I. **Univariate Calculus**

Given two sets $X$ and $Y$, a function is a rule that associates each member of $X$ with exactly one member of $Y$. That is, some $x$ goes in, and some $y$ comes out. These notations are used to describe functions:

$$f : X \to Y$$
$$y = f(x)$$

In these cases, $x$ is called the **independent variable** and $y$ the **dependent variable**. That is, we can pick any value of $x$ we want to stick into the function, but we can't really pick what value $y$ takes on—that depends on $x$.

When $f$ is a function from $X$ into $Y$, the set $X$ is called the **domain** of the function and $Y$ is called the **range**. The domain is the set of permissible values to stick into the function, and the value that the function takes must be somewhere within the range. Most functions used are **real-valued functions**: functions whose domains and ranges are the set of real numbers or some subset thereof. We use the letter $\mathbb{R}$ to denote real numbers, and we write a real-valued function as $f : \mathbb{R} \to \mathbb{R}$. (You put a real number in, and you get a real number out.)

In economics, the amount of a good $x$ demanded is a function of the price of that good. In other words,

$$x = x(p)$$

This is called a **demand function**. Sometimes the same letter will be used to denote the function as the dependent variable.

Functions are sometimes referred to as **mappings**. Really, functions are only a specific type of mappings: those in which the output is a single element. Sometimes we want to divide functions into several types:

- **One-to-one (injective):** Each member of the range comes from at most one element in the domain. (There are not two $x$es that give the same $y$.)

- **Many-to-one:** There are multiple $x$es that give the same value of $y$.

- **Onto (surjective):** Each member of the range gets used at least once.

- **One-to-one and onto (bijective):** Each member of the range gets used exactly once (by exactly one $x$ value).

**Example:** The function is $f(x) = x$, with the domain given as $(-\infty, +\infty)$ and the range also as $(-\infty, +\infty)$. This is a one-to-one mapping, since there are not two $x$ values that give you the same $f(x)$. It is also onto, since each $y$ between $-\infty$ and $+\infty$ gets used. Therefore, it is a bijective mapping.
Example: The function is \( f(x) = x^3 \), with the domain given as \((-\infty, +\infty)\) and the range also as \((-\infty, +\infty)\). This is not a one-to-one mapping, since there are multiple \( x \) values that give you the same \( f(x) \); for example, \( f(-2) = 4 = f(2) \). Nor is it onto, since negative values of \( y \) are never given (there is no value of \( x \) that gives you \( f(x) = -12 \), at least not among real numbers—which is what we defined the domain as). Therefore, it is simply a many-to-one function.

Example: The function is \( f(x) = x^2 \), with the domain given as \([0, +\infty)\) and the range also as \([0, +\infty)\). This is a one-to-one mapping, since there are not multiple \( x \) values that give you the same \( f(x) \). It is also onto, since all numbers in the range of \( y \) are used by some \( x \) value. Since it is both one-to-one and onto, it is bijective.

The moral: whether a function is injective, surjective, both, or neither depends on the function, the domain, and the range together. We are most likely to care about whether a function is bijective, since this means that the function is invertible. If \( f : X \to Y \) is invertible, this means that there exists another function \( f^{-1} : Y \to X \) such that:

\[
y = f(x) \quad \text{is equivalent to} \quad x = f^{-1}(y).
\]

Even if you don’t remember the vocabulary, keep in mind that these two conditions are necessary to guarantee that the inverse of a function is itself a function.

The demand function tells us how much a person wants to buy at a certain price. If a business knows this, it might ask the question, “given that I would like to get people to buy \( x \) units, what price should I charge?” The business would simply find what is called the inverse demand function:

\[
x = x(p) \quad \text{is equivalent to} \quad p = x^{-1}(x(p)) = p(x).
\]

However, this is only solvable provided the demand function meets the two requirements given above. Does it, though?

Some functions go up and down and all over the place. A fairly boring function doesn’t, and we call these boring functions monotonic. Here are those definitions:

- **(Weakly) increasing:** \( x_1 > x_2 \Rightarrow f(x_1) \geq f(x_2) \).
- **Strictly increasing:** \( x_1 > x_2 \Rightarrow f(x_1) > f(x_2) \).
- **(Weakly) decreasing:** \( x_1 > x_2 \Rightarrow f(x_1) \leq f(x_2) \).
- **Strictly decreasing:** \( x_1 > x_2 \Rightarrow f(x_1) < f(x_2) \).
- **(Weakly) monotonic:** Either weakly increasing or weakly decreasing.
- **Strictly monotonic:** Either strictly increasing or strictly decreasing.

Example: The function is \( f(x) = x \), with domain of \((-\infty, +\infty)\) and range of \((-\infty, +\infty)\). This is a strictly increasing function, since a larger value of \( x \) means a larger value of \( f(x) \). (Of course, you can also call it weakly increasing — technically it satisfies this definition, though it’s
a bit silly to mention “weakly” when we know it is strictly—and it is also strictly monotonic.) This makes it invertible, and \( f^{-1}(y) = y \).

**Example:** The function is \( f(x) = x \), with domain of \((-\infty, +\infty)\) and range of \((-\infty, +\infty)\). This is *not* a monotonic function, since it sometimes it decreases and sometimes it increases. The function is not invertible.

**Example:** The function is \( f(x) = x \), with domain of \([0, +\infty)\) and range of \([0, +\infty)\). On this domain, the function is strictly increasing, so it is also strictly monotonic. This function is invertible, and \( f^{-1}(y) = \sqrt{y} \). (On the last example, it would be wrong to say that an inverse of \( f^{-1}(y) = \pm \sqrt{y} \) exists, since a function can take on only one value. A “correspondence” is a type of mapping that can, but we won’t meet those until later in the course.)

**Example:** The function is \( f(x) = x \), with domain of \((-\infty, +\infty)\) and range of \((-\infty, +\infty)\). This function satisfies the definition of (weakly) increasing and (weakly) decreasing, so it is also (weakly) monotonic. However, since it is not strictly monotonic, no inverse exists.

Back to the demand curve: provided that \( x = p(x) \) is strictly monotonic, an inverse demand function exists.

Another important property of functions is continuity. A continuous function is one that can be drawn with a single, continuous brushstroke. Technically, a function \( f \) is **continuous at a point** \( x \) if for any \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that if a point \( y \) is less than distance \( \delta \) from \( x \), then \( f(y) - f(x) \) is less \( \varepsilon \). The function is **continuous** if it is continuous at every point in its domain.\(^1\)

Less stringent than continuity is **piecewise continuity**. This describes a function that has only a finite number of points of discontinuity within any finite interval.

At a point where a function is continuous, you can take a derivative to see how the value of \( f(x) \) changes when \( x \) changes. The **derivative** of \( f \) at a point \( x \) is:

\[
\lim_{\delta \downarrow 0} \frac{f(x+\delta) - f(x)}{\delta}
\]

The derivative of \( f \) might be denoted by \( f'(x) \) or, when \( y = f(x) \), by \( dy/dx \). The function is **continuously differentiable at a point** \( x \) when its derivative is a

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\(^1\) Often in mathematical and economic analysis, the Greek letter \( \varepsilon \) is used to mean “a really, really small amount.” While the definition above doesn’t preclude large \( \varepsilon \), the interest is when \( \varepsilon \) gets really small. In this case, we’re saying that the function never has any jumps, certainly not of several inches, not even of a billionth of an inch, not even a billionth of a billionth of an inch. The letter \( \delta \) usually means “a change.” Given these conventions, when you see a definition like the one for continuity, try to translate it into your own words.
continuous function at that point. When it is continuously differentiable at all points, it is called a **continuously differentiable** function.

Sometimes you will see the notation \( f \in C \) or \( f \in C_X \) (where \( X \) is the domain of \( f \)) to indicate that \( f \) is a continuous function. \( C \) is the set of continuous functions. If \( f \) is continuously differentiable, this is written \( f \in C^1 \). If it is **twice continuously differentiable**, \( f \in C^2 \), and so forth for higher-order derivatives. Being twice continuously differentiable implies being (once) continuously differentiable, which in turn implies being continuous.

**Example:** \( f(x) = x^2 \) is a continuous, continuously differentiable function.

**Example:** \( f(x) = 1/x \) is not continuous at \( x = 0 \). (Furthermore, its derivative is also not continuous at that point.)

**Example:** \( f(x) = |x| \) is a continuous function that is not continuously differentiable: its derivative is \(-1\) when \( x \) is negative, \(+1\) when \( x \) is positive, and undefined at \( x = 0 \). Because it's still fairly well behaved, you might say that it is piecewise continuously differentiable if you really wanted to.

In economics, the term **marginal** means the effect of a small change in one thing on something else, like the **marginal utility of consumption** or the **marginal product of labor**. Looking back, we see that this fits the definition of a derivative quite well. You'll probably become familiar with these:

- **Utility function:** \( U = U(c) \)
  - Marginal utility of consumption: \( dU/dc \) or \( U'(c) \)

- **Production function:** \( Y = F(L) \)
  - Marginal product of labor: \( dF/dL \) or \( p \cdot F'(L) \)

The derivative can also be interpreted as the slope of a line tangent to the function at that point. Think back to diagrams of total cost and marginal cost curves.

Calculating the derivative of a function using the proper definition can be very tedious. The quick and easy way is to recall the **power rule**:

\[
f(x) = a \cdot x^n \quad \Rightarrow \quad f'(x) = a \cdot n \cdot x^{n-1}
\]

Because you can often break down functions (like polynomials) into several terms of this form, you can take most derivatives easily using this. Here are some other rules to follow for taking derivatives:
Addition rule: \[ h(x) = f(x) + g(x) \]
\[ h'(x) = f'(x) + g'(x) \]

Product rule: \[ h(x) = f(x)g(x) \]
\[ h'(x) = f'(x)g(x) + f(x)g'(x) \]

Quotient rule: \[ h(x) = \frac{f(x)}{g(x)} \]
\[ h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \]

Chain rule: \[ h(x) = f(g(x)) \]
\[ h'(x) = f'(g(x)) \cdot g'(x) \]

Inverse rule: \[ g(x) = f^{-1}(x) \]
\[ g'(f(x)) = \frac{1}{f'(x)} \]

(The quotient rule and the inverse rule require that the term in the denominator is not zero, obviously.) The function \( h \) in the chain rule is called a **composite function**; that is, \( h \) is not directly a function of \( x \), but it is a function \( f \) of \( g(x) \). If that sounds confusing, think of this example: your utility is not actually a function of prices. However, prices do affect how much you can afford to buy, and that affects your utility. In economics, this is called an individual’s **value function**: 

\[ V(p) = U(x(p)) \]

In other words, the “value” of facing prices \( p \) is the utility you get from your optimal demand \( x(p) \) when facing these prices.

The inverse rule is also very useful for getting information from inverse functions. For example, a consequence of utility maximization is an equation like “the marginal utility of consuming some good equals the marginal cost (that is, price) of that good”:

\[ U'(x) = \lambda p \]

(\( \lambda \) is some constant, the multiplier from the utility maximization problem—ignore it for now.) This gives us an inverse demand function for \( x \) very easily:

\[ p = \frac{1}{\lambda} U'(x) \]

(Ignore the constant \( \lambda \).) We might want to know how demand changes when prices change—in other words, what is the derivative of the demand function, \( dx/dp \)? This inverse demand function tells the opposite derivative:

\[ \frac{dp}{dx} = \frac{1}{\lambda} U''(x) \]

We can use the inverse rule to determine what we want:
\[
\frac{dx}{dp} = \left( \frac{dp}{dx} \right)^{-1} = \frac{\lambda}{U''(x)}
\]

Let’s actually take this one step further, and try to establish the price elasticity of demand. For small changes in prices, this is defined as

\[
\eta = \frac{dx/x}{dp/p} = \frac{dx}{dp} \cdot \frac{p}{x}
\]

We have a formula for \( \frac{dx}{dp} \), so we can stick this into the equation. We also have the condition that \( p = \lambda U'(x) \), so we’ll substitute that in as well. The final answer is:

\[
\eta = \frac{dx/x}{dp/p} = \frac{dx}{dp} \cdot \frac{p}{x} = \frac{U'(x)}{xU''(x)}
\]

This formula always works (more or less). If we know \( x \) and we know the utility function, we can always calculate the price elasticity this way.

As a final note, Some interesting functions have \( x \) as an exponent or take the logarithm of \( x \). (Two functions that show up frequently are exponential \( e \) and the natural logarithm \( \ln \).) Remember these rules for exponents and logarithms:

\begin{align*}
    a^x \cdot a^y &= a^{x+y} & a^{-x} &= \frac{1}{a^x} \\
    a^x / a^y &= a^{x-y} & (a^x)^y &= a^{xy} \\
    \log(x \cdot y) &= \log x + \log y & \log(1/x) &= -\log x \\
    \log(x/y) &= \log x - \log y & \log x^y &= y \log x \\
    a^0 &= 1 & \log 1 &= 0
\end{align*}

The power rule makes it easy to take the derivatives of most functions. However, these “interesting” functions—like sine, cosine, and logarithm—have derivatives that aren’t so simple. For natural logarithms and exponentials, here are the rules:

\begin{align*}
    f(x) &= e^x & \Rightarrow & f'(x) &= e^x \\
    f(x) &= \ln x & \Rightarrow & f'(x) &= 1/x
\end{align*}

If you encounter a trigonometric function or something stranger and need its derivative, consult Chapter 3 of Sydsæter, Strøm, and Berck (or another math book) for a list.