- Beautiful Connection
- The First Fundamental Theorem of Calculus
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Beautiful Connection

Thus far we have learned about two very important limits in the study of calculus. That is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\int_a^b f(x)dx = \lim_{|P| \to 0} \sum_{i=1}^{n} f(\bar{x}_i)\Delta x_i$$

It turns out that these two limits have a very beautiful connection to one another. This connection is so important to the study of calculus, that it is known as **The First Fundamental Theorem of Calculus**. (Just given the name, one should quickly pick up on the importance of this theorem.)

Let us first try to establish the connection intuitively. Let $v(t)$ be the velocity of an object at time t. The total distance travelled from time t_1 to t_2 is the area under the velocity curve between $t = t_1$ and $t = t_2$. Here's why....

Thus we can see that the displacement from time $t = 0$ to $t = x$ is $s(x) = \int_0^x v(t)dt$. We know that $s'(x) = v(x)$. This implies that

$$\boxed{\frac{d}{dx}s(x) = \frac{d}{dx}\int_0^x v(t)dt = v(x)}$$

This is in essence the First Fundamental Theorem of Calculus (1st FTC).

First Fundamental Theorem of Calculus Let f be continuous on a closed interval $[a, b]$ and let x be a (variable) point in (a, b). Then

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Examples Determine the following derivatives

1. $\dfrac{d}{dx}\displaystyle\int_1^x t^2\sin(2t)dt$

2. $\dfrac{d}{dx}\displaystyle\int_x^5 \sqrt{s^3+3\tan(\pi s)+6}ds$

3. $\dfrac{d}{dx}\displaystyle\int_0^{x^2+x} (\cos(5t)+t)dt$ (Chain Rule)

4. $\dfrac{d}{dx}\displaystyle\int_0^x x^2(t+1)dt$ (Product Rule)

Accumulation Function

$A(x) = \displaystyle\int_0^x f(t)dt$ is the accumulation function for $f(x)$, in that it represents the accumulated area with respect to x.

Graph the accumulation function $A(x)$ that is equal to the indicated area. (Begin by graphing the function in the integrand).

1. $A(x) = \displaystyle\int_1^x (t-1)dt$

2. $A(x) = \int_0^x \sin(t)dt$

Properties of the Definite Integral

Comparison Property If f and g are integrable on $[a, b]$ and if $f(x) \leq g(x)$ for all x in $[a, b]$, then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

Boundedness Property If f is integrable on $[a, b]$ and $m \leq f(x) \leq M$ for all x in $[a, b]$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

Linearity of the Definite Integral Suppose that f and g are integrable on $[a, b]$ and that k is a constant. Then kf and $f + g$ are integrable and

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx$$

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

Example Suppose that $\int_0^1 f(x)dx = 2, \int_1^2 f(x)dx = 3, \int_0^1 g(x)dx = -1, \int_0^2 g(x)dx = 4.$ Determine the following.

1. $3\int_0^2 f(x)dx$

2. $\int_2^1 [2f(t) + 6g(t)]dt$

Let $G(x) = \int_0^x \sin(t)dt$

1. Write the differential equation $\frac{dy}{dx} = G'(x)$, and solve for y. Assume $y = G(0)$ when $x = 0$.

2. Show that $\int_0^\pi \sin(t)dt = 2$