The Kontsevich integral for $(2, n)$-type torus knots.

S. D. Tyurina*

Kolomna Pedagogical Institute
e-mail: tyurina@mccme.ru

Abstract

Vassiliev's invariants seem to be a very promising set of knot invariants to classify knot types. All Vassiliev's invariants may be obtained from the Kontsevich integral calculating weights of chord diagrams.

In 1997 an explicit formula for the universal Vassiliev invariant of the trivial knot was obtained [5].

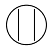
To find an analogues formula for an arbitrary knot is an open problem.

We discuss a formula to compute the Kontsevich integral of a $(2, n)$-type torus knot.

1 Introduction

1.1 Chord diagrams

A chord diagram D_n of order n is a circle with a distinguished set of n unordered pairs of points connected by chords regarded up to orientation preserving diffeomorphisms of the circle.
Denote $\mathcal{D}_n$ the vector space over $\mathbf{Q}$ generated by chord diagrams of order n.
For example, $\mathcal{D}_2$ is generated by 2 diagrams:

$\mathcal{D}_2 =<$  ,  $>$.

To code a diagram with n chords we will use the following usual notation. We numerate ends of chors according to their first passing and write a sequence of $2n$ numbers. For example, two chord diagrams above is coding by [1122] and [1212].
Consider the following relations in $\mathcal{D}_n$:

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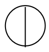
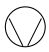
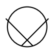
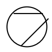
1-term relation:  $= 0$

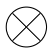
4-term relation:  $-$  $+$  $-$  $= 0$

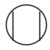
Consider the quotient space:

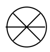
$$\mathcal{A}_n = \mathcal{D}_n/_{\{1-,4-t.relations\}}.$$

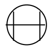
1.2 Weight systems

A linear function $W_n : \mathcal{A}_n \longrightarrow \mathbf{Q}$ is called a weight system of order n.
We give examples of weight systems of order 2, 3, and 4.

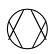
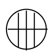
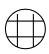
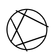
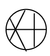
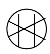
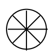
$W_2($  $)=1$

$W_2($  $)=0$

$W_3($  $)=2$

$W_3($  $)=1$

$W_3 = 0$ in other cases.

							
$W_{4(1)}$	0	0	1	0	0	1	1
$W_{4(2)}$	1	-1	2	0	-1	1	0
$W_{4(3)}$	0	1	-3	1	2	-2	0

Elsewhere $W_{4(1)} = W_{4(2)} = W_{4(3)} = 0$.

Denote $\bar{\mathcal{A}}$ the graded completion of the direct sum of spaces $\mathcal{A}_n : \bar{\mathcal{A}} = \oplus_{j=0}^{\infty}\mathcal{A}_j$.

1.3 Tangles

Let $M = [a, b] \times \mathbf{C} \subset \mathbf{R^3}$. A tangle is a 1-dimensional compact oriented piecewise smooth submanifold $T \subset M$ lying between two horizontal planes $b \times \mathbf{C}$ and $a \times \mathbf{C}$, called the top plane and the bottom plane of T, such that every boundary point of T is lying in the top or the bottom plane.

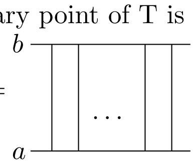
$T =$ 

Fig. 1

Two tangles T and T' are called equivalent if there is an isotopy of $\mathbf{R^3}$ sending T to T' and the top (resp. the bottom) plane of T to the top (resp. the bottom) plane of T'. When the top boundary of a tangle T_2 coincides with the bottom one of other tangle T_1 one can define the product (or the composition) of tangles $T_1 \cdot T_2$ by putting T_1 above T_2. Any tangle can be represented as a composition of the elementary tangles:

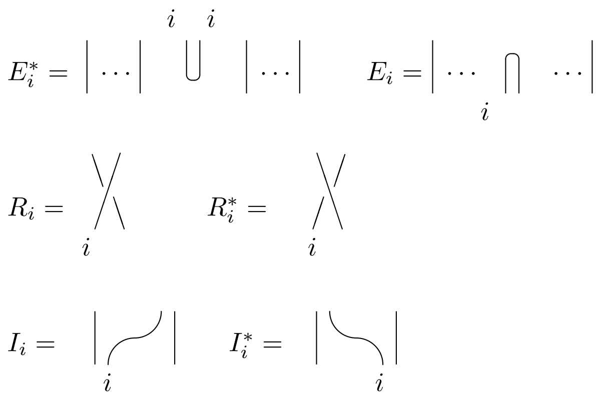


Fig. 2

The trivial tangle will be denoted I_i.

Any knot can be represented as a tangle with empty boundary. For an oriented knot we need to indicate the orientation of the tangle strands. We equip every boundary point with the sign $r = \pm 1$ that is defined according to the picture:

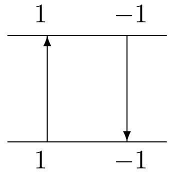
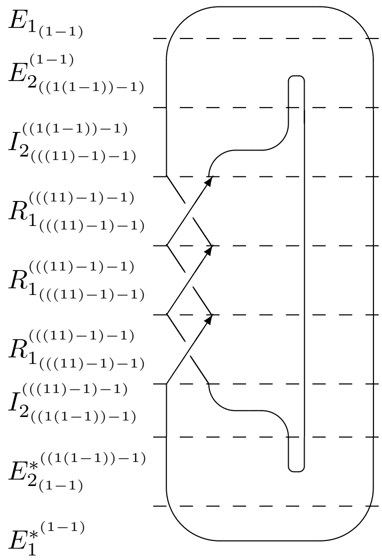


Fig. 3

We consider the so-called q-tangles [8], which are a generalization of tangles with non-associative words of two numbers 1 and -1. A word w with brackets is called a non-associative word of 1's and -1's if w is equal to 1, -1 or (w_1, w_2), where w_1 and w_2 are non-associative words of 1's and -1's. The support of w is the sequence of 1's and -1's obtained from w by removing all brackets. For example, there are two non-associative words ((11)1) and (1(11)) with support (1,1,1). Let r (resp. s) be words consist of top (ressp. bottom) boundary points of a tangle T, u and v be non-associative words with support r and s respectively. The triple (T, u, v) is called a q-tangle. As for usual tangles, we can consider a category of q-tangles. The product of two q-tangles (T, u, v) and (T', u', v') is defined by $(T, u, v)(T', u', v') = (TT', u, v'))$, if $v = u'$.

Positions of brackets are illustrated in the following picture:



knot 3_1

Fig. 4

Thus the knot 3_1 (the trefoil or the $(2,3)$-type torus knot) has the following decomposition:

$$3_1 = E_1 \cdot E_2 \cdot I_2^* \cdot R_1 \cdot R_1 \cdot R_1 \cdot I_2 \cdot E_2^* \cdot E_1^*. \quad (1)$$

1.4 A chord diagram on a tangle

A tangled chord diagram Ω_{ij} is the trivial tangle with a chord connecting i-th and j-th strands:

$\Omega_{ij} =$

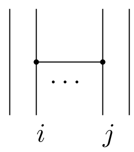


Fig. 5

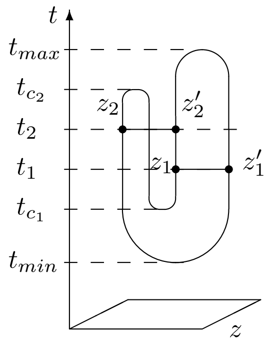
The product (or composition) of tangled chord diagrams is defined as the product of tangles.

1.5 Torus knots

By definition a torus knot is a simple closed curve on the 2-dimensional torus T^2. It is well known that a $(2,n)$-type torus knot is represented as a closed braid $(b_1)^n, n \geq 3, n$ is odd.

2 The Kontsevich integral

Represent $\mathbf{R^3}$ as the direct product of the complex line with coordinate z and of the real line with coordinate t. Consider a knot K embedded in $\mathbf{R}^3 = \mathbf{C}_z \times \mathbf{R}_t$ in such a way that the coordinate t is a Morse function on K. (All critical points are nondegenerated and all critical levels are different).



$\bigcup$-knot
Fig. 6

Denote the maximum and the minimum of a function t on K by t_{min} and t_{max}. Consider m different noncritical levels $t_{min} < t_1 < \ldots < t_m < t_{max}$. In the picture (see Fig. 6) $m = 2$. Choose two points z_i, z_i' of the intersection of the knot with the horizontal plane $\{t = t_i\}$. The knot branches define locally the smooth functions $z_i(t), z_i'(t)$. The m-times iterated integral is the integral with values in $\mathcal{A}_m$:

$$Z_m(K) = \int_{\substack{t_{min}<t_1<\ldots<t_m<t_{max} \\ t_i are \quad noncritical}} \sum_{P=\{(z_i,z_i')|i=1,\ldots,m\}} (-1)^{\downarrow(P)} D_P \bigwedge_{i=1}^{m} \frac{dz_i - dz_i'}{z_i - z_i'},$$

where sum is over all choices of pairs $(z_i(t), z_i'(t))$ for all levels, D_P is chord diagram formed by chords connected points from each pair, and the symbol "$\downarrow$" denotes the number of points z_i or z_i' from P in which the knot orientation is downward.

The Kontsevich integral is the integral with values in $\bar{\mathcal{A}}$:

$$Z(K) = \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} Z_m(K).$$

The Kontsevich integral is an invariant under the deformation in the class of knots with nondegenerate critical points and the same number of critical points. Unfortunately, the Kontsevich integral is not invariant under deformations that change the number of critical points. Denote by $c(K)$ the number of critical points of the function t on K and consider modified Kontsevich integral:

$$I(K) = \frac{Z(K)}{Z(\bigcup)^{c(K)/2}}.$$

$I(K)$ is an isotopy knot invariant. It is called the universal Vassiliev invariant.

3 Drinfeld's associator

Consider the linear differential equation

(2) $$G'(z) = \frac{1}{2\pi i}(\frac{A}{z} + \frac{B}{z-1})G(z),$$

where $G(z)$ is a formal series in two non-commutating variables A and B with coefficients which are analytic functions in the complex variable z. This equation has a singularity at 0 and 1. Set $h = \frac{1}{2\pi i}$. There exist unique solutions G_0 and G_1 of (2) in a neighborhoods U_0 and U_1 of 0 and 1 with asymptotics

$$G_0(z) \sim_{z\to 0} z^{hA}, \qquad G_1(z) \sim_{z\to 1} (1-z)^{hB},$$

$U_0 \cap U_1 \neq \emptyset$. Solutions $G_0(z)$ and $G_1(z)$ are connected by an invertible element $\Phi(A,B)$, which is defined by $G_0(z) = G_1(z)\Phi(A,B)$ and called *Drinfeld's associator.* For the first terms of $\log\Phi(A,B)$ we have the following expression (see [6]):
$\log\Phi = (\frac{AB}{48} - \frac{8[AAAB]+[ABAB]}{11520}$
$+\frac{96[AAAAAB]+4[AAABAB]+65[AABBAB]+68[ABAAAB]+4[ABABAB]}{5806080}$
$-(interchange A \leftrightarrow B)$,
where $[A_1 \dots A_m]$ is a short for the iterated bracket $[A_1,[A_2,\dots,[A_{m-1},A_m]\dots]]$. Let $A = \Omega_{12}, B = \Omega_{23}$ and consider $\Phi(\Omega_{12},\Omega_{23})$, where Ω_{ij} is a tangled chord diagram on the 3-strands trivial tangle. Close each tangle to $\cup$-knot (see Fig. 6). Closed associator $\tilde{\Phi}$ and its inversion $\tilde{\Phi}^{-1}$ have the following forms:

(3) $$\tilde{\Phi} = 1 + \frac{1}{24}d_2 + \frac{1}{5760}(d_{4(1)} + 3d_{4(2)} - 8d_{4(3)}) + \dots,$$

(4) $$\tilde{\Phi}^{-1} = 1 - \frac{1}{24}d_2 + \frac{1}{5760}(-d_{4(1)} + 7d_{4(2)} - 2d_{4(3)})$$
$$+\frac{2}{5806080}(2d_{6(1)} - 8d_{6(2)} + 22d_{6(3)} - 13d_{6(4)} - 60d_{6(5)}$$
$$+52d_{6(6)} + 24d_{6(7)} - 44d_{6(8)} + 19d_{6(9)}) + \dots,$$

where $d_{i(j)}$ are the following notations for chord diagrams: $d_2 = [1212]$, $d_{4(1)} = [12342143]$, $d_{4(2)} = [12314342]$, $d_{4(3)} = [12341432]$, $d_{6(1)} = [123142563456]$, $d_{6(2)} = [123142536456]$, $d_{6(3)} = [121342536456]$, $d_{6(4)} = [121324536456]$, $d_{6(5)} = [121324354656]$, $d_{6(6)} = [121323454656]$, $d_{6(7)} = [121234536456]$, $d_{6(8)} = [121234354656]$, $d_{6(9)} = [121234345656]$.

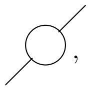
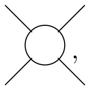
Here the product of chord diagrams is defined as connected sum of diagrams on circles. It is easy to see that $\tilde{\Phi}^{-1}$ is equal to $I(\cup)$.

4 Known results

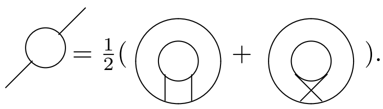
In 1997 Bar-Natan obtained an expression for the universal Vassiliev invariant of the trivial knot (see [5], [7]) in the following form:

$$I(O) = \exp \sum_{n=0}^{\infty} b_{2n} w_{2n} = 1 + (\sum_{n=0}^{\infty} b_{2n} w_{2n}) + \frac{1}{2} (\sum_{n=0}^{\infty} b_{2n} w_{2n})^2 + \ldots,$$

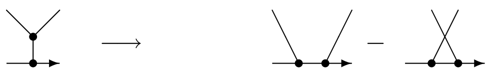
where b_{2n} are modified Bernouli numbers (which are coefficients of the following series: $\sum_{n=0}^{\infty} b_{2n} x^{2n} = \frac{1}{2} \ln \frac{e^{x/2} - e^{-x/2}}{x/2}$, $(b_2 = 1/48, b_4 = -1/5760, \ldots)$); w_{2n} are diagrams of the type:

$w_2 =$ , $\quad w_4 =$ , $\quad \ldots,$

The product $w_{2k} w_{2m}$ is the disjoint union of w_{2k} and w_{2m}. Any diagram w_{2m} is equal to a linear combination of $(2n)!$ Feynman's diagrams with coefficients $\frac{1}{(2n)!}$:



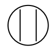
Any Feynman diagram may be reduced to linear combination of usual chord diagrams by using the following rule:



In 1995 T.Q.T.Le, J.Murakami [8] computed the universal Vassiliev invariant of the trefoil: starting from its decomposition (see (1)) and putting
$I(R_i) = e^{r_i r_i \Omega_{ii}/2} P_{ii}$,
$I(R_i^*) = e^{-r_i r_i \Omega_{ii}/2} P_{ii}$,
$I(E_i^*) = E_i^*$,
$I(E_i) = Z(\cup)^{-1} E_i$,
$I(I_j) = \Phi(\sum_{i_0 \le p \le i_1 - 1, i_1 \le q \le i_2 - 1} r_p r_q \Omega_{pq}, \sum_{i_1 \le p \le i_2 - 1, i_2 \le q \le i_3 - 1} r_p r_q \Omega_{pq})$,
we have:

$$I(2,3) = \tilde{\Phi}^{-2} E_1 E_2 \cdot \Phi(\Omega_{12}, -\Omega_{23}) \cdot e^{3\Omega_{12}/2} P_{12} \cdot \Phi(-\Omega_{23}, \Omega_{12}) E_1^* E_2^*,$$

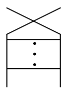
where
– P_{ii} is the operator that interchanges i-th and i-st strands of a tangle,
– $e^{3\Omega_{12}/2} = 1 + \frac{3}{2}\Omega_{12} + \frac{9}{8}\Omega_{12}^2 + \frac{9}{16}\Omega_{12}^3 + \frac{27}{128}\Omega_{12}^4 + \frac{81}{1280}\Omega_{12}^5 + \frac{81}{5120}\Omega_{12}^6 + \ldots,$
– $\Phi(\Omega_{ij}, \Omega_{kl})$ is Drinfeld's associator, and $\tilde{\Phi}$ is it closure.

Finally Le, Murakami obtained the following expression for the first terms of the universal Vassiliev invariant for the trefoil: $I(3_1) = 1 + \frac{23}{24}$  $+ \frac{1}{6}$  $\ldots$

5 The universal Vassiliev invariant for the $(2,n)$-type torus knots

We have a formula for the universal Vassiliev invariant of a $(2,n)$-type torus knot:

$$(5)\quad I(2,n) = \tilde{\Phi}^{-2} E_1 E_2 \cdot \Phi(\Omega_{12}, -\Omega_{23}) \cdot \left(\sum_{m=0}^{\infty} \left(\frac{n}{2}\right)^m \cdot \frac{D_m}{m!}\right) \cdot \Phi(-\Omega_{23}, \Omega_{12}) E_1^* E_2^*,$$

where $D_m =$  $\}m$ chords.

By formula (5) one can compute terms of the universal Vassiliev invariant of order ≤ 6 because there is no complete formula for the associator.

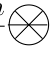
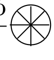
Theorem. *The following formula gives the universal Vassiliev invariant for a $(2,n)$-type torus knot up to terms of order ≤ 6:*

$$(6)\quad I(2,n) = 1 + \frac{3n^2-4}{24} d_2 + \frac{n^3-n}{48} d_3 + \frac{15n^4-16}{5760} d_{4(4)} - 8\frac{15n^2-16}{5760} d_{4(2)}$$
$$+\frac{1}{11520}((12n^5-12n)d_{5(1)} + (-6n^5-20n^3+26n)d_{5(2)} + (-40n^3+40n)d_{5(3)}$$
$$+(20n^3-20n)d_{5(4)}) + \frac{1}{46080} n^6 d_{6(10)} + \frac{1}{1451520}((64-378n^2)d_{6(1)} - 256 d_{6(2)}$$
$$+(809+1764n^2-1260n^4)d_{6(3)} + (-521-630n^2+630n^4)d_{6(4)} + (-1920+1764n^2)d_{6(5)}$$
$$+(1769-1512n^2)d_{6(6)} + (453-882n^2+315n^4)d_{6(7)} + (-568+2016n^2-630n^4)d_{6(8)}$$
$$+(188-882n^2+315n^4)d_{6(9)}) + \ldots,$$

where $d_2 = [1212]$, $d_3 = [123123]$, $d_{4(1)} = [12342143]$, $d_{4(2)} = [12314342]$, $d_{4(3)} = [12341432]$, $d_{4(4)} = [12341234]$, $d_{5(1)} = [1231425345]$, $d_{5(2)} = [1213425345]$, $d_{5(3)} = [1213243545]$, $d_{5(4)} = [1212343545]$, $d_{6(1)} = [123142563456]$, $d_{6(2)} = [123142536456]$, $d_{6(3)} = [121342536456]$, $d_{6(4)} = [121324536456]$, $d_{6(5)} = [121324354656]$, $d_{6(6)} = [121323454656]$, $d_{6(7)} = [121234536456]$, $d_{6(8)} = [121234354656]$, $d_{6(9)} = [121234345656]$, $d_{6(10)} = [123456123456]$.

The proof of formula (6) is given by direct calculation with using (3), (4), (5).

Consider the difference between (6) and (4). We have

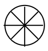
$\hat{I}(2,n) = I(2,n) - I(O) = \frac{3n^2-3}{24}$  $+ \frac{n^3-n}{48}$  $+ \frac{15n^4-15}{5760}$  $- \frac{n^2-1}{48}$  $\ldots$

Let $W_k : \mathcal{A}_k \longrightarrow \mathbf{Q}$ be a weight system of order k. Extend W_k to $\bar{\mathcal{A}}$ by

$$W_k^{\infty}(D) = \begin{cases} W_k(D), & \text{if } D \in \mathcal{A}_k; \\ 0, & \text{if } D \notin \mathcal{A}_k. \end{cases}$$

It is easy to see that $W_k^{\infty}(I(2,n))$ and $W_k^{\infty}(\hat{I}(2,n))$ are Vassiliev's invariants of order k. For the weight systems W_2, W_3 and $W_{4(1)}$ (see the example above) we have: $V_2(2,n) = W_2^{\infty}(\hat{I}(2,n)) = \frac{n^2-1}{8}, V_3(2,n) = W_3^{\infty}(\hat{I}(2,n)) = \frac{n^3-n}{48}$.

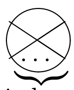
For W_4^{∞} consider the following condition: $W_4^{\infty}(I(3,2)) = 0$. It equivalent to the next relation between the 4-order chord diagrams:

(*)  $= \frac{4}{5}$ 

Let $\tilde{\mathcal{A}}_k = \mathcal{D}_k/_{\{1-,4-t.relations\ and\ relations\ '*'\}}$, where relations '*' are the relations in $\mathcal{D}_k, k \geq 4$ corresponding to conditions: $V_k(2,n) = 0$ for any $n < k$. Then for $k = 4$ we have $V_{4(1)}(2,n) = \tilde{W}_{4(1)}^{\infty}(I(2,n)) = \frac{n^4-10n^2}{384}$, $V_{4(2)}(2,n) = V_{4(3)}(2,n) = 0$.

We have the following formula for the universal Vassiliev invariant with values in $\oplus_{i=0}^{\infty}\tilde{\mathcal{A}}_i$ for the $(2,n)$-type torus knots:

$$I(2,n) = \exp \sum_{l=0}^{\infty} b_{2l} w_{2l} \sum_{j=2}^{\infty} V_j(2,n) D_j,$$

where $D_j =$  ,

j chords

$$V_k(n) = \frac{(n-k+1)(n-k+3)\dots(n-1)(n+1)\dots(n+k-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot 2k} \qquad \text{for even } k$$

and

$$V_k(n) = \frac{(n-k+2)(n-k+4)\dots(n-1)(n+1)\dots(n+k-2)n}{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2k-2) \cdot k} \qquad \text{for odd } k.$$

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