4.2 Monotonicity and Concavity

- Increasing and Decreasing Functions
- Monotonicity Theorem
- Concavity Theorem
- Inflection Points

Definition

Let f, be defined on an interval I (open, closed, or neither). We say that:

i.) f is increasing on I if, for every pair of numbers x_1 and x_2 in I,

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

ii.) f is decreasing on I if, for every pair of numbers x_1 and x_2 in I,

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

iii.) f is strictly monotonic on I if it is either increasing on I or decreasing on I.

Often times in applications, one is interested in knowing when a function is increasing or decreasing. For example, a business manager is interested in knowing when his/her profits are increasing. Or a biologist is interested in when the number of infected organisms is decreasing. How do we know where a function is increasing and decreasing (other than by looking at a graph)? Recall from our introduction to the derivative, we saw that when the tangent line has a positive slope the function is increasing, and when the tangent line has a negative slope the function is decreasing. (A horizontal tangent line represents a stationary point–neither increasing nor decreasing.) This notion is formalized in the following theorem.

Monotonicity Theorem

Let f be continuous on an interval I and differentiable at every interior point of I.

(i) If $f'(x) > 0$ for all x interior to I, then f is increasing on I.

(ii) If $f'(x) < 0$ for all x interior to I, then f is decreasing on I.

Example: Where is $h(x) = \dfrac{x}{1+x^2}$ increasing and decreasing?

Whether or not a function is increasing or decreasing is an important characteristic, as is how a function is increasing or decreasing. Does the function increase very fast or very slowly or at a constant rate? We can determine these characteristics by looking at the second derivative of the function. The first derivative represents a rate of change, while the second derivative represents the rate of change of the rate of change. The second derivative tells us how fast a function is changing and gives us a measure of **concavity**.

Definition

Let f be differentiable on an open interval I. We say that f (as well as its graph) is **concave up** on I if f' is increasing, and we say that f is **concave down** on I if f' is decreasing on I.

A function that is concave up can be either increasing or decreasing (same for concave down). Fill in the following chart to display this.

Increasing, Concave Up	Decreasing, Concave Up
Increasing, Concave Down	Decreasing, Concave Down

Concavity Theorem

Let f be twice differentiable on the open interval I.

(i) If $f''(x) > 0$ for all x in I, then f is concave up on I.

(ii) If $f'(x) < 0$ for all x in I, then f is concave down on I.

Example 1: Where is $s(t) = 3t^5 - 5t^3 + 1$ increasing, decreasing, concave up, and concave down? Sketch a graph.

Example 2: Where is $h(x) = \dfrac{x}{1+x^2}$ increasing, decreasing, concave up, and concave down? Sketch a graph.

Inflection Points

An inflection point of $f(x)$ is a point $(c, f(c))$, such that f is concave up on one side of c and concave down on the other side of c.

Points where $f''(x) = 0$ or $f''(x)$ does not exist are *candidates* for inflection points. (This means, that inflection points will only occur at these points, but does not have to necessarily occur.)

Example 1: Find all inflection points of $y = x^4$.

Example 2: Find all inflection points of $h(t) = t^2 - \dfrac{1}{t^2}$

Example 3: Draw a graph of a function that has the following characteristics.

$f(0) = f(4) = 1$; $f(2) = 2$; $f(6) = 0$;

$f'(x) > 0$ on $(0, 2)$; $f'(x) < 0$ on $(2, 4) \cup (4, 6)$;

$f'(2) = f'(4) = 0$; $f''(x) > 0$ on $(0, 1) \cup (3, 4)$;

$f''(x) < 0$ on $(1, 3) \cup (4, 6)$