IX. Analysis: Functions, Correspondences, and Fixed Points

Compact sets are nice because any function is guaranteed a maximum and a minimum on one. Today we start off by talking about another nice property of some sets. A set \( S \subseteq \mathbb{R}^n \) is convex if for any \( x, y \in S \), it is also the case that all points of the form \( \alpha x + (1 - \alpha) y \in S \), where \( \alpha \in [0, 1] \). In other words, if you connect any two points with a line segment, that segment stays within the set.

Most states in the U.S. are not convex sets. There are a few, like Wyoming and Colorado, that have the nice property that you can get in your jeep in any town and drive in a straight line to any other town in the state without crossing any state lines.

If two sets, \( S \subseteq \mathbb{R}^n \) and \( T \subseteq \mathbb{R}^n \), are both convex sets, then their intersection \( S \cap T \) is convex, as well as their sum, \( \alpha S + \beta T \), where \( \alpha, \beta \in \mathbb{R} \). The “sum” of sets is defined as:

\[
\alpha S + \beta T = \left\{ x \in \mathbb{R}^n : \exists s \in S, \exists t \in T \text{ s.t. } x = \alpha s + \beta t \right\}
\]

The set \( S \cup T \) is every possible combination of a vector in \( S \) with a vector in \( T \). It is usually thought of as “tracing” one set around the edge of another. The set \( \alpha S \) comes from shrinking or enlargening the set \( S \) to \( \alpha \) of its original size; and also moving the set closer or further from the original by \( \alpha \). (These are concepts that are easier to draw than to explain.)

Given two vectors \( x, y \in \mathbb{R}^n \), any vector of the form \( \alpha x + (1 - \alpha) y \), \( \alpha \in [0, 1] \), is called a convex combination of \( x \) and \( y \). Given \( k \) vectors \( x_1, x_2, \ldots, x_k \in \mathbb{R}^n \), any other vector \( x \) of the form \( \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_k x_k \), with \( \sum_{i=1}^k \alpha_i = 1 \), is a convex combination of those vectors. For a set \( S \subseteq \mathbb{R}^n \), the convex hull of \( S \) is the set of all convex combinations of vectors in \( S \). In other words, we fill in all the holes in the set as well as any places where there are inlets in the boundary. The convex hull, denoted by \( \text{co}(S) \) or \( \text{conv}(S) \), is also the smallest convex set containing \( S \). I imagine the convex hull of a set as being what you’d get if you put a giant (multi-dimensional) rubber band around the set, and then filled in the inside.

Convex sets should not be confused with convex functions. It’s almost the same word being used to describe two different properties, though there is a relationship. This is a fourth characterization of concave and convex functions.
Theorem: A function \( f : X \to \mathbb{R} \), \( X \subseteq \mathbb{R}^n \), is convex if and only the epigraph of the function, \( \{(x,y) \in \mathbb{R}^{n+1} : y \geq f(x)\} \), is a convex set.

Theorem: A function \( f : X \to \mathbb{R} \), \( X \subseteq \mathbb{R}^n \), is concave if and only if the subgraph of the function, \( \{(x,y) \in \mathbb{R}^{n+1} : y \leq f(x)\} \), is a convex set.

The epigraph consists of the graph of the function all points lying above this (everything on top of the graph, as the epidermis is the top layer of the skin). The subgraph consists of the graph and everything below it.

Two similar properties are quasi-convexity and quasi-concavity. These are defined in terms of the sets of points in \( X \) that give you at most or at least a certain value.

Definition: A function \( f : X \to \mathbb{R} \), \( X \subseteq \mathbb{R}^n \), is quasi-convex if its lower contour sets, \( \{x \in X : f(x) \leq t\} \), are convex sets.

Definition: A function \( f : X \to \mathbb{R} \), \( X \subseteq \mathbb{R}^n \), is quasi-concave if its upper contour sets, \( \{x \in X : f(x) \geq t\} \), are convex sets.

These are weird terms and weird concepts. Think of contour lines on a topological map. Initially, you’re looking at a flat plain at sea level with a mountain rising to the height of 100 meters in the middle. If you pick a given altitude, say, 73 meters, all the points that are at least this high constitute a slice of the mountain. This is a convex set. The same is true if you look at 54 meters. If you pick 0 meters, all points are least this high—again a convex set. The upper contours sets are all convex. The map with a mountain on it is a quasi-concave function. This is good, since we typically think of concave functions as looking sort of like this: \( \cap \). A topological map of the Grand Canyon would illustrate what is meant by a quasi-convex function.

All concave functions are quasi-concave, but so are some convex functions. In fact, any monotonic function is both quasi-concave and quasi-convex. What quasi-concavity essentially eliminates are functions which decrease and then increase, like the typical \( \cup \)-shaped convex function. This is another way of characterizing quasi-concave and quasi-convex functions. There is also first derivative test.

Theorem: A function \( f : X \to \mathbb{R} \), \( X \subseteq \mathbb{R}^n \), is quasi-convex if and only if \( \forall x, y \in X \) and \( \forall \alpha \in [0,1] : f(\alpha x + (1-\alpha)y) \leq \max\{f(x), f(y)\} \).
Theorem: A function $f : X \to \mathbb{R}$, $X \subseteq \mathbb{R}^n$, is quasi-concave if and only if $\forall x, y \in X$ and $\forall \alpha \in [0,1]: f(\alpha x + (1-\alpha)y) \geq \min \{ f(x), f(y) \}$.

Theorem: A function $f : X \to \mathbb{R}$, $X \subseteq \mathbb{R}^n$, is quasi-concave if and only if $\frac{\partial f(x)}{\partial x} \cdot (y-x) \geq 0$ for all $\forall x, y \in X$ such that $f(y) \geq f(x)$.

Theorem: A function $f : X \to \mathbb{R}$, $X \subseteq \mathbb{R}^n$, is quasi-convex if and only if $\frac{\partial f(x)}{\partial x} \cdot (y-x) \leq 0$ for all $\forall x, y \in X$ such that $f(y) \geq f(x)$.

There is also a test for quasi-concavity and quasi-convexity of functions which involves the signs of the principal minors of the bordered Hessian matrix of the function. If you ever need this, go look it up in Simon and Blume (Chapter 21).

What’s the relation between concavity and quasi-concavity? Well, as mentioned, all concave functions are also quasi-concave, though the converse is not true. The monotonic transformation of any concave function is also a quasi-concave function (though it may no longer be concave). Both properties describe how the first derivative of a function changes. While concavity measures the magnitude of changes in the first derivative, quasi-concavity looks only at the direction of the changes. In fact, equivalent to the definitions of these terms are (in the single variable case), for $x \geq y$:

**Concave:** $f'(x) \geq f'(y)$

**Quasi-concave:** $\text{sgn}(f'(x)) \geq \text{sgn}(f'(y))$

Convex sets are nice for a reason other than their relation to functions. Imagine two sets in the $\mathbb{R}^2$ plane. You want to be able to split the plane in half with a straight line so that all of one set is on one side of the line; all of the other set is on the other side. Whenever these sets are disjoint convex sets, you are guaranteed to be able to separate them, by something called the separating hyperplane theorem.

What is a hyperplane, a supersonic jet? No, a **hyperplane** is an $(n-1)$ dimensional affine set in $\mathbb{R}^n$. An affine set is a flat surface that stretches off in all directions. A hyperplane in $\mathbb{R}^2$ is called a **line**, and in $\mathbb{R}^3$ a **plane**, and the analogy continues into higher dimensions. Note that a line is not a hyperplane in $\mathbb{R}^3$, since a line is one-dimensional, not $(n-1) = (3-1)$ dimensional.
The graph of a line in $\mathbb{R}^2$ can always be described in the form $\{(x_1, x_2) \in \mathbb{R}^2 : a_1 x_1 + a_2 x_2 = b\}$. (Typically, you think of $x_2$ as $y$ and set $a_2 = 1$, and you have what we call an affine function, right?) Any plane in $\mathbb{R}^3$ can be written as the set of points $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : a_1 x_1 + a_2 x_2 + a_3 x_3 = b\}$. Similarly, any hyperplane in $\mathbb{R}^n$ can be represented in the form:

$$\{x \in \mathbb{R}^n : a \cdot x = b\}$$

by some vector $a \in \mathbb{R}^n$ and some constant $b$.

A hyperplane separates $\mathbb{R}^n$ into two **halfspaces**. For example, the line $y = 0$ splits $\mathbb{R}^2$ into the positive values of $y$ and the negative values of $y$. There are other ways to divide up $\mathbb{R}^2$: the line $y = x$ divides it into the area where $y$ is greater than $x$ and where $y$ is less than $x$. In fact, there are exactly as many ways to divide a space into halfspaces as there are hyperplanes. The hyperplane described by $a \cdot x = b$ identifies two **closed halfspaces**:

$$\{x \in \mathbb{R}^n : a \cdot x \leq b\} \quad \text{and} \quad \{x \in \mathbb{R}^n : a \cdot x \geq b\}$$

as well as the two **open halfspaces**:

$$\{x \in \mathbb{R}^n : a \cdot x < b\} \quad \text{and} \quad \{x \in \mathbb{R}^n : a \cdot x > b\}$$

The distinction is, of course, just whether you want to include the hyperplane itself in the halfspace. When $X$ and $Y$ are the halfspaces which are separated by the hyperplane $H$, it is said that $H$ **generates** the (closed or open) halfspaces $X$ and $Y$.

My buddy Oliver and I are now two pioneers, settling out in some uninhabited region of the American west, and we’re trying to divide up the land. A hyperplane is like a fence that runs in a straight line, dividing the space into two people’s territories. The **supporting hyperplane theorem** says that if I have a blob of land that I want to ensure is in my territory, and I pick any tree stump on the boundary of this blob of land or even somewhere not in the blob; then provided the blob of land is a convex set, I can use that tree stump as a fence post and not leave any of this nice pasture on the wrong side of the fence.

**Theorem:** Suppose that $X \subset \mathbb{R}^n$ is convex and that $x \notin \text{int}(X)$. Then there exists $a \in \mathbb{R}^n$, with $a \neq \mathbf{0}$, such that $a \cdot x \geq a \cdot y$ for all $y \in X$. 

— Fall 2007 math class notes, page 69
Oliver has also picked out a blob of land that he likes. The question is whether we can put up a fence that separates them, so that his pasture is on one side, and mine on the other. Fortunately, it doesn’t overlap mine or we’d have real problems separating them; provided that they are both convex sets, the **separating hyperplane theorem** guarantees that we can run a fence between them.

**Theorem:** Suppose that \( X, Y \subset \mathbb{R}^n \) are convex, disjoint (that is, \( X \cap Y = \emptyset \)) sets. Then there exist \( a \in \mathbb{R}^n \) (with \( a \neq 0 \)) and \( b \in \mathbb{R} \) such that \( a \cdot x \geq b \) for all \( x \in X \), and \( a \cdot y \leq b \) for all \( y \in Y \). In other words, the sets \( X \) and \( Y \) are contained within separate closed halfspaces generated by some hyperplane.

This concept is used for examining the existence of equilibria in economies. If I have time, I’ll show an example.

At the start of class we talked about functions. Some functions, like \( f(x) = x^2 \), cannot be inverted into another function because multiple \( x \) take on the same value. However, they might be sensible, well-behaved mappings that take on multiple values. In would make sense to say that the inverse of \( f(x) = x^2 \) is:

\[
G(y) = \{-\sqrt{y}, +\sqrt{y}\}
\]

which is well defined on \( \mathbb{R}_+ \), except that it takes on multiple values. A **correspondence** or a **multivaluent function** \( F \) is a mapping from a set \( X \) into subsets of a set \( Y \). (A function is a mapping from \( X \) into *elements* of \( Y \).) A correspondence might be denoted by:

\[
F : X \mapsto Y \quad \text{or} \quad F : X \to 2^Y
\]

Some people use the same notation for a function as for a correspondence \((F : X \to Y)\). Even the first notation I list for a correspondence is sometimes used for a function as well. The second one is unambiguous, though. Given a set \( Y \), the **power set** of \( Y \) is the set of all possible subsets of \( Y \). It is denoted by \( 2^Y \). For instance, given the set:

\[
S = \{x, y, z\}
\]

The power set of \( S \) is then:

\[
2^S = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}
\]
Note that each of these elements is a set itself, though some of the sets contain only one (or zero!) elements. Also note that given a set of three elements, there are \(8 = 2 \cdot 2 \cdot 2\) elements in the set’s power set.

A correspondence is a mapping from a set \(X\) into subsets of the set \(Y\). Another way to think about a correspondence is as a function from \(X\) into \(2^Y\) (since it gives you a single element of \(2^Y\) — elements which happen to be sets themselves).

Sometimes the individual utility maximization problem will not have a single solution. For instance, consider the utility function and budget constraint:

\[
U(x_1, x_2) = x_1 + x_2, \quad x_1 p_1 + x_2 p_2 \leq w
\]

The two goods are perfect substitutes for each other. Indifference curves are straight lines. If we solve the problem, we find that the person spends all his money on the cheaper of the two goods, unless the prices are equal. In that case, he’ll spend any share (he doesn’t care) of his income on either good. In other words,

\[
x(\vec{p}, w) = (x_1(\vec{p}, w), x_2(\vec{p}, w)) = \begin{cases} 
\left\{ \left( \frac{w}{p_1}, 0 \right) \right\} & \text{if } p_1 < p_2 \\
\left\{ \left( \alpha \frac{w}{p_1}, (1 - \alpha) \frac{w}{p_2} \right) : \alpha \in [0, 1] \right\} & \text{if } p_1 = p_2 \\
\left\{ \left( 0, \frac{w}{p_1} \right) \right\} & \text{if } p_1 > p_2 
\end{cases}
\]

This is now called a demand correspondence. You will also encounter lots of best-response correspondences in game theory. While correspondences might seem like a very foreign idea, you’re probably familiar with some already. The budget set can be represented as a correspondence mapping prices and wealth into affordable bundles. This is the budget correspondence:

\[
B(p, w) \equiv \left\{ x \in X, X_i \subseteq \mathbb{R}^L_+ : \sum_{t=1}^{L} x_{it} p_t \leq w_i \right\}
\]

An advantage of thinking of this as a correspondence, with price and wealth as independent variables, is that we can talk about continuity. Draw a picture of a budget set in \(\mathbb{R}^2\), and then change one of the prices a little bit. The budget set changes only a little bit. This seems more or less “continuous”—there are no big jumps most of the time. However, when a price goes to zero, the budget set can suddenly “explode.” This seems like a discontinuity, in a way.
Because correspondences are more complicated than functions, there are a couple of ways that they can be “almost continuous.”

**Definition:** The correspondence $F : X \subseteq \mathbb{R}^n \mapsto \mathbb{R}^m$ is **lower hemi-continuous** at $\bar{x}$ if, for all $\bar{y} \in F(\bar{x})$ and each neighborhood $U = B_{\varepsilon}(\bar{y})$ of $\bar{y}$, there exists a neighborhood $N = B_{\delta}(\bar{x})$ such that $F(x) \cap U \neq \emptyset$ for all $x \in N \cap X$.

**Definition:** The correspondence $F : X \subseteq \mathbb{R}^n \mapsto \mathbb{R}^m$ is **upper hemi-continuous** at $\bar{x}$ if for every open set $U \supseteq F(\bar{x})$, there exists a neighborhood $N = B_{\delta}(\bar{x})$ such that $F(x) \subseteq U$ for all $x \in N \cap X$.

**Definition:** The correspondence $F : X \subseteq \mathbb{R}^n \mapsto \mathbb{R}^m$ is **continuous** at $\bar{x}$ if it is both upper and lower hemi-continuous.

If we plot the demand for $x_1$ as a function of relative prices in the previous example, it looks like a large letter Z with right angle bends. There is a sort of “wall” in this correspondence, a vertical line. My working definition of upper and lower hemi-continuity are these:

**Quasi-dfn.:** The correspondence $F : X \subseteq \mathbb{R}^n \mapsto \mathbb{R}^m$ is lower hemi-continuous if any walls are not a part of the graph of $F$ (that is, the graph of the correspondence is like an open set at these walls.)

**Quasi-dfn.:** The correspondence $F : X \subseteq \mathbb{R}^n \mapsto \mathbb{R}^m$ is upper hemi-continuous if any walls are a part of the graph of $F$ (that is, the graph of the correspondence is like a closed set at these walls.)

Some pictures might explain what this means.

If we have a function or a correspondence mapping some set $X$ into itself, sometimes we wonder if there is any $x$ which is also equal to $f(x)$. This relates to the idea of an equilibrium. Think about this: quantity produced is a function of price. Price is a function of quantity demanded (remember the inverse demand curve):

$$q_s = q_s(p), \quad q_d = q_d(p) \iff p = p(q_d)$$

Then quantity produced can be written as a composite function of quantity demanded:
\[ q_s(p) = q_s(p(q_d)) \]

We have an equilibrium when the value of \( q \) that goes into the function is the same as the value of \( q \) that comes out. For a function \( f : X \to X \), a **fixed point** of \( f \) is any \( x^* \) such that \( x^* = f(x^*) \). For a correspondence \( F : Y \rightharpoonup Y \), a **fixed point** of \( F \) is any \( y^* \) such that \( y^* \in F(y^*) \).

Do all functions have fixed points? Definitely not—the function \( f(x) = x + 1 \) has none. Under what circumstances can we guarantee that a function has a fixed point? **Brouwer's Fixed Point Theorem** tells us one set of conditions, and **Tarski's Fixed Point Theorem** another.

**Theorem:** Suppose \( X \) is a non-empty, convex, compact subset of \( \mathbb{R}^n \), and let \( f : X \to X \) be a continuous function. Then \( f \) has a fixed point.

**Theorem:** Suppose \( X \) is a non-empty, convex, compact subset of \( \mathbb{R}^n \), and let \( f : X \to X \) be a nondecreasing function. Then \( f \) has a fixed point.

There are similar theorems for correspondences. **Kakutani's Fixed Point Theorem** is analogous to Brouwer for correspondences.

**Theorem:** Suppose \( X \) is a non-empty, convex, compact subset of \( \mathbb{R}^n \), and let \( F : X \rightharpoonup X \) be an upper semi-continuous correspondence with non-empty, closed, convex images. Then \( F \) has a fixed point.

According to **Browder's Fixed Point Theorem**, derived from a principle known as **Michael's Selection Theorem**, the same is true of lower semi-continuous correspondences meeting the same criteria.