I. UNIVARIATE CALCULUS

Given two sets $X$ and $Y$, a function is a rule that associates each member of $X$ with exactly one member of $Y$. That is, some $x$ goes in, and some $y$ comes out. These notations are used to describe functions:

$$f: X \to Y$$
$$y = f(x)$$

In these cases, $x$ is called the independent variable and $y$ the dependent variable. That is, we can pick any value of $x$ we want to stick into the function, but we can’t really pick what value $y$ takes on—-that depends on $x$.

When $f$ is a function from $X$ into $Y$, the set $X$ is called the domain of the function and $Y$ is called the range. The domain is the set of permissible values to stick into the function, and the value that the function takes must be somewhere within the range. Most functions used are real-valued functions: functions whose ranges are the set of real numbers or some subset thereof, $Y \subseteq \mathbb{R}$. The domain is usually also a subset of the real numbers.

In economics, the amount of a good $x$ demanded is a function of a person’s wealth and the price of that good. In other words,

$$x = x(p, w)$$

This is called a demand function. Sometimes the same letter will be used to denote the function as the dependent variable.

Functions are sometimes referred to as mappings. Really, functions are only a specific type of mappings: those in which the output is a single element. Sometimes we want to divide functions into two types. Given a certain domain $X$ and a range $Y$, a function $f$ from $X$ into $Y$ is called one-to-one or injective if each member of the range comes from at most one element in the domain. That is, if $x \neq y$, then $f(x) \neq f(y)$. Functions which are not one-to-one are often called many-to-one. Take the domain and the range to both be all the real numbers:

**Example:** $f(x) = x$ is a one-to-one mapping.

**Example:** $f(x) = x^2$ is a many-to-one mapping, since $f(-2) = 4 = f(2)$. (However, if the domain is only the positive real numbers, it is one-to-one.)

If every member of the range gets used by some member of the domain, the function is called surjective or onto. The test here is to pick any element in the range, and ask what value of the domain could possibly give you that. If you can always answer this question, the function is surjective. Again,
Example: \( f(x) = x \) is a surjective function.

Example: \( f(x) = x^2 \) is not surjective. If \( f(x) = -12 \), what is \( x \)? You can’t answer this question. In this case, if the range were only positive real numbers, this problem wouldn’t arise.

A function with is both injective and surjective is called bijective. This is a useful property, since any bijective function have an inverse function: that is, there is a function that associated each element of \( y \) with an element of \( x \). Usually the inverse of a function \( f \) is denoted by \( f^{-1} \).

\[
  y = f(x) \quad \text{is equivalent to:} \quad x = f^{-1}(y).
\]

Even if you don’t remember the vocabulary, keep in mind that these two conditions are necessary to guarantee that the inverse of a function is itself a function.

The demand function tells us how much a person wants to buy at a certain price. (Let’s forget about wealth for now.) If a business knows this, it might ask the question, “given that I would like to get the person to buy \( x \) units, what price will produce this?” The business would simply find what is called the inverse demand function:

\[
  x = x(p) \quad \text{is equivalent to:} \quad p = x^{-1}(x(p)) = p(x).
\]

However, this is only solvable provided the demand function meets the two requirements given above. Does it, though?

Some functions go up and down and all over the place. A fairly boring function doesn’t. A monotonic function is one which is either always (weakly) increasing or always (weakly) decreasing. Of course, a function \( f \) is increasing or weakly increasing or nondecreasing if for any \( x \) and \( y \) with \( x \) less than \( y \), then \( f(x) \) is less than or equal to \( f(y) \). The definition of decreasing, weakly decreasing, and nonincreasing is analogous.

The function is strictly increasing when this holds with strict inequality; that is, when \( f(x) \) is strictly less than \( f(y) \). The definition of strictly decreasing is similar. A function which is either one of these is called strictly monotone. Whenever a function is strictly monotone, it is one-to-one.

Question: Thinking back to a question I missed on my final exam in intro undergrad micro, I know now that demand curves are always downward sloping. As we might say in grad school, the demand function is strictly decreasing. This implies that it is strictly monotone and hence one-to-one or injective. If we define sensibly domains and ranges for the demand function, it is also surjective. Yeah! The demand function is in fact a bijective function, so the inverse demand function is well-defined!
Another property frequently used is continuity. A continuous function is one which can be drawn with a single, continuous brushstroke. Technically, a function $f$ is **continuous at a point** $x$ if for any $\epsilon > 0$ there exists a $\delta > 0$ such that if a point $y$ is less than distance $\delta$ from $x$, then $f(y) - f(x)$ is less $\epsilon$. The function is **continuous** if it is continuous at every point in its domain.

Often in mathematical and economic analysis, the Greek letter $\epsilon$ is used to mean “a really, really small amount.” While the definition above doesn’t preclude large $\epsilon$, the interest is when $\epsilon$ gets really small. In this case, we’re saying that the function never has any jumps, certainly not of several inches, not even of a billionth of an inch, not even a billionth of a billionth of an inch. The letter $\delta$ usually means “a change.” Given these conventions, when you see a definition like the one for continuity, try to translate it into your own words.

 Less stringent than continuity is **piecewise continuity**. This describes a function that has only a finite number of points of discontinuity within any finite interval.

At a point where a function is continuous, you can take a derivative to see how the value of $f(x)$ changes when $x$ changes. The **derivative** of $f$ at a point $x$ is:

$$ \lim_{\delta \to 0} \frac{f(x + \delta) - f(x)}{\delta} $$

The derivative of $f$ might be denoted by $f'(x)$ or, when $y = f(x)$, by $dy/dx$. The function is **continuously differentiable at a point** $x$ when its derivative is a continuous function at that point. When it is continuously differentiable at all points, it is called a **continuously differentiable** function.

Sometimes you will see the notation $f \in C$ or $f \in C_x$ (where $X$ is the domain of $f$) to indicate that $f$ is a continuous function. $C$ is the set of continuous functions. If $f$ is continuously differentiable, this is written $f \in C'$. If it is **twice continuously differentiable**, $f \in C''$, and so forth for higher-order derivatives. Being twice continuously differentiable implies being continuously differentiable, which in turn implies being continuous.

**Example:** $f(x) = x^2$ is a continuous, cont. diff’ble function.

**Example:** $f(x) = 1/x$ is not continuous at $x = 0$. (Furthermore, its derivative is also not continuous at that point.)

**Example:** $f(x) = |x|$ is a continuous function that is not continuously differentiable: its derivative is $-1$ when $x$ is negative, $+1$ when $x$ is positive, and undefined at $x = 0$. Because it’s still fairly well behaved, you might say that it is piecewise cont. diff’ble if you really wanted to.
In economics, the term **marginal** means the effect of a small change in one thing on something else, like the **marginal utility of consumption** or the **marginal revenue product of labor**. Looking back, we see that this fits the definition of a derivative quite well. You’ll probably become familiar with these:

- **Utility function:** \( U = U(c) \)
- **Marginal utility of consumption:** \( dU / dc \) or \( U'(c) \)

- **Revenue function:** \( R = p \cdot Y = p \cdot F(L) \)
- **Marginal revenue product of labor:** \( p \cdot dF / dL \) or \( p \cdot F'(L) \)

The derivative can also be interpreted as the slope of a line tangent to the function at that point. Think back to diagrams of total cost and marginal cost curves.

Calculating the derivative of a function using the proper definition can be very tedious. The quick and easy way is to recall the **power rule**:

\[
f'(x) = a \cdot x^n \quad \Rightarrow \quad f''(x) = a \cdot n \cdot x^{n-1}
\]

Because you can often break down functions (like polynomials) into several terms of this form, you can take most derivatives easily using this. Here are some other rules to follow for taking derivatives:

- **Addition rule:** \( h(x) = f(x) + g(x) \) \( \Rightarrow \) \( h'(x) = f'(x) + g'(x) \)
- **Product rule:** \( h(x) = f(x)g(x) \) \( \Rightarrow \) \( h'(x) = f'(x)g(x) + f(x)g'(x) \)
- **Quotient rule:** \( h(x) = \frac{f(x)}{g(x)} \) \( \Rightarrow \) \( h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \)
- **Chain rule:** \( h(x) = f(g(x)) \) \( \Rightarrow \) \( h'(x) = f'(g(x)) \cdot g'(x) \)
- **Inverse rule:** \( g(x) = f^{-1}(x) \) \( \Rightarrow \) \( g'(f(x)) = \frac{1}{f'(x)} \)

(The quotient rule and the inverse rule require that the term in the denominator is not zero, obviously.) The function \( b \) in the chain rule is called a **composite function**; that is, \( b \) is not directly a function of \( x \), but it is a function \( f \) of \( g(x) \). If that sounds confusing, think of this example: your utility is not actually a function of prices. However, prices **do** affect how much you can afford to buy, and **that** affects your utility. During the next year, you will often see an individual’s **value function**. This is the composite function I just described:

\[
V(p,w) = U(x(p,w))
\]

This is the value of having wealth \( w \) when facing prices \( p \). It is simply the utility you get from your optimal demand \( x(p,w) \) at these prices with this wealth.
The inverse rule is also very useful for getting information from inverse functions. For example, a consequence of utility maximization is an equation like “the marginal utility of consuming some good equals the marginal cost (that is, price) of that good”:

\[ U'(x) = \lambda p \]

(\( \lambda \) is some constant, the multiplier from the utility maximization problem—ignore it for now.) This gives us an inverse demand function for \( x \) very easily:

\[ p = \frac{1}{\lambda} U'(x) \]

(Ignore the constant \( \lambda \).) What if we want to know how \( x \) changes when \( p \) changes?—what is the derivative of the demand function, \( dx/dp \)? What about the price elasticity of \( x \)? The inverse demand function gives us the opposite of that:

\[ \frac{dp}{dx} = \frac{1}{\lambda} U''(x) \]

The inverse rule then tells us that:

\[ \frac{dx}{dp} = \left( \frac{dp}{dx} \right)^{-1} = \frac{\lambda}{U''(x)} \]

And that tells us how the optimal bundle will change when the price changes. (The idea is that the change in \( x \) resulting from a change in \( p \) depends on the curvature of the utility function.) As for point elasticity, the familiar “percent change in this with respect to a percent change in that” takes the form of:

\[ \varepsilon = \frac{dx}{dp} \cdot \frac{x}{p} = \frac{dx}{dp} \cdot \frac{p}{x} \]

We found what \( dx/dp \) is. Using this and the original optimality condition, we find this formula for elasticity:

\[ \varepsilon = \frac{dx}{dp} \cdot \frac{p}{x} = \frac{\lambda p}{xU''(x)} = \frac{U'(x)}{xU''(x)} \]

The last term here show up again this next year as something similar to a measure of relative risk aversion.

As a final note, Some interesting \( f(x) \) have \( x \) as an exponent or take the logarithm of \( x \). Two functions that show up frequently are exponential \( e \) and the natural logarithm \( \ln \). Remember these rules for exponents and logarithms:

\[
\begin{align*}
    a^x \cdot a^y &= a^{x+y} \\
    a^x / a^y &= a^{x-y} \\
    (a^x)^y &= a^{xy} \\
    \log(x \cdot y) &= \log x + \log y \\
    \log(x/y) &= \log x - \log y \\
    a^0 &= 1 \\
    \log 1 &= 0
\end{align*}
\]
The power rule makes it easy to take the derivatives of most functions. However, these “interesting” functions—like sine, cosine, and logarithm—have derivatives that aren’t so simple. For natural logarithms and exponentials, here are the rules:

\[ f'(x) = e^x \quad \Rightarrow \quad f'(x) = e^x \]
\[ f'(x) = \ln x \quad \Rightarrow \quad f'(x) = 1/x \]

If you encounter a trigonometric function or something stranger and need its derivative, consult Chapter 3 of Sydsæter, Strøm, and Berck (or another math book) for a list.

**References:**

Simon and Blume: Chapters 2-3.
Sydsæter, Strøm, and Berck: Chapters 2-3.
Salas and Hille: Chapters 1-3.