OUTSOURCING AMONG COMPETING VENDORS

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The Situation

- Single Manufacturer: has a fixed number of items under warranty that need to be serviced.

- n Vendors: each has a fixed service capacity to repair the items.

The Manufacturer's Problem

How to allocate the items among the vendors?

The Vendor's Problem

How to price their services competitively?

Market Assumptions

- Vendors compete against each other.
- They do not (or are not allowed to) form coalitions.
- No vendor is indispensable.
- Vendor signs a contract with the manufacturer to service each warranty claim for a fixed charge. This covers the vendor's labor, the manufacturer covers the parts cost.

Market Assumption (cont.)

- Vendors' maximum service capacities are open information to the manufacturer and vendors as well.
- The manufacturer incurs a goodwill cost whenever an item under warranty fails and has to be out of service.
- The manufacturer chooses an allocation of the items under warranty to the vendors so as to minimize his repair costs and goodwill costs.
- The vendors choose contract prices to maximize their revenues.

Mathematical Model

- c_i : cost per repair at vendor i, $1 \leq i \leq n$.
- μ_i: service capacity at vendor i, in number of repairs per unit time.
- λ: total failure rate of all items under warranty. (Number of items times individual failure rate.)
- λ_i: total failure rate assigned to vendor i. (λ_i/λ = fraction of items allocated to vendor i.)
- $g_i(\lambda_i)$ = Vendor i's contribution to the goodwill cost if vendor i gets allocation λ_i.

Assumptions

- There are enough vendors:

$$\sum_{i=1}^{n} \mu_i > \lambda.$$

- No vendor is indispensable:

$$\sum_{i \neq j} \mu_i > \lambda, \quad \text{for each} \quad 1 \leq j \leq n.$$

- $g_i(\lambda_i)$ is defined for $0 \leq \lambda_i \leq \mu_i$ and satisfies:

$$\frac{d^k g_i(x)}{dx^k} \geq 0 \quad \text{for} \quad k = 1, 2, 3, 4, \;\; 0 \leq x \leq \mu_i.$$

Game Theoretic Formulation

- The vendors are the leaders, i.e. ith vendor selects a contract value c_i.
- The manufacturer responds to the contract value vector $c = [c_1, c_2, \cdots, c_n]$ by deciding the allocation $\bar{\lambda}_i(c)$ for vendor i, $1 \leq i \leq n$.

Manufacturer's Problem

Find $\lambda_i = \bar{\lambda}_i(c)$ that minimizes

$$\sum_{i=1}^{n} [g_i(\lambda_i) + c_i \lambda_i]$$

$$s.t. \qquad \sum_{i=1}^{n} \lambda_i = \lambda$$

$$0 \leq \lambda_i \leq \mu_i, \;\; i = 1, ..., n$$

Vendor's Problem

Vendor i receives the following revenue:

$$R_i(c) = c_i \bar{\lambda}_i(c).$$

Let

$$c_{-i} = [c_1, c_2, \cdots, c_{i-1}, c_{i+1}, \cdots, c_n].$$

If c_{-i} is fixed, vendor i will choose c_i that maximizes R_i. However, each vendor will behave this way, thus generating an n-person non-zero sum non-cooperative game.

Solution to the Manufacturer's Problem

Let $\lambda_i = \lambda_i(k)$ be the solution to the following equation:

$$\frac{dg_i(\lambda_i)}{d\lambda_i} = g_i'(\lambda_i) = k, \quad g_i'(0) \leq k \leq g_i'(\mu_i)$$

Define

$$\lambda_i(k) = 0 \quad \text{for} \quad k < g_i'(0), \quad \text{and} \quad \lambda_i(k) = \mu_i \quad \text{for} \quad k > g_i'(\mu_i).$$

Solution to the Manufacturer's Problem

Property: $\lambda_i(k)$ is monotone nondecreasing function with k.

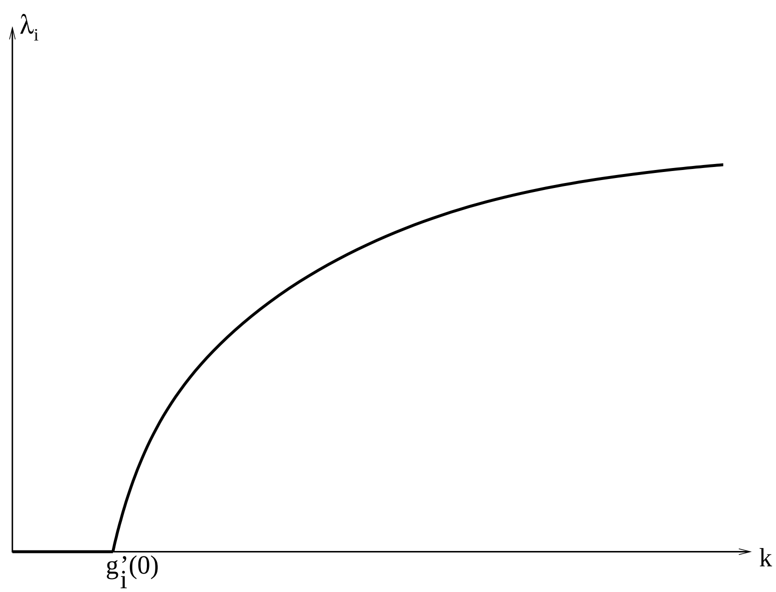


Figure 1: $\lambda_i(k)$ as a function of k.

Solution to the Manufacturer's Problem (cont.)

Theorem 1. *The optimal allocation function* $\bar{\lambda}_i(c)$ *is given by*

$$\bar{\lambda}_i(c) = \lambda_i(\bar{k}(c) - c_i),$$

where $\bar{k}(c)$ *is the unique value of* k *satisfying*

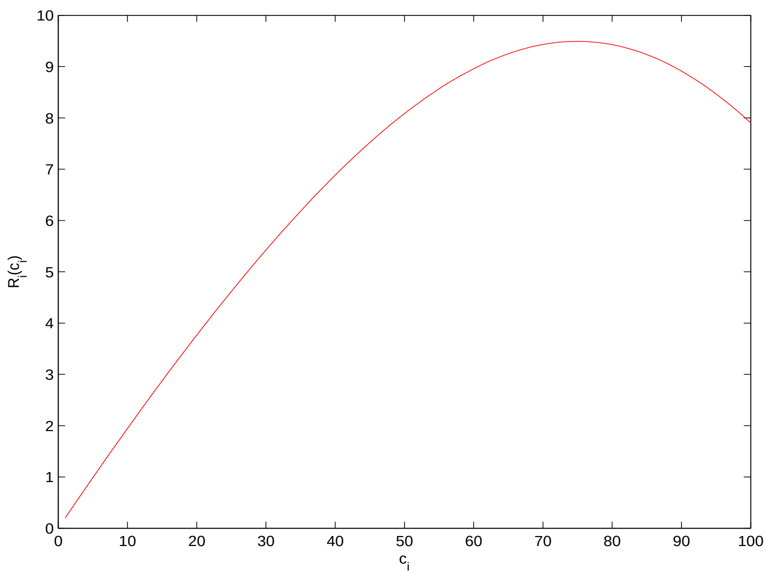
$$\sum_{i=1}^{n} \lambda_i(k - c_i) = \lambda.$$

Property: The optimal allocation function $\bar{\lambda}_i(c)$ is a non-increasing function of c_i, and is a non-decreasing function of c_j for $j \neq i$, for $i = 1, 2, \cdots, n$.

Solution to the Vendor's Problem

Vendor i assumes that $c_{-i} = [c_1, c_2, \cdots, c_{i-1}, c_{i+1}, \cdots, c_n]$ is given, and determines c_i that maximizes

$$R_i(c) = c_i \bar{\lambda}_i(c).$$



Solution to the Vendor's Problem

Results: R_i is a concave function of c_i when $g_i(\lambda_i)$ is a quadratic function for $i = 1, 2, \cdots, n$.

Difficulty: R_i need not be a concave function of c_i in general.

Numerical Observation: $R_i(c)$ is a unimodal function of c_i.

From now on we assume that R_i achieves its maximum at $c_i = \alpha_i(c_{-i})$ that satisfies the first order condition

$$\frac{dR_i}{dc_i} = c_i \frac{\partial \bar{\lambda}_i(c)}{\partial c_i} + \bar{\lambda}_i(c) = 0.$$

$c_i = \alpha_i(c_{-i})$ is the response function for the ith vendor.

Nash Equilibrium

Let $c_i^*, \quad 1 \le i \le n$ simultaneously solve the first order condition for $i = 1, 2, \cdots, n$. That is,

$$c_i^* = \alpha_i(c_{-i}^*), \quad i = 1, 2, \cdots, n.$$

Then $c^* = [c_1^*, c_2^*, \cdots, c_n^*]$ is a Nash equilibrium for the game.

Thus no player will unilaterally deviate from this solution.

Nash Equilibrium

Theorem 2. *Let k^*, m^*, and c^* be a solution to the following set of $n+2$ equations:*

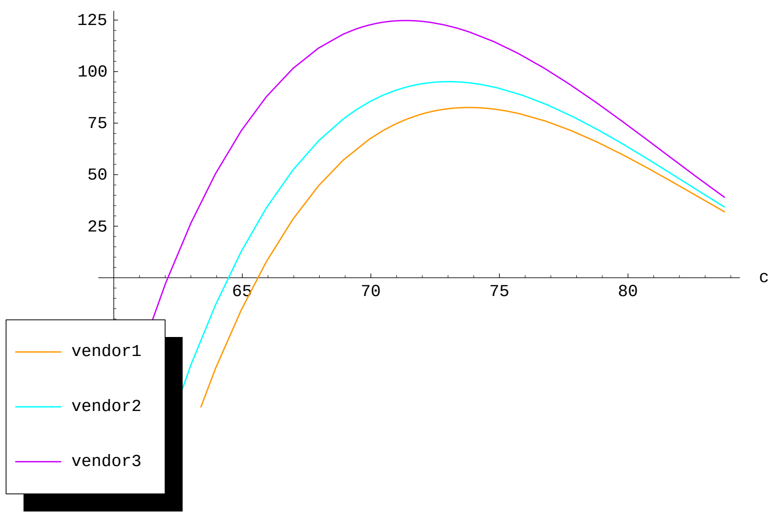
$$
\begin{aligned}
\frac{\lambda_i(k-c_i)}{c_i\lambda_i'(k-c_i)} + m\lambda_i'(k-c_i) &= 1 \text{ for } i = 1, 2, \cdots, n, \\
\sum_{i=1}^{n} \frac{\lambda_i}{c_i\lambda_i'(k-c_i)} &= n-1, \\
\sum_{i=1}^{n} \lambda_i(k-c_i) &= \lambda.
\end{aligned}
$$

Then c^* is a Nash equilibrium, i.e., the optimal contract vector for all the vendors from which no vendor will deviate unilaterally.

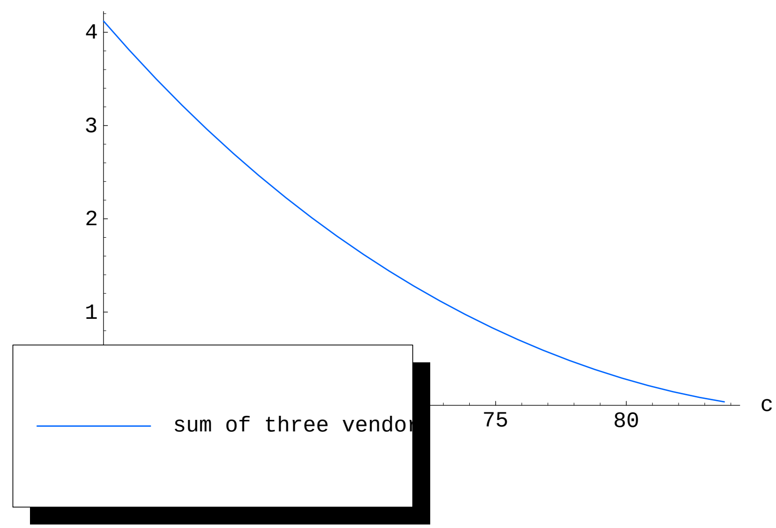
Nash Equilibrium

Property: For a given value of k there can be at most n solutions to the above system of equations.

$$\frac{1}{\lambda_i'} - \frac{\lambda_i}{c_i(\lambda_i')^2}$$



$$\sum_{i=1}^{n} \frac{\lambda_i}{c_i \lambda_i'(k-c_i)}$$



Nash Equilibrium

Results: There is a unique Nash Equilibrium when

- n=2 for general cost functions $g_i(\lambda_i)$ or
- $g_i(\lambda_i)$ is a quadratic function for all $i = 1, 2, \cdots, n$, i.e.,

$$g_i(\lambda_i) = a_i(\frac{\lambda_i}{\mu_i}) + b_i(\frac{\lambda_i}{\mu_i})^2$$

  where a_i and b_i are two positive parameters for $0 \leq \lambda_i \leq \mu_i$.

Numerical Observation: There is a unique Nash Equilibrium.

Example: Identical Vendors,Quadratic Costs

The goodwill cost function for vendor i, with service capacity μ, is determined by two positive parameters a and b as follows:

$$g_i(\lambda_i) = a\frac{\lambda_i}{\mu} + b(\frac{\lambda_i}{\mu})^2, \quad 0 \leq \lambda_i \leq \mu.$$

we get

$$\lambda_i^* = \lambda/n, \quad \text{and} \quad c_i^* = \frac{2b\lambda}{\mu^2(n-1)} \text{ for } i = 1, \cdots, n$$

Identical Vendors,Quadratic Costs (cont.)

Let $r =$ the minimum revenue needed for any vendor to join the market. The maximum number of vendors the market will support

$$n^* = \frac{1 + \sqrt{1 + \frac{8b\lambda^2}{r\mu^2}}}{2}$$

Manufacturer's optimal cost is a decreasing function of n, and hence the manufacturer will use all the vendors.

Example: Identical Vendors, M/M/1 Costs

The goodwill cost function for vendor i, with service capacity μ, is determined by a single parameter h as follows:

$$g_i(\lambda_i) = h\frac{\lambda_i}{\mu - \lambda_i}, \quad 0 \leq \lambda_i < \mu.$$

This is also an increasing, strictly convex and differentiable function. We think of the ith vendor as an $M/M/1$ queue with arrival rate λ_i, service rate μ and holding cost rate h per item per unit time. Then g_i is the holding cost per unit time in steady state. The unique Nash equilibrium is given by

$$c_i^* = \frac{2\lambda h\mu}{(\mu - \frac{\lambda}{n})^3(n-1)} \quad \text{and} \quad \lambda_i^* = \lambda/n, \quad 1 \leq i \leq n.$$

Effects of Coalition Forming

Case I: Three identical vendors with service capacity μ, and identical quadratic holding cost function.

$$F^*(I) = \frac{a\lambda}{\mu} + \frac{4b\lambda^2}{3\mu^2} \quad \text{and} \quad R_i^*(I) = \frac{b\lambda^2}{3\mu^2}, \text{ for } i = 1, 2, 3.$$

Case II: Two of them merge to form new vendor with service capacity 2μ, the third firm remains alone.

$$F^*(II) = \frac{1}{180}\left(\frac{132a\lambda}{\mu} + \frac{279b\lambda^2}{\mu^2} - \frac{a^2}{b}\right) \quad \text{and}$$

$$R_1^*(II) = \frac{(a\mu + 9b\lambda)^2}{90b\mu^2}, \quad R_2^*(II) = \frac{(a\mu - 6b\lambda)^2}{90b\mu^2}.$$

Effects of Coalition Forming(cont.)

Comparison:

- $F^*(II) \le F^*(I)$, if $\frac{\lambda}{\mu} \le \frac{a(\sqrt{\frac{205}{3}}+8)}{13b}$.
- $R_1^*(II) > R_1^*(I) + R_2^*(I)$.
- $R_2^*(II) \ge R_3^*(I)$, if

$$\frac{\lambda}{\mu} \le \frac{a}{b}(1 - \sqrt{\frac{5}{6}}) \quad \text{or} \quad \frac{\lambda}{\mu} \ge \frac{a}{b}(1 + \sqrt{\frac{5}{6}}).$$

Effect of Service Capacity

Consider 10 vendors with total service capacity $=22$ and 10 scenarios:

Scenario k: $\mu_i = \frac{22-0.4(10-k)}{k}, \quad i = 1, 2, \cdots, k,$ and

$$\mu_i = 0.4, \quad i = k+1, \cdots, 10, \qquad h_i = 8, \quad i = 1, 2, \cdots, 10.$$

- When $0 < \lambda \leq 0.038$
  scenario 1 is the best
- When $0.038 < \lambda \leq 3.6$
  scenario 2 is the best.