2.8 Limits at Infinity, Infinite Limits

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Limits at Infinity

Definition Limit as $x \to \infty$

Let f be defined on $[c, \infty)$ for some number c. We say that $\lim_{x\to\infty} f(x) = L$ if for each $\epsilon > 0$ there is an M such that

$$x > M \Rightarrow |f(x) - L| < \epsilon$$

Definition Limit as $x \to -\infty$

Let f be defined on $(-\infty, c]$ for some number c. We say that $\lim_{x\to-\infty} f(x) = L$ if for each $\epsilon > 0$ there is an M such that

$$x < M \Rightarrow |f(x) - L| < \epsilon$$

A very fundamental property of limits at infinity is the following

For every $k > 0$, $\lim_{x\to\infty} \frac{1}{x^k} = 0$

This very fundamental idea will allow us to find limits at $\pm\infty$ for many functions.

Examples

Determine the following limits at $\pm\infty$. Hint: Use the standard trick of dividing the numerator and denominator by the highest power of x in the denominator.

1. $\lim_{x\to\infty} \dfrac{2x - 5}{3x^2 + 5x - 4}$

2. $\lim_{x\to-\infty} \dfrac{x + 3}{x - 1}$

By this point you should be able to look at limits to $\pm\infty$ and quickly pick out the limit.

3. $\lim_{x\to\infty} \dfrac{1-2x^3}{5+x^2+6x^3}$

4. $\lim_{x\to-\infty} \sqrt{\dfrac{\pi x^4+5x^2}{3x^3+4x^4}}$

5. $\lim_{x\to\infty} \dfrac{3x^2}{(2x+1)(4x-1)}$

6. $\lim_{x\to\infty} \dfrac{(x-1)(x+2)}{(x-10)^3}$

Infinite Limit

1. $\lim_{x\to2^+} \dfrac{3}{x-2}$

2. $\lim_{x\to2^-} \dfrac{3}{x-2}$

3. $\lim_{x\to2} \dfrac{3}{x-2}$

4. $\lim_{x\to3^+} \dfrac{-9x}{(x-3)^2}$

5. $\lim_{x\to3^-} \dfrac{-9x}{(x-3)^2}$

6. $\lim_{x\to3} \dfrac{-9x}{(x-3)^2}$

7. $\lim_{y\to0} \dfrac{1}{y}$

Relation to Asymptotes

Vertical Asymptotes The vertical line $x = c$ is a vertical asymptote of $y = f(x)$ if any of the following are true:

$$\lim_{x \to c^+} f(x) = \infty \quad \lim_{x \to c^+} f(x) = -\infty$$

$$\lim_{x \to c^-} f(x) = \infty \quad \lim_{x \to c^-} f(x) = -\infty$$

Horizontal Asymptotes The horizontal line $y = b$ is a horizontal asymptote for $y = f(x)$ if either of the following is true:

$$\lim_{x \to \infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b$$

Examples: Determine the vertical and horizontal asymptotes of the following functions (if they exist). Sketch a graph of the function.

1. $F(x) = \dfrac{3x + 1}{x - 2}$

2. $g(x) = \dfrac{\sin x}{x}$

3. $h(x) = \dfrac{1}{x^2 - 2}$