

Sensitivity of Variational Inequalities on Perturbed Polyhedral Convex Sets: Analysis and Implementation

Shu Lu

Department of Industrial and Systems Engineering
University of Wisconsin-Madison

Feb 7, 2007

From continuous optimization to variational inequalities

From continuous optimization to variational inequalities

Let $F : \mathbb{R}^n \rightarrow \mathbb{R}$ be a differentiable function, and S be a subset of \mathbb{R}^n . Consider the constrained optimization problem

$$\begin{aligned} & \text{minimize } F(x) \\ & \text{subject to } x \in S. \end{aligned} \tag{1}$$

From continuous optimization to variational inequalities

Let $F : \mathbb{R}^n \rightarrow \mathbb{R}$ be a differentiable function, and S be a subset of \mathbb{R}^n . Consider the constrained optimization problem

$$\begin{aligned} & \text{minimize } F(x) \\ & \text{subject to } x \in S. \end{aligned} \tag{1}$$

Any local minimizer x of this problem must satisfy

$$x \in S \quad \text{and} \quad \langle \nabla F(x), v \rangle \geq 0 \text{ for each } v \in T_S(x). \tag{2}$$

$T_S(x)$, the **tangent cone**, is the set of all limits of the form $\lim_{k \rightarrow \infty} \frac{x_k - x}{\lambda_k}$, where $x_k \in S$ and $\lambda_k > 0$ for each k , and $\lim_{k \rightarrow \infty} \lambda_k = 0$.

From continuous optimization to variational inequalities

Let $F : \mathbb{R}^n \rightarrow \mathbb{R}$ be a differentiable function, and S be a subset of \mathbb{R}^n . Consider the constrained optimization problem

$$\begin{aligned} & \text{minimize } F(x) \\ & \text{subject to } x \in S. \end{aligned} \tag{1}$$

Any local minimizer x of this problem must satisfy

$$x \in S \quad \text{and} \quad \langle \nabla F(x), v \rangle \geq 0 \text{ for each } v \in T_S(x). \tag{2}$$

$T_S(x)$, the **tangent cone**, is the set of all limits of the form $\lim_{k \rightarrow \infty} \frac{x_k - x}{\lambda_k}$, where $x_k \in S$ and $\lambda_k > 0$ for each k , and $\lim_{k \rightarrow \infty} \lambda_k = 0$.

When S is convex, Problem (2) is equivalent to

$$x \in S \quad \text{and} \quad \langle \nabla F(x), x' - x \rangle \geq 0 \text{ for each } x' \in S. \tag{3}$$

Definition of variational inequalities

In general, a variational inequality in \mathbb{R}^n is of the form

$$x \in S \quad \text{and} \quad \langle f(x), x' - x \rangle \geq 0 \text{ for each } x' \in S, \quad (4)$$

where S is a convex subset of \mathbb{R}^n and f is a function from S to \mathbb{R}^n .

Definition of variational inequalities

In general, a variational inequality in \mathbb{R}^n is of the form

$$x \in S \quad \text{and} \quad \langle f(x), x' - x \rangle \geq 0 \text{ for each } x' \in S, \quad (4)$$

where S is a convex subset of \mathbb{R}^n and f is a function from S to \mathbb{R}^n .

For each $x \in S$ define the **normal cone** $N_S(x)$ by

$$N_S(x) := \{y \in \mathbb{R}^n \mid \langle y, x' - x \rangle \leq 0 \text{ for each } x' \in S\},$$

and let $N_S(x) = \emptyset$ for $x \notin S$. Then (4) is equivalent to

$$0 \in f(x) + N_S(x). \quad (5)$$

Definition of variational inequalities

In general, a variational inequality in \mathbb{R}^n is of the form

$$x \in S \quad \text{and} \quad \langle f(x), x' - x \rangle \geq 0 \quad \text{for each } x' \in S, \quad (4)$$

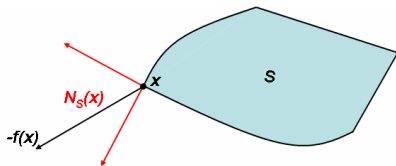
where S is a convex subset of \mathbb{R}^n and f is a function from S to \mathbb{R}^n .

For each $x \in S$ define the **normal cone** $N_S(x)$ by

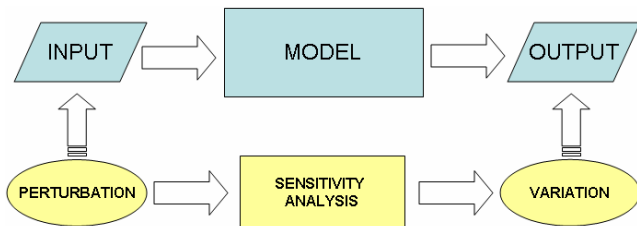
$$N_S(x) := \{y \in \mathbb{R}^n \mid \langle y, x' - x \rangle \leq 0 \text{ for each } x' \in S\},$$

and let $N_S(x) = \emptyset$ for $x \notin S$. Then (4) is equivalent to

$$0 \in f(x) + N_S(x). \quad (5)$$



Usefulness of sensitivity analysis



Sensitivity analysis for variational inequalities...

- Helps to identify conditions under which one could have a good prospect of successful numerical solution of a variational inequality
- Provides tools for analyzing convergence rates of algorithms
- Can be embedded in bi-level programming

Outline of topics

1 Variational Inequalities over Perturbed Polyhedral Convex Sets

Outline of topics

- 1 Variational Inequalities over Perturbed Polyhedral Convex Sets
- 2 Sensitivity of Traffic Network Equilibria

Outline of topics

- 1 Variational Inequalities over Perturbed Polyhedral Convex Sets
- 2 Sensitivity of Traffic Network Equilibria
- 3 Summary

1 Variational Inequalities over Perturbed Polyhedral Convex Sets

2 Sensitivity of Traffic Network Equilibria

3 Summary

Linear variational inequality over polyhedral convex sets

Consider the variational inequality in x

$$0 \in L(x) + w + N_{S(u)}(x). \quad (6)$$

Linear variational inequality over polyhedral convex sets

Consider the variational inequality in x

$$0 \in L(x) + w + N_{S(u)}(x). \quad (6)$$

Here,

- L is an affine map from \mathbb{R}^n to \mathbb{R}^n defined by $L(x) = Mx + b$;

Linear variational inequality over polyhedral convex sets

Consider the variational inequality in x

$$0 \in L(x) + w + N_{S(u)}(x). \quad (6)$$

Here,

- L is an affine map from \mathbb{R}^n to \mathbb{R}^n defined by $L(x) = Mx + b$;
- $w \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are two parameters;

Linear variational inequality over polyhedral convex sets

Consider the variational inequality in x

$$0 \in L(x) + w + N_{S(u)}(x). \quad (6)$$

Here,

- L is an affine map from \mathbb{R}^n to \mathbb{R}^n defined by $L(x) = Mx + b$;
- $w \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are two parameters;
- $S(u)$ is a polyhedral convex set in \mathbb{R}^n defined as

$$S(u) = \{x \in \mathbb{R}^n \mid Ax \leq u\} = \{x \in \mathbb{R}^n \mid \langle a_i, x \rangle \leq u_i, i = 1, \dots, m\} \quad (7)$$

where A is an $m \times n$ matrix whose rows are a_1, \dots, a_m . Write $\text{dom } S = \{u \in \mathbb{R}^m \mid S(u) \neq \emptyset\}$.

Linear variational inequality over polyhedral convex sets

Consider the variational inequality in x

$$0 \in L(x) + w + N_{S(u)}(x). \quad (6)$$

Here,

- L is an affine map from \mathbb{R}^n to \mathbb{R}^n defined by $L(x) = Mx + b$;
- $w \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are two parameters;
- $S(u)$ is a polyhedral convex set in \mathbb{R}^n defined as

$$S(u) = \{x \in \mathbb{R}^n \mid Ax \leq u\} = \{x \in \mathbb{R}^n \mid \langle a_i, x \rangle \leq u_i, i = 1, \dots, m\} \quad (7)$$

where A is an $m \times n$ matrix whose rows are a_1, \dots, a_m . Write $\text{dom } S = \{u \in \mathbb{R}^m \mid S(u) \neq \emptyset\}$.

Let $u_0 \in \text{dom } S$; suppose that $x_0 \in \mathbb{R}^n$ satisfies $0 \in L(x_0) + N_{S(u_0)}(x_0)$.

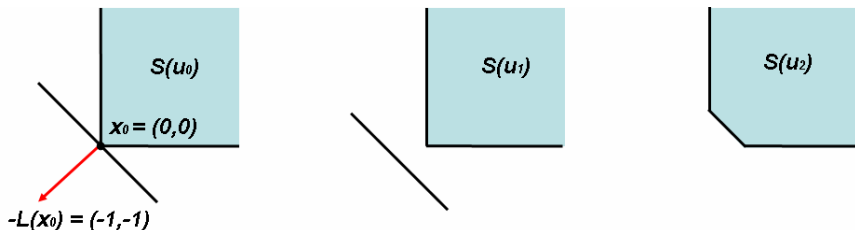
We analyze solutions of (6) near x_0 for $(u, w) \in \text{dom } S \times \mathbb{R}^n$ near $(u_0, 0)$.



Example 1: let $n = 2$, $m = 3$ with

$$A = \begin{bmatrix} -1 & -1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, M = \begin{bmatrix} -1 & 1 \\ 1 & 4 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and take $u_0 = (0, 0, 0)^T$, $u_1 = (\epsilon, 0, 0)^T$ and $u_2 = (-\epsilon, 0, 0)^T$ for some positive number ϵ . The figure below shows $S(u_i)$ for $i = 1, 2, 3$.



The origin of \mathbb{R}^2 , x_0 , satisfies $0 \in L(x_0) + N_{S(u_0)}(x_0)$.

Variational inequalities and normal maps

For each $u \in \text{dom } S$, define the **normal map** $L_{S(u)} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$L_{S(u)}(z) = L(\Pi_{S(u)}(z)) + z - \Pi_{S(u)}(z), \quad (8)$$

where $\Pi_{S(u)}(z)$ is the Euclidean projection of z on $S(u)$.

Variational inequalities and normal maps

For each $u \in \text{dom } S$, define the **normal map** $L_{S(u)} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$L_{S(u)}(z) = L(\Pi_{S(u)}(z)) + z - \Pi_{S(u)}(z), \quad (8)$$

where $\Pi_{S(u)}(z)$ is the Euclidean projection of z on $S(u)$.

The normal map (8) is related to the variational inequality (6) by

$$\begin{array}{|l} 0 \in L(x) + w + N_{S(u)}(x) \\ z = x - L(x) - w \end{array} \begin{array}{l} \implies \\ \iff \end{array} \begin{array}{|l} L_{S(u)}(z) = -w \\ x = \Pi_{S(u)}(z) \end{array}$$

Variational inequalities and normal maps

For each $u \in \text{dom } S$, define the **normal map** $L_{S(u)} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$L_{S(u)}(z) = L(\Pi_{S(u)}(z)) + z - \Pi_{S(u)}(z), \quad (8)$$

where $\Pi_{S(u)}(z)$ is the Euclidean projection of z on $S(u)$.

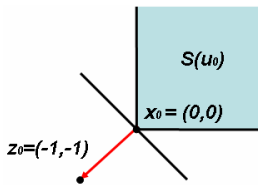
The normal map (8) is related to the variational inequality (6) by

$$\begin{array}{l} 0 \in L(x) + w + N_{S(u)}(x) \\ z = x - L(x) - w \end{array} \iff \begin{array}{l} L_{S(u)}(z) = -w \\ x = \Pi_{S(u)}(z) \end{array}$$

Define $z_0 = x_0 - L(x_0)$; we have

$$L_{S(u_0)}(z_0) = 0 \text{ and } \Pi_{S(u_0)}(z_0) = x_0.$$

In Example 1, $z_0 = x_0 - L(x_0) = (-1, -1)$.



Faces of the polyhedral convex set $S(u)$

For each $u \in \text{dom } S \dots$

- Let $\mathcal{F}[S(u)]$ be the family of faces of $S(u)$

$$\mathcal{F}[S(u)] = \{\emptyset\} \cup \{\operatorname{argmax}_{S(u)} \langle \cdot, y \rangle : y \in \mathbb{R}^n\}$$

Faces of the polyhedral convex set $S(u)$

For each $u \in \text{dom } S \dots$

- Let $\mathcal{F}[S(u)]$ be the family of faces of $S(u)$

$$\mathcal{F}[S(u)] = \{\emptyset\} \cup \{\operatorname{argmax}_{S(u)} \langle \cdot, y \rangle : y \in \mathbb{R}^n\}$$

- Each F in $\mathcal{F}[S(u)] \setminus \{\emptyset\}$ has an active index set I_F : the collection of indices i such that $\langle a_i, x \rangle = u_i$ for every point x of F

Faces of the polyhedral convex set $S(u)$

For each $u \in \text{dom } S \dots$

- Let $\mathcal{F}[S(u)]$ be the family of faces of $S(u)$

$$\mathcal{F}[S(u)] = \{\emptyset\} \cup \{\operatorname{argmax}_{S(u)} \langle \cdot, y \rangle : y \in \mathbb{R}^n\}$$

- Each F in $\mathcal{F}[S(u)] \setminus \{\emptyset\}$ has an active index set I_F : the collection of indices i such that $\langle a_i, x \rangle = u_i$ for every point x of F
- Let $\mathcal{I}(u)$ be the family of active index sets of $S(u)$

Faces of the polyhedral convex set $S(u)$

For each $u \in \text{dom } S \dots$

- Let $\mathcal{F}[S(u)]$ be the family of faces of $S(u)$

$$\mathcal{F}[S(u)] = \{\emptyset\} \cup \{\operatorname{argmax}_{S(u)} \langle \cdot, y \rangle : y \in \mathbb{R}^n\}$$

- Each F in $\mathcal{F}[S(u)] \setminus \{\emptyset\}$ has an active index set I_F : the collection of indices i such that $\langle a_i, x \rangle = u_i$ for every point x of F
- Let $\mathcal{I}(u)$ be the family of active index sets of $S(u)$
- Each $I \subset \{1, \dots, m\}$ defines a face for $S(u)$

$$F(I, u) = \{x \in \mathbb{R}^n \mid \langle a_i, x \rangle = u_i, i \in I, \langle a_j, x \rangle \leq u_j, j \in \{1, \dots, m\} \setminus I\}$$

Faces of the polyhedral convex set $S(u)$

For each $u \in \text{dom } S \dots$

- Let $\mathcal{F}[S(u)]$ be the family of faces of $S(u)$

$$\mathcal{F}[S(u)] = \{\emptyset\} \cup \{\text{argmax}_{S(u)} \langle \cdot, y \rangle : y \in \mathbb{R}^n\}$$

- Each F in $\mathcal{F}[S(u)] \setminus \{\emptyset\}$ has an active index set I_F : the collection of indices i such that $\langle a_i, x \rangle = u_i$ for every point x of F
- Let $\mathcal{I}(u)$ be the family of active index sets of $S(u)$
- Each $I \subset \{1, \dots, m\}$ defines a face for $S(u)$

$$F(I, u) = \{x \in \mathbb{R}^n \mid \langle a_i, x \rangle = u_i, i \in I, \langle a_j, x \rangle \leq u_j, j \in \{1, \dots, m\} \setminus I\}$$

$\mathcal{F}[S(u)] \setminus \{\emptyset\}$	F	\longrightarrow	I_F		$\mathcal{I}(u)$
	$F(I, u)$	\longleftarrow	I		

Normal manifold of the polyhedral convex set $S(u)$

Normal manifold of the polyhedral convex set $S(u)$

For each $I \in \mathcal{J}(u)$, $N_{S(u)}(x)$ takes a constant value on $\text{ri} F(I, u)$:

$$N_{S(u)}[F(I, u)] = \text{pos}\{a_i, i \in I\}.$$

Normal manifold of the polyhedral convex set $S(u)$

For each $I \in \mathcal{I}(u)$, $N_{S(u)}(x)$ takes a constant value on $\text{ri} F(I, u)$:

$$N_{S(u)}[F(I, u)] = \text{pos}\{a_i, i \in I\}.$$

The normal manifold of $S(u)$ is the collection of n -cells

$$C(I, u) = F(I, u) + N_{S(u)}(F(I, u)) = F(I, u) + \text{pos}\{a_i, i \in I\}$$

for all $I \in \mathcal{I}(u)$.

Normal manifold of the polyhedral convex set $S(u)$

For each $I \in \mathcal{I}(u)$, $N_{S(u)}(x)$ takes a constant value on $\text{ri} F(I, u)$:

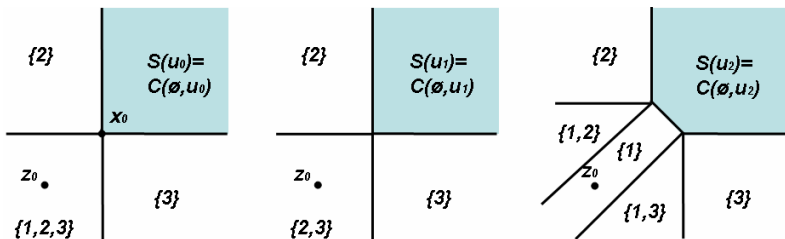
$$N_{S(u)}[F(I, u)] = \text{pos}\{a_i, i \in I\}.$$

The normal manifold of $S(u)$ is the collection of n -cells

$$C(I, u) = F(I, u) + N_{S(u)}(F(I, u)) = F(I, u) + \text{pos}\{a_i, i \in I\}$$

for all $I \in \mathcal{I}(u)$.

The figure below shows normal manifolds of sets in Example 1.



Homeomorphism theorem under coherent orientation

For a given $u \in \text{dom } S...$

- $\Pi_{S(u)}$ and $L_{S(u)}$ are piecewise affine on the normal manifold of $S(u)$

¹The section of M in a subspace E is $Q^T M Q$ given that columns of Q form an orthonormal basis for E .

Homeomorphism theorem under coherent orientation

For a given $u \in \text{dom } S \dots$

- $\Pi_{S(u)}$ and $L_{S(u)}$ are piecewise affine on the normal manifold of $S(u)$
- The affine map representing $L_{S(u)}$ on $C(I, u)$ has the same determinant as the section¹ of the matrix M in $\ker A_I$

¹The section of M in a subspace E is $Q^T M Q$ given that columns of Q form an orthonormal basis for E .

Homeomorphism theorem under coherent orientation

For a given $u \in \text{dom } S$...

- $\Pi_{S(u)}$ and $L_{S(u)}$ are piecewise affine on the normal manifold of $S(u)$
- The affine map representing $L_{S(u)}$ on $C(I, u)$ has the same determinant as the section¹ of the matrix M in $\ker A_I$
- $L_{S(u)}$ is a Lipschitzian homeomorphism from \mathbb{R}^n to \mathbb{R}^n iff it is **coherently oriented** - the determinants of the affine maps representing it have the same nonzero sign [Robinson, 1992]

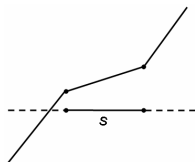
¹The section of M in a subspace E is $Q^T M Q$ given that columns of Q form an orthonormal basis for E .

Homeomorphism theorem under coherent orientation

For a given $u \in \text{dom } S \dots$

- $\Pi_{S(u)}$ and $L_{S(u)}$ are piecewise affine on the normal manifold of $S(u)$
- The affine map representing $L_{S(u)}$ on $C(I, u)$ has the same determinant as the section¹ of the matrix M in $\ker A_I$
- $L_{S(u)}$ is a Lipschitzian homeomorphism from \mathbb{R}^n to \mathbb{R}^n iff it is **coherently oriented** - the determinants of the affine maps representing it have the same nonzero sign [Robinson, 1992]

A piecewise affine map from \mathbb{R} to \mathbb{R} is coherently oriented iff it is either strictly increasing or strictly decreasing



¹The section of M in a subspace E is $Q^T M Q$ given that columns of Q form an orthonormal basis for E .

Global coherent orientation condition

The following are equivalent:

- The variational inequality $0 \in L(x) + w + N_{S(u)}(x)$ has a unique solution for each $(u, w) \in \text{dom } S \times \mathbb{R}^n$
- $L_{S(u)}$ is a global Lipschitzian homeomorphism for each $u \in \text{dom } S$

Global coherent orientation condition

The following are equivalent:

- The variational inequality $0 \in L(x) + w + N_{S(u)}(x)$ has a unique solution for each $(u, w) \in \text{dom } S \times \mathbb{R}^n$
- $L_{S(u)}$ is a global Lipschitzian homeomorphism for each $u \in \text{dom } S$
- For each $u \in \text{dom } S$, the determinants of the sections of the matrix M in $\ker A_I$ for all $I \in \mathcal{I}(u)$ have the same nonzero sign

Global coherent orientation condition

The following are equivalent:

- The variational inequality $0 \in L(x) + w + N_{S(u)}(x)$ has a unique solution for each $(u, w) \in \text{dom } S \times \mathbb{R}^n$
- $L_{S(u)}$ is a global Lipschitzian homeomorphism for each $u \in \text{dom } S$
- For each $u \in \text{dom } S$, the determinants of the sections of the matrix M in $\ker A_I$ for all $I \in \mathcal{I}(u)$ have the same nonzero sign
- The determinants of the sections of the matrix M in $\ker A_I$ for all $I \in \bigcup_{u \in \text{dom } S} \mathcal{I}(u)$ have the same nonzero sign

Global coherent orientation condition

The following are equivalent:

- The variational inequality $0 \in L(x) + w + N_{S(u)}(x)$ has a unique solution for each $(u, w) \in \text{dom } S \times \mathbb{R}^n$
- $L_{S(u)}$ is a global Lipschitzian homeomorphism for each $u \in \text{dom } S$
- For each $u \in \text{dom } S$, the determinants of the sections of the matrix M in $\ker A_I$ for all $I \in \mathcal{I}(u)$ have the same nonzero sign
- The determinants of the sections of the matrix M in $\ker A_I$ for all $I \in \bigcup_{u \in \text{dom } S} \mathcal{I}(u)$ have the same nonzero sign

These particularly hold when M is positive definite (possibly asymmetric)
- the section of a positive definite matrix in a subspace is still positive definite, and has strictly positive determinant.

Question

Can we reduce the collection of index sets that need to be considered, provided that we are interested only in local solutions under local perturbations?

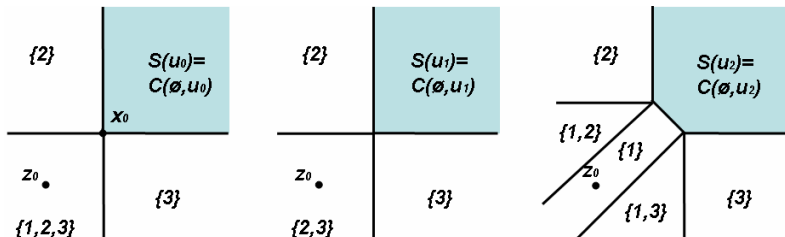
Question

Can we reduce the collection of index sets that need to be considered, provided that we are interested only in local solutions under local perturbations?

[Lu and Robinson, 2006]: yes

Branching phenomenon

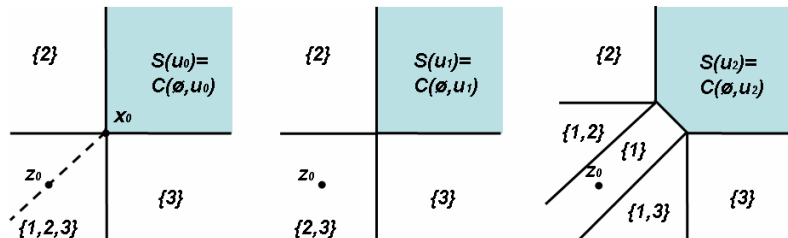
A closer look at the faces and normal manifolds of $S(u)$ in Example 1...



Branching phenomenon: a face of $S(u_0)$ splits into several new faces as u_0 changes to u_1 and u_2 .

Branching phenomenon

A closer look at the faces and normal manifolds of $S(u)$ in Example 1...



Branching phenomenon: a face of $S(u_0)$ splits into several new faces as u_0 changes to u_1 and u_2 .

Moreover, the n -cells in the normal manifold of $S(u_1)$ or $S(u_2)$ can be approximated by parts of the n -cells in the normal manifold of $S(u_0)$.

We can show...

- If we define $l_0 := \{i \in \{1, \dots, m\} \mid \langle a_i, x_0 \rangle = (u_0)_i\}$
- And define for each $u \in \text{dom } S$ a subset of $\mathcal{I}(u)$

$$\mathcal{I}_0(u) := \{I \in \mathcal{I}(u) \mid I \subset l_0 \text{ and } z_0 - x_0 \in \text{pos}\{a_i, i \in I\}\}$$

- Then there exists a ball $\text{int } \mathbb{B}(z_0, \gamma)$ around z_0 , and a neighborhood U_1 of u_0 in $\text{dom } S$, such that for each $u \in U_1$

$$\begin{aligned} I \in \mathcal{I}_0(u) &\Rightarrow \text{int } \mathbb{B}(z_0, \gamma) \cap C(I, u) \neq \emptyset \\ I \in \mathcal{I}(u) \setminus \mathcal{I}_0(u) &\Rightarrow \text{int } \mathbb{B}(z_0, 3\gamma) \cap C(I, u) = \emptyset \end{aligned}$$

In Example 1, we have...

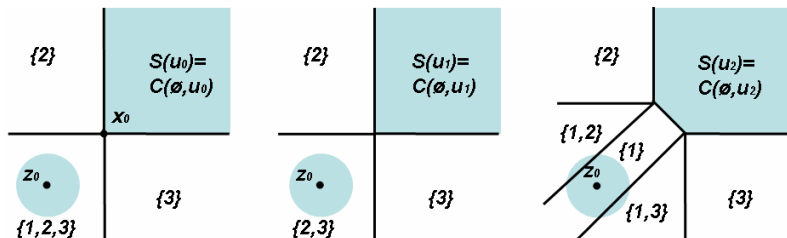
$$I_0 = \{1, 2, 3\}$$

$$\mathcal{I}_0(u_0) = \{\{1, 2, 3\}\}$$

$$\mathcal{I}_0(u_1) = \{\{2, 3\}\}$$

$$\mathcal{I}_0(u_2) = \{\{1\}, \{1, 2\}, \{1, 3\}\}$$

The ball $\text{int } \mathbb{B}(z_0, \gamma)$ looks like...



We can further show...

For each $u \in U_1$, the set $\mathcal{J}_0(u)$ has a minimum element $I_c(u)$

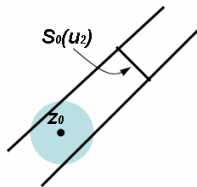
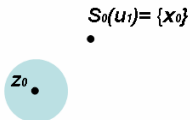
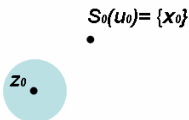
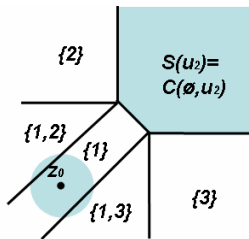
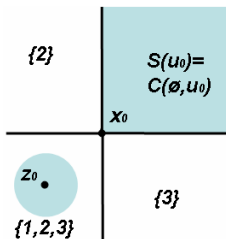
If we define a polyhedral convex set $S_0(u)$ for each $u \in U_1$ by

$$S_0(u) = \{x \in \mathbb{R}^n \mid \langle a_i, x \rangle = u_i, i \in I_c(u), \langle a_i, x \rangle \leq u_i, i \in I_0 \setminus I_c(u)\}$$

Then there is a neighborhood U_2 of u_0 in U_1 such that for $u \in U_2$...

- The structure of normal manifolds of $S_0(u)$ and $S(u)$ coincide in $\text{int } \mathbb{B}(z_0, \gamma)$
- The normal manifold of $S_0(u)$ contains no more n -cells than those meeting $\text{int } \mathbb{B}(z_0, \gamma)$
- And the normal maps $L_{S(u)}$ and $L_{S_0(u)}$ coincide in $\text{int } \mathbb{B}(z_0, \gamma)$

In Example 1...



If we assume...

that the determinants of the sections of the matrix M in $\ker A_l$ for all $l \in \bigcup_{u \in \text{dom } S} \mathcal{I}_0(u)$ have the same nonzero sign

If we assume...

that the determinants of the sections of the matrix M in $\ker A_l$ for all $l \in \bigcup_{u \in \text{dom } S} \mathcal{I}_0(u)$ have the same nonzero sign

Then for u in U_2 ...

- $L_{S_0(u)}$ is coherently oriented, hence a global homeomorphism

If we assume...

that the determinants of the sections of the matrix M in $\ker A_l$ for all $l \in \bigcup_{u \in \text{dom } S} \mathcal{I}_0(u)$ have the same nonzero sign

Then for u in U_2 ...

- 1 $L_{S_0(u)}$ is coherently oriented, hence a global homeomorphism
- 2 $L_{S_0(u)}$ is a local homeomorphism on $\text{int } \mathbb{B}(z_0, \gamma)$

If we assume...

that the determinants of the sections of the matrix M in $\ker A_l$ for all $l \in \bigcup_{u \in \text{dom } S} \mathcal{I}_0(u)$ have the same nonzero sign

Then for u in U_2 ...

- 1 $L_{S_0(u)}$ is coherently oriented, hence a global homeomorphism
- 2 $L_{S_0(u)}$ is a local homeomorphism on $\text{int } \mathbb{B}(z_0, \gamma)$
- 3 $L_S(u)$ is a local homeomorphism on $\text{int } \mathbb{B}(z_0, \gamma)$

If we assume...

that the determinants of the sections of the matrix M in $\ker A_l$ for all $l \in \bigcup_{u \in \text{dom } S} \mathcal{I}_0(u)$ have the same nonzero sign

Then for u in U_2 ...

- 1 $L_{S_0(u)}$ is coherently oriented, hence a global homeomorphism
- 2 $L_{S_0(u)}$ is a local homeomorphism on $\text{int } \mathbb{B}(z_0, \gamma)$
- 3 $L_{S(u)}$ is a local homeomorphism on $\text{int } \mathbb{B}(z_0, \gamma)$
- 4 The image of $\text{int } \mathbb{B}(z_0, \gamma)$ under $L_{S(u)}$ contains a fixed ball in \mathbb{R}^n , $\text{int } \mathbb{B}(0, \delta)$

If we assume...

that the determinants of the sections of the matrix M in $\ker A_l$ for all $l \in \bigcup_{u \in \text{dom } S} \mathcal{J}_0(u)$ have the same nonzero sign

Then for u in U_2 ...

- 1 $L_{S_0(u)}$ is coherently oriented, hence a global homeomorphism
- 2 $L_{S_0(u)}$ is a local homeomorphism on $\text{int } \mathbb{B}(z_0, \gamma)$
- 3 $L_{S(u)}$ is a local homeomorphism on $\text{int } \mathbb{B}(z_0, \gamma)$
- 4 The image of $\text{int } \mathbb{B}(z_0, \gamma)$ under $L_{S(u)}$ contains a fixed ball in \mathbb{R}^n , $\text{int } \mathbb{B}(0, \delta)$
- 5 For each $u \in U_2$ and $w \in \text{int } \mathbb{B}(0, \delta)$, there exists a unique $z \in \text{int } \mathbb{B}(z_0, \gamma)$ such that $L_{S(u)}(z) = w$

If we assume...

that the determinants of the sections of the matrix M in $\ker A_l$ for all $l \in \bigcup_{u \in \text{dom } S} \mathcal{I}_0(u)$ have the same nonzero sign

Then for u in U_2 ...

- 1 $L_{S_0(u)}$ is coherently oriented, hence a global homeomorphism
- 2 $L_{S_0(u)}$ is a local homeomorphism on $\text{int } \mathbb{B}(z_0, \gamma)$
- 3 $L_{S(u)}$ is a local homeomorphism on $\text{int } \mathbb{B}(z_0, \gamma)$
- 4 The image of $\text{int } \mathbb{B}(z_0, \gamma)$ under $L_{S(u)}$ contains a fixed ball in \mathbb{R}^n , $\text{int } \mathbb{B}(0, \delta)$
- 5 For each $u \in U_2$ and $w \in \text{int } \mathbb{B}(0, \delta)$, there exists a unique $z \in \text{int } \mathbb{B}(z_0, \gamma)$ such that $L_{S(u)}(z) = w$

Recall the relation between normal maps and variational inequalities...

Sensitivity of variational inequality: a theorem

Let L , S , u_0 and x_0 be as defined. The following are equivalent:

- 1 For each $I \in \bigcup_{u \in \text{dom } S} \mathcal{I}_0(u)$ the determinant of the section of the matrix M in $\ker A_I$ has the same nonzero sign.
- 2 There exist neighborhoods U' of u_0 in $\text{dom } S$, W' of the origin and X' of x_0 in \mathbb{R}^n , and a single-valued, Lipschitz continuous function $x : U' \times W' \rightarrow \mathbb{R}^n$ such that for each $(u, w) \in U' \times W'$, $x(u, w)$ is the unique solution in X' of the variational inequality

$$0 \in L(x) + w + N_{S(u)}(x).$$

Sensitivity of variational inequality: a theorem

Let L , S , u_0 and x_0 be as defined. The following are equivalent:

- 1 For each $I \in \bigcup_{u \in \text{dom } S} \mathcal{I}_0(u)$ the determinant of the section of the matrix M in $\ker A_I$ has the same nonzero sign.
- 2 There exist neighborhoods U' of u_0 in $\text{dom } S$, W' of the origin and X' of x_0 in \mathbb{R}^n , and a single-valued, Lipschitz continuous function $x : U' \times W' \rightarrow \mathbb{R}^n$ such that for each $(u, w) \in U' \times W'$, $x(u, w)$ is the unique solution in X' of the variational inequality

$$0 \in L(x) + w + N_{S(u)}(x).$$

- Can replace $\text{dom } S$ by any convex subset U_0 of $\text{dom } S$ containing u_0
- Can extend to nonlinear variational inequalities over perturbed polyhedral convex sets

In Example 1...

	$\bigcup_{u \in \text{dom } S} \mathcal{I}(u)$							
	$\bigcup_{u \in \text{dom } S} \mathcal{I}_0(u)$							
	{1, 2, 3}	{2, 3}	{1, 2}	{1, 3}	{1}	{2}	{3}	\emptyset
$\ker A_I$	{0}	{0}	{0}	{0}	$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \mathbb{R}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbb{R}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbb{R}$	\mathbb{R}^2
Q	[]	[]	[]	[]	$\begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	l_2
$Q^T M Q$	[]	[]	[]	[]	[1/2]	[4]	[-1]	M
Det	1	1	1	1	1/2	4	-1	-5

1 Variational Inequalities over Perturbed Polyhedral Convex Sets

2 Sensitivity of Traffic Network Equilibria

3 Summary

Notation

Consider a network with

- arcs $a \in \mathcal{A}$ with the **arc flow** $(x = [x_a]) \in \mathbb{R}^\alpha$
- routes $p \in \mathcal{P}$ with the **route flow** $(q = [q_p]) \in \mathbb{R}^\pi$
- origin-destination pairs $w \in \mathcal{W}$ with the **travel demand** $(d = [d_w]) \in \mathbb{R}^\omega$

We use incidence matrices

- $\Lambda \in \mathbb{R}^{\omega \times \pi}$ (OD-route): $\Lambda_{wp} = 1$ if p connects w
- $\Delta \in \mathbb{R}^{\alpha \times \pi}$ (arc-route): $\Delta_{ap} = 1$ if a is on p

and the **arc cost function** $f : \mathbb{R}^\alpha \rightarrow \mathbb{R}^\alpha$ such that

- cost on arc a is $f_a(x)$
- cost on route p is $c_p(x) = (\Delta_{(\cdot, p)})^T f(x)$

Traffic equilibria

Traffic equilibria

Given the travel demand d and the arc cost function f

- The network flow (q, x) is **feasible** if

$$q \geq 0, \quad d = \Lambda q, \quad x = \Delta q$$

Traffic equilibria

Given the travel demand d and the arc cost function f

- The network flow (q, x) is **feasible** if

$$q \geq 0, \quad d = \Lambda q, \quad x = \Delta q$$

- A feasible network flow attains a **user equilibrium** if each route p with $q_p > 0$ satisfies

$$c_p(x) \leq c_{p'}(x) \text{ for each } p' \text{ connecting the same OD pair as } p$$

Traffic equilibria

Given the travel demand d and the arc cost function f

- The network flow (q, x) is **feasible** if

$$q \geq 0, \quad d = \Lambda q, \quad x = \Delta q$$

- A feasible network flow attains a **user equilibrium** if each route p with $q_p > 0$ satisfies

$$c_p(x) \leq c_{p'}(x) \text{ for each } p' \text{ connecting the same OD pair as } p$$

- We say that x is an equilibrium arc flow if there exists $q \in \mathbb{R}^\pi$ such that (q, x) is feasible and attains a user equilibrium

Traffic equilibria

Given the travel demand d and the arc cost function f

- The network flow (q, x) is **feasible** if

$$q \geq 0, \quad d = \Lambda q, \quad x = \Delta q$$

- A feasible network flow attains a **user equilibrium** if each route p with $q_p > 0$ satisfies

$$c_p(x) \leq c_{p'}(x) \text{ for each } p' \text{ connecting the same OD pair as } p$$

- We say that x is an equilibrium arc flow if there exists $q \in \mathbb{R}^\pi$ such that (q, x) is feasible and attains a user equilibrium

[Smith, 1979] and [Dafermos, 1980] showed that x is an equilibrium arc flow iff

$$0 \in f(x) + N_S(x)$$

where $S = \{x \in \mathbb{R}^\alpha \mid \text{for some } q \in \mathbb{R}_+^\pi, \Lambda q = d, \Delta q = x\}$.

Parametric traffic equilibria

- Treat d as a parameter; let it take values in an open set D in \mathbb{R}^ω
- Introduce another parameter u , which takes values in an open set U in \mathbb{R}^m , into the arc cost function to get $f : U \times \mathbb{R}^\alpha \rightarrow \mathbb{R}^\alpha$
 $(u, x) \rightarrow f(u, x)$

Parametric traffic equilibria

- Treat d as a parameter; let it take values in an open set D in \mathbb{R}^ω
- Introduce another parameter u , which takes values in an open set U in \mathbb{R}^m , into the arc cost function to get $f : U \times \mathbb{R}^\alpha \rightarrow \mathbb{R}^\alpha$

$$(u, x) \rightarrow f(u, x)$$

Given the parameter (u, d) , x is an equilibrium arc flow iff

$$0 \in f(u, x) + N_{S(d)}(x) \quad (9)$$

where $S(d)$ is the set of feasible arc flows,

$$S(d) = \{x \in \mathbb{R}^\alpha \mid \text{for some } q \in \mathbb{R}_+^\pi, \Lambda q = d, \Delta q = x\}.$$

Parametric traffic equilibria

- Treat d as a parameter; let it take values in an open set D in \mathbb{R}^ω
- Introduce another parameter u , which takes values in an open set U in \mathbb{R}^m , into the arc cost function to get $f : U \times \mathbb{R}^\alpha \rightarrow \mathbb{R}^\alpha$
 $(u, x) \rightarrow f(u, x)$

Given the parameter (u, d) , x is an equilibrium arc flow iff

$$0 \in f(u, x) + N_{S(d)}(x) \quad (9)$$

where $S(d)$ is the set of feasible arc flows,

$$S(d) = \{x \in \mathbb{R}^\alpha \mid \text{for some } q \in \mathbb{R}_+^\pi, \Lambda q = d, \Delta q = x\}.$$

Define $H(d) := \{q \in \mathbb{R}_+^\pi : \Lambda q = d\}$ to be the set of feasible route flows; we then have $S(d) = \Delta H(d)$.

Sensitivity of generalized equations: a theorem³

Let G be a polyhedral multifunction² from $\mathbb{R}^k \times \mathbb{R}^n \rightarrow \mathbb{R}^n$,
 (d^0, x^0) a point of $\mathbb{R}^k \times \mathbb{R}^n$,
 D_0 a convex subset of \mathbb{R}^k containing u^0 ,
 X_0 a polyhedral convex neighborhood of x^0 in \mathbb{R}^n ,
 U_0 a neighborhood of a point u^0 in \mathbb{R}^j ,
 f a Lipschitz continuous function from $U_0 \times X_0$ to \mathbb{R}^n strictly
differentiable in x at x^0 uniformly on U_0 .

Suppose that the values u^0 , x^0 and d^0 satisfy the generalized equation

$$0 \in f(u, x) + G(d, x), \quad (10)$$

and define for $x \in X_0$

$$L(x) := f(u^0, x^0) + d_x f(u^0, x^0)(x - x^0). \quad (11)$$

²A polyhedral multifunction is a set-valued function with its graph a finite union of polyhedral convex sets.

³From [Lu and Robinson, 2006]

Sensitivity of generalized equations: cont'd

Assume that there is a neighborhood Ω_0 of the origin in \mathbb{R}^n such that the linearized generalized equation

$$w \in L(x) + G(d, x)$$

has a unique solution for each $(d, w) \in D_0 \times \Omega_0$.

Then there are neighborhoods U' of u^0 in U_0 , D' of d^0 in D_0 , and X' of x^0 in \mathbb{R}^n , and a single-valued, Lipschitzian function $x : U' \times D' \rightarrow X'$, such that for each $(u, d) \in U' \times D'$ the point $x(u, d)$ is the unique solution in X' of (10).

Furthermore, if $f(\cdot, \cdot)$ is strictly differentiable at (u^0, x^0) and $u^0 \in \text{int}(U_0)$, then the function $x(\cdot, \cdot)$ is semidifferentiable at (u^0, d^0) and its semiderivative $dx(u^0, d^0)(r, s)$ coincides locally at the origin of $\mathbb{R}^j \times \mathbb{R}^k \times \mathbb{R}^n$ with

$$[L(\cdot) + G(d^0 + s, \cdot)]^{-1}(-d_u f(u^0, x^0)(r)) - x^0.$$

Apply this theorem to traffic equilibria...

The theorem deals with $0 \in f(u, x) + G(d, x)$

Traffic equilibria in the form $0 \in f(u, x) + N_{S(d)}(x)$

- The operator $N_{S(d)}(x)$ here is a polyhedral multifunction of (d, x)

Apply this theorem to traffic equilibria...

The theorem deals with $0 \in f(u, x) + G(d, x)$

Traffic equilibria in the form $0 \in f(u, x) + N_{S(d)}(x)$

- The operator $N_{S(d)}(x)$ here is a polyhedral multifunction of (d, x)
- Difficulty: the theorem would require the linearized variational inequality

$$w \in L(x) + N_{S(d)}(x)$$

to have a locally unique solution for any w near the origin and any d near d^0

Apply this theorem to traffic equilibria...

The theorem deals with $0 \in f(u, x) + G(d, x)$

Traffic equilibria in the form $0 \in f(u, x) + N_{S(d)}(x)$

- The operator $N_{S(d)}(x)$ here is a polyhedral multifunction of (d, x)
- Difficulty: the theorem would require the linearized variational inequality

$$w \in L(x) + N_{S(d)}(x)$$

to have a locally unique solution for any w near the origin and any d near d^0

- [Lu, 2006]: use special structure of the set $S(d)$

Variation of polyhedral convex sets: a proposition

- Let D be a subset of \mathbb{R}^k
- For each $d \in D$ define sets
$$H(d) = \{q \in \mathbb{R}^l \mid Vd + Wq \leq s\}$$
 and $S(d) = \Delta H(d)$ where V , W and Δ are $m \times k$, $m \times l$ and $n \times l$ matrices respectively
- Let $d^0 \in D$, $q^0 \in H(d^0)$, $x^0 = \Delta q^0$ and $A = \{i : V_i d^0 + W_i q^0 = s_i\}$

Assume the *constraint nondegeneracy condition*

$$\text{range } W_A \supseteq V_A(d_0 - D).$$

Then there exists an affine function $\tilde{q} : D \rightarrow \mathbb{R}^l$ satisfying $\tilde{q}(d^0) = 0$ and $W_A \tilde{q}(d) = V_A(d^0 - d)$. For each such \tilde{q} , there exist neighborhoods D_2 of d_0 in D and X_2 of x_0 in \mathbb{R}^n such that for any $d \in D_2$,

$$S(d) \cap X_2 = (S(d_0) + \Delta \tilde{q}(d)) \cap X_2.$$

Sets of feasible arc flows

Suppose that the parameter (u^0, d^0) is given and we have

- $x^0 \in S(d^0)$ solving $0 \in f(u^0, x) + N_{S(d^0)}(x)$
- $q^0 \in H(d^0)$ with $x^0 = \Delta q^0$

Suppose further that the base travel demand $d^0 > 0$, then...

- The constraint nondegeneracy condition in the previous proposition holds
- There exist neighborhoods X_2 of x^0 in \mathbb{R}^α and D_2 of d^0 in D , and an affine function $\tilde{q} : D \rightarrow \mathbb{R}^\pi$ such that $\tilde{q}(d^0) = 0$ and

$$S(d) \cap X_2 = (S(d^0) + \Delta \tilde{q}(d)) \cap X_2$$

for each $d \in D_2$

Sensitivity of traffic equilibria: a summary

Assume the notation for the traffic equilibrium problem.

Define a map $L : \mathbb{R}^\alpha \rightarrow \mathbb{R}^\alpha$ by $L(x) := f(u^0, x^0) + d_x f(u^0, x^0)(x - x_0)$.

Define $K := T_{S(d^0)}(x^0) \cap f(u^0, x^0)^\perp$ to be the critical cone.

Suppose

- 1** $f(\cdot, \cdot)$ is continuously differentiable around (u^0, x^0) ;
- 2** the base travel demand $d^0 > 0$;
- 3** the normal map L_K is a global homeomorphism from \mathbb{R}^α to \mathbb{R}^α .

Then there exist neighborhoods U' of u^0 in U , D' of d^0 in D and X' of x^0 in \mathbb{R}^α , and a single-valued, Lipschitz continuous, semidifferentiable function $x : U' \times D' \rightarrow X'$, such that for each $(u, d) \in U' \times D'$, the point $x(u, d)$ is the unique solution in X' of the traffic equilibrium problem

$$0 \in f(u, x) + N_{S(d)}(x).$$

More about the critical cone

More about the critical cone

A certain partition of the route set \mathcal{P} by $\mathcal{P}^{00} \cup \mathcal{P}^{01} \cup \mathcal{P}^1 \cup \mathcal{P}^2$ gives

$$\begin{aligned}
 K &= \Delta \{ q \in \mathbb{R}^\pi \mid \Lambda q = 0, q_p \geq 0 \text{ for } p \in \mathcal{P}^{00}, q_p = 0 \text{ for } p \in \mathcal{P}^{01}, \\
 &\quad q_p \in \mathbb{R} \text{ for } p \in \mathcal{P}^1 \cup \mathcal{P}^2 \} \\
 &= (\Delta_{00} - \Delta_1 \Lambda_{00}) \mathbb{R}_+^{\pi_{00}} + (\Delta_2 - \Delta_1 \Lambda_2) \mathbb{R}^{\pi_2}
 \end{aligned}$$

More about the critical cone

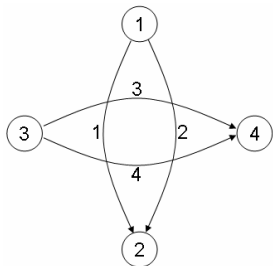
A certain partition of the route set \mathcal{P} by $\mathcal{P}^{00} \cup \mathcal{P}^{01} \cup \mathcal{P}^1 \cup \mathcal{P}^2$ gives

$$\begin{aligned} K &= \Delta \{ q \in \mathbb{R}^\pi \mid \Lambda q = 0, q_p \geq 0 \text{ for } p \in \mathcal{P}^{00}, q_p = 0 \text{ for } p \in \mathcal{P}^{01}, \\ &\quad q_p \in \mathbb{R} \text{ for } p \in \mathcal{P}^1 \cup \mathcal{P}^2 \} \\ &= (\Delta_{00} - \Delta_1 \Lambda_{00}) \mathbb{R}_+^{\pi_{00}} + (\Delta_2 - \Delta_1 \Lambda_2) \mathbb{R}^{\pi_2} \end{aligned}$$

We can then...

- Use the second expression to check the homeomorphism condition, to check if K is a subspace by solving a linear programming problem, and to compute the derivatives when K is a subspace
- Use the first expression to compute the semiderivatives by solving a linearized traffic equilibrium problem with zero travel demands and upper and lower bounds on the route flow

Numerical example



OD 1: Node 1 -> Node 2

OD 2: Node 3 -> Node 4

Route 1: Node 1 -> Node 2 via Arc 1

Route 2: Node 1 -> Node 2 via Arc 2

Route 3: Node 3 -> Node 4 via Arc 3

Route 4: Node 3 -> Node 4 via Arc 4

$$f(u, x) = \begin{bmatrix} 1 & 0.5 & 1 & 0 \\ 0.5 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0.5 \\ 0 & 1 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 20 \\ 20 \\ 20 \\ 20 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

Given $u^0 = [0, 0, 0, 0]^T$ and $d^0 = [20, 20]^T \dots$

- 1 Solve the base problem to obtain

$$x^0 = [10, 10, 10, 10]^T, q^0 = [10, 10, 10, 10]^T$$

- 2 Partition the route set to have

$$\mathcal{P}^{00} = \mathcal{P}^{01} = \emptyset, \mathcal{P}^1 = \{1, 3\}, \mathcal{P}^2 = \{2, 4\}$$

- 3 Write down the critical cone K : it is a subspace in this example

- 4 Check the homeomorphism condition: it holds here

- 5 Compute the derivatives $\nabla_{x(u,d)}(u^0, d^0) =$

$$\begin{matrix} & u_1 & u_2 & u_3 & u_4 & d_1 & d_2 \\ x_1 & \left(\begin{array}{cccccc} 1/3 & -1/3 & -2/3 & 2/3 & 1/2 & 0 \\ -1/3 & 1/3 & 2/3 & -2/3 & 1/2 & 0 \\ -2/3 & 2/3 & 1/3 & -1/3 & 0 & 1/2 \\ 2/3 & -2/3 & -1/3 & 1/3 & 0 & 1/2 \end{array} \right) \end{matrix}$$

Larger examples

<i>Dimensions</i>	Sioux Falls, SD	Anaheim, CA
Nodes	24	416
OD pairs	528	1406
Arcs	76	914
User optimal routes	770	1710
Unused user optimal routes	63	86
Matrix for inversion	29	42

The equilibrium arc flow is differentiable in both examples.

Summary

- The first characterization of good behavior of solutions of linear variational inequalities when the right-hand sides of the constraints change
- A consequence: sufficient condition for good behavior of solutions of general (nonlinear) linearly-constrained variational inequalities when the right-hand sides of the constraints change
- Applications to a wide variety of optimization and equilibrium problems occurring in practice
- Complete analysis of the fixed-demand traffic network equilibria under perturbations of the demand and the arc cost function

Acknowledgment

- The research presented here was done under the supervision of Dr. Stephen M. Robinson; the first part of this talk is based on a submitted paper coauthored with him
- The work on the second part benefited from discussion with Dr. David Boyce
- We used the origin-based traffic assignment algorithm by Dr. Hillel Bar-Gera in solving the unperturbed traffic equilibrium problems
- The research presented here was sponsored in part by the National Science Foundation under Grant DMS-0305930, in part by the Air Force Research Laboratory under agreement number FA9550-04-1-0192, and in part by the U. S. Army Research Laboratory and the U. S. Army Research Office under Grant DAAD19-01-1-0502

Questions and Comments...



Böhm, V. (1975).

On the continuity of the optimal policy set for linear programs.
SIAM Journal on Applied Mathematics, 28:303–306.



Cho, H.-J., Smith, T. E., and Friesz, T. L. (2000).

A reduction method for local sensitivity analyses of network equilibrium arc flow.
Transportation Research Part B, 34:31–51.



Cottle, R. W., Pang, J.-S., and Stone, R. E. (1992).

The Linear Complementarity Problem.
Academic Press, Inc., Boston, MA.



Dafermos, S. (1980).

Traffic equilibrium and variational inequalities.
Transportation Science, 14:42–54.



Dafermos, S. (1988).

Sensitivity analysis in variational inequalities.

Mathematics of Operations Research, 13:421–433.



Dafermos, S. and Nagurney, A. (1984).

Sensitivity analysis for the asymmetric network equilibrium problem.

Mathematical Programming, 28:174–184.



Dontchev, A. L. and Rockafellar, R. T. (1996).

Characterizations of strong regularity for variational inequalities over polyhedral convex sets.

SIAM Journal on Optimization, 6:1087–1105.



Facchinei, F. and Pang, J.-S. (2003).

Finite-Dimensional Variational Inequalities and Complementarity Problems.

Springer Series in Operations Research. Springer-Verlag, New York.

Published in two volumes, paginated continuously.



Fiacco, A. V. (1976).

Sensitivity analysis for nonlinear programming using penalty methods.

Mathematical Programming, 10:287–311.



Hall, M. A. (1978).

Properties of the equilibrium state in transportation networks.

Transportation Science, 12:208–216.



Klatte, D. and Kummer, B. (2002).

Nonsmooth Equations in Optimization: Regularity, Calculus, Methods and Applications.

Kluwer Academic Publishers, Dordrecht.



Liu, J. (1995).

Strong stability in variational inequalities.

SIAM Journal on Control and Optimization, 33:725–749



Lu, S. (2006).

Sensitivity of static traffic equilibria with perturbations in arc cost function and travel demand.

Preprint 2006, submitted for publication.



Lu, S. and Robinson, S. M. (2006).

Variational inequalities over perturbed polyhedral convex sets.

Preprint 2006, submitted for publication.



Luo, Z.-Q., Pang, J.-S., and Ralph, D. (1996).

Mathematical Programs with Equilibrium Constraints.

Cambridge University Press.



Mangasarian, O. L. and Shiau, T.-H. (1987).

Lipschitz continuity of solutions of linear inequalities, programs and complementarity problems.

SIAM Journal on Control and Optimization, 25:583–595.



Outrata, J. V. (1997).

On a special class of mathematical programs with equilibrium constraints.

In Gritzmann, P., Horst, R., Sachs, E., and Tichatschke, R., editors, *Recent Advances in Optimization: Proceedings of the Eighth French-German Conference on Optimization*, number 452 in Lecture Notes in Economics and Mathematical Systems, pages 246–260, Berlin. Springer-Verlag.



Patriksson, M. (2004).

Sensitivity analysis of traffic equilibria.

Transportation Science, 38:258–281.



Patriksson, M. and Rockafellar, R. T. (2003).

Sensitivity analysis of aggregated variational inequality problems, with application to traffic equilibrium.

Transportation Science, 37:56–68.



Qiu, Y. and Magnanti, T. L. (1989).

Sensitivity analysis for variational inequalities defined on polyhedral sets.

Mathematics of Operations Research, 14:410–432.



Ralph, D. (1993).

A new proof of Robinson's homeomorphism theorem for PL-normal maps.

Linear Algebra and Its Applications, 178:249–260.



Ralph, D. (1994).

On branching numbers of normal manifolds.






Nonlinear Analysis: Theory, Methods, and Applications, 22:1041–1050.



Robinson, S. M. (1992).

Normal maps induced by linear transformations.

Mathematics of Operations Research, 17:691–714.

-  Robinson, S. M. (2003).
Constraint nondegeneracy in variational analysis.
Mathematics of Operations Research, 28:201–232.
-  Scholtes, S. (1996).
A proof of the branching number bound for normal manifolds.
Linear Algebra and its Applications, 246:83–95.
-  Smith, M. J. (1979).
The existence, uniqueness and stability of traffic equilibria.
Transportation Research Part B, 13:295–304.
-  Tobin, R. L. (1986).
Sensitivity analysis for variational inequalities.
Journal of Optimization Theory and Applications, 48:191–204.
-  Tobin, R. L. and Friesz, T. L. (1988).
Sensitivity analysis for equilibrium network flow.

Transportation Science, 22:242–250.



Yang, H. and Bell, M. G. (2005).

Sensitivity analysis of network traffic equilibria revisited: The corrected approach.

Preprint 2005, submitted for publication.



Yen, N. D. (1995).

Lipschitz continuity of solutions of variational inequalities with a parametric polyhedral constraint.

Mathematics of Operations Research, 20:695–708.