u^n,d^n)\}$ in $U \times \operatorname{dom} S$ converging to (u,d) satisfies that $G(u^n,d^n)$ converges to $G(u,d)$. In view of this and the continuity of H, it is readily verified that Q is continuous. □

Theorem 2 has the following two implications. (1) If sequences $\{(u^n, d^n)\}$ and $\{q^n\}$ converge to (u, d) in $U \times \operatorname{dom} S$ and q in $\mathbb{R}^\pi$ respectively, with q^n being a UE route flow under (u^n, d^n) for each n, then q is a UE route flow under (u, d). (2) If the sequence $\{(u^n, d^n)\}$ belongs to $U \times \operatorname{dom} S$ and converges to (u, d) in $U \times \operatorname{dom} S$, and q is a UE route flow under (u, d), then there exists a sequence $\{q^n\}$ converging to q, with q^n being a UE route flow under (u^n, d^n) for each n.

3.4 Continuity of the maximum entropy UE route flow

This section proves stability of the MEUE route flow. Before defining the entropy function, we first define a function e from a subset of $\mathbb{R}^2$, $\{(x, y) \mid y \geq x \geq 0\}$, to $\mathbb{R}$ by

$$e(x,y) = \begin{cases} x \ln(x/y) & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

We will need the following lemma.

Lemma 3. *The function e is continuous on the set $\{(x, y) \mid y \geq x \geq 0\}$. For each fixed $y \in \mathbb{R}_+$, the function $e(\cdot, y)$ is strictly convex with respect to x on the interval $[0, y]$.*

Proof. Define two functions

$$a(x) = \begin{cases} x \ln x & \text{if } x > 0 \\ 0 & \text{if } x = 0, \end{cases} \qquad b(x,y) = \begin{cases} x \ln y & \text{if } y \geq x > 0 \\ 0 & \text{if } y \geq x = 0. \end{cases}$$

Note that $e(x, y) = a(x) - b(x, y)$ wherever it is defined. Clearly, the function a is continuous at each $x > 0$; by L'Hospital's rule it is also continuous at 0. The function b is clearly continuous at each (x, y) with $y > 0$; it is also continuous at $(0, 0)$, because its absolute value is bounded from above by $|y \ln y|$ on its entire domain. This proves the continuity of e.

The strict convexity of $e(\cdot, y)$ with respect to x is trivial when $y = 0$. Let $y > 0$ be fixed. For each $x > 0$, the partial derivative of e with respect to x at (x, y) is $1 + \ln(x/y)$, which strictly increases as x increases. Thus, $e(\cdot, y)$ is strictly convex on $[0, y]$, see, e.g., (Rockafellar & Wets 1998, Lemma 2.12). □

Below, we define the entropy function E as a function from a set $\operatorname{dom} E \subset \mathbb{R}^\omega \times \mathbb{R}^\pi$ to $\mathbb{R}$. The set $\operatorname{dom} E$ is defined as

$$\operatorname{dom} E := \{(d, q) \in \mathbb{R}^\omega \times \mathbb{R}^\pi \mid d_w \geq q_p \geq 0 \text{ for each } w \in \mathcal{W} \text{ and each } p \in \mathcal{P}_w\},$$

and for each $(d, q) \in \operatorname{dom} E$ we define the entropy function as

$$E(d,q) = -\sum_{w \in \mathcal{W}} \sum_{p \in \mathcal{P}_w} e(q_p, d_w). \tag{8}$$

The following corollary is an immediate result of Lemma 3.

Corollary 4. *The function E is continuous on* $\operatorname{dom} E$*, and is strictly concave with respect to q for each fixed $d \in \mathbb{R}^{\omega}_{+}$.*

The maximum entropy user equilibrium problem is to find the UE route flow that maximizes the entropy function. For each $(u, d) \in U \times \operatorname{dom} S$, the problem is

$$\begin{array}{ll} \max & E(d, q) \\ \text{s.t.} & q \in Q(u, d), \end{array} \tag{9}$$

where the set $Q(u, d)$ is defined in Theorem 2 and is the set of all UE route flows under parameter (u, d).

The following theorem shows that the UE route flow selected to maximize an objective function with certain properties is a continuous function with respect to (u, d). We will apply this theorem to obtain stability results on the maximum entropy UE route flow.

Theorem 5. *Assume the hypotheses in Theorem 1. Let F be a single-valued function from* $\operatorname{dom} F \subset \mathbb{R}^{\omega} \times \mathbb{R}^{\pi}$ *to* $\mathbb{R}$*, such that*

$$(d, q) \in \operatorname{dom} F$$

whenever $d \in$ $\operatorname{dom} S$ *and $q \in Q(u, d)$ for some $u \in U$. Suppose that F is a continuous function on* $\operatorname{dom} F$*, and is strictly concave with respect to q for each fixed d. For each $(u, d) \in U \times$* $\operatorname{dom} S$*, the problem*

$$\begin{array}{ll} \max & F(d, q) \\ \textit{s.t.} & q \in Q(u, d), \end{array} \tag{10}$$

has a unique solution, denoted by $q(u, d)$, and the function q is continuous on $U \times$ $\operatorname{dom} S$*.*

Proof. For each $(u, d) \in U \times \operatorname{dom} S$, the set $Q(u, d)$ is nonempty, closed, convex and bounded, where the non-emptiness follows from Theorem 2, the closedness and convexity follows from its definition in (6), and the boundedness follows from the fact that $0 \leq q_p \leq d_w$ for each $p \in \mathcal{P}_w$ and $w \in \mathcal{W}$. Moreover, the multifunction Q is continuous on $U \times \operatorname{dom} S$, as proved in Theorem 2.

Because $F(d, \cdot)$ is strictly concave, the problem (10) has a unique solution for each $(u, d) \in U \times \operatorname{dom} S$, which we denote by $q(u, d)$. To prove the continuity of q, let $\{(u^n, d^n)\}, n = 1, 2, 3, \ldots,$ be a sequence in $U \times \operatorname{dom} S$ converging to $(u, d) \in U \times \operatorname{dom} S$, and let $q^n = q(u^n, d^n)$. We need to prove $\lim_{n \to \infty} q^n = q(u, d)$.

Clearly, the sequence $\{q^n\}$ is bounded. Because $q^n \in Q(u^n, d^n)$, the continuity of Q ensures that all limit points of $\{q^n\}$ belong to $Q(u, d)$. Let q be one such limit point; it suffices to prove that $q = q(u, d)$.

Let $\tilde{q} \in Q(u, d)$. By the continuity of Q, there exists a sequence $\{\tilde{q}^n\}$ converging to $\tilde{q}$, with $\tilde{q}^n \in Q(u^n, d^n)$ for each n. The definition of q^n implies that

$$F(d^n, q^n) \geq F(d^n, \tilde{q}^n)$$

Table 3: The MEUE route-flow solution for the example

	Base	Scenario I	Scenario II	Scenario III
Route		O-D pair 2-13		
1	800.9399	868.4355	728.3017	853.1028
2	434.3997	471.0067	395.0034	462.6908
3	1576.695	1633.92	1665.708	1535.575
4	855.1403	886.1768	903.4176	832.838
5	864.145	739.4246	847.7702	853.1028
6	468.6798	401.0361	459.7986	462.6908

for each n. Passing to a subsequence and taking limits, we obtain

$$F(d, q) \geq F(d, \tilde{q}).$$

Thus, q solves (10) under (u, d). This proves that $q = q(u, d)$. □

The way we defined the entropy function E ensures that $\operatorname{dom} E$ includes every pair (d, q) satisfying $d \in \operatorname{dom} S$ and $q \in Q(u, d)$ for some $u \in U$. In view of Corollary 4, we can apply Theorem 5 with E in place of F to obtain the following corollary. It justifies the existence and uniqueness of the MEUE route flow for each $(u, d) \in U \times \operatorname{dom} S$, and obtains its continuity with respect to (u, d).

Corollary 6. *Assume the hypotheses in Theorem 1. For each $(u, d) \in U \times \operatorname{dom} S$, the problem* (9) *has a unique solution, denoted by $q(u, d)$, and the function q is continuous on $U \times \operatorname{dom} S$.*

4 Discussion

Having proven the stability of the MEUE route flow, we now revisit the example in Section 2. We note that MEUE route flows in this example are relatively easy to compute since the routes can be enumerated and the interaction between the two origins is minor. For details the reader is referred to (Bar-Gera & Boyce 1999). The MEUE route solutions for the four scenarios are reported in Table 3. As expected, these results suggest a generally continuous pattern of change in perturbed cases.

Table 4 gives the flow on route 3 and the portion of the volume on link 5-7 contributed by O-D pair 1-13. Similarly, the change of both route flows and select link analysis results demonstrate more stable and reasonable behavior under perturbation. In the capacity expansion scenario (Scenario II), the flow on route 3 is increased by about 5.6%, which is expected since that route uses the expanded link. For the select link analysis, the maximum entropy result suggests 25 (1.9%) travelers from origin 1 would move to link 5-6 from 5-7. Considering the total flow leaving link 5-7 is about 50, another 25 flow shift would have to be contributed by origin 2. Thus, the expansion will equally impact both origins.

Table 4: Comparison of route flow and select link analysis: MEUE solution

Flow	Base		Scenario I		Scenario II		Scenario III	
	flow	%	flow	%	flow	%	flow	%
$q_3^{1,13}$	1576.695	100	1633.92	3.6	1665.708	5.6	1689.132	7.1
$x_{5-7}^{1,13}$	1332.825	100	1140.461	-14.4	1307.569	-1.9	1315.79	-1.3

We emphasize that Theorem 5 is given in a very general setting, which includes the maximum entropy UE route flow as a special case. Indeed, one can use that theorem as a guide to choose objective functions for the purpose of selecting UE route flows. More specifically, to guarantee the existence, uniqueness and stability of the UE route flows, one should choose an objective function that satisfies the conditions given in the theorem. To establish the relationship between a particular functional form and the behavioral realism is an interesting direction for further investigation. Until that relationship is well understood, the entropy function is still, arguably, the most intuitive, justifiable and computationally feasible choice that bears all the desired properties.

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