When we solve a constrained optimization problem, the value function $V$ is the objective function $f$ evaluated at the optimal $x$ (little stars denote optimal values):

$$V(b) \equiv \max_{x \in X} f(\bar{x}) \quad \text{s.t.} \quad \bar{g}(\bar{x}) = \bar{b}$$

$$V(b) \equiv f(x^*) \quad \text{where} \quad x^* = \arg \max_{x \in X} f(\bar{x}) \quad \text{s.t.} \quad \bar{g}(\bar{x}) = \bar{b}$$

Value functions are usually is stated in terms of the constraining $b$ or other parameters of the optimization problem (wealth, for instance, in the utility maximization problem). Provided that the constrained optimization problem has a unique solution and the objective function is continuous, the value function is also continuous— a change in wealth doesn’t cause a huge jump in happiness. But how much happier does a little more money make you?

Here’s the most interesting thing about value functions and Lagrange multipliers:

$$D_b V(b) = \lambda, \quad \text{or equivalently:} \quad \partial V/\partial b_i = \lambda_i, \quad i = 1, 2, \ldots, k$$

In the utility maximization problem, that means that the Lagrange multiplier measures, in terms of utility, the value of just a little more wealth. It can also be interpreted as the most utility you would trade for a little more money. Because of this, it is known as the shadow price of wealth. The multiplier on a scarce resource is called its shadow price.

Let’s also stick some other parameters, a vector that I’ll call $\tilde{\alpha}$, into the value function. Whereas $\bar{b}$ includes the variables that bind (wealth, in the utility maximization problem), $\tilde{\alpha}$ includes things that affect the outcome of the problem, but are not the binding factors themselves (like prices). Provided certain continuity and uniqueness conditions hold, the envelope theorem says that:
\[ \frac{dV}{d\alpha_i} = \frac{\partial V}{\partial \alpha_i} = \frac{\partial L}{\partial \alpha_i} \]

This means that you don’t have to worry about re-optimizing the problem, figuring out how much choice variables change, when taking the derivative of the value function. If you did take into account these changes, you would find that they all cancel out. Consider the value function of a utility maximization problem:

\[ V(\bar{p}, w) = U(x^*(\bar{p}, w)) + \lambda(w - \bar{p} \cdot x^*(\bar{p}, w)) \]

(Why do I include the multiplier and the constraint in the value problem? Really it should be there, but as it equals zero, it usually gets omitted.) According to the envelope theorem,

\[ \frac{dV}{d\bar{p}_i} = -\lambda x^*_i(\bar{p}, w) \]

But it would seem to me that since all the \( x \) are also functions of \( p \), shouldn’t we take these into account as well? Did the envelope theorem leave out some important terms? I take the total derivative in order to find out:

\[ \frac{dV}{d\bar{p}_i} = -\lambda x^*_i + \sum_{i=1}^{l} \frac{\partial U}{\partial x^*_i} \frac{\partial x^*_i}{\partial \bar{p}_i} - \lambda \bar{p}_i \frac{\partial x^*_i}{\partial \bar{p}_i} = -\lambda x^*_i + \sum_{i=1}^{l} \frac{\partial x^*_i}{\partial \bar{p}_i} \left( \frac{\partial U}{\partial x^*_i} - \lambda \bar{p}_i \right) \]

But wait! At an optimum, the term inside parentheses is zero! All of these cancel out, and it ends up that the envelope theorem was telling the truth all along.

Example: Consider the cost minimization problem facing a firm. It needs to produce a given level of output, \( \hat{q} \), using a vector \( \hat{y} \) of inputs. There is a production function \( f \):

\[ \min_{\hat{y}}(\bar{p} \cdot \hat{y}) \text{ subject to: } \hat{q} = f(\hat{y}) \]

Find the first order conditions to this optimization problem. Let \( \hat{y}^\ast \) denote the solution to this problem, the cost minimizing way of producing the required quantity. Let

\[ c(\hat{q}, \bar{p}) = \bar{p} \cdot \hat{y}^\ast + \lambda(\hat{q} - f(\hat{y}^\ast)) \]
be the least cost of producing this quantity. What does the envelope theorem tell you $\frac{dc}{dp}$ should be? Take a total derivative and verify this.

When playing around with things derived from first order conditions, you might take a derivative and find out that a lot of terms cancel, because of the envelope theorem. Keep this in mind in macro classes.