Managing Warranties and Warranty Reserves

V. G. Kulkarni[1]

Joint Work with Shau-Shiang Ja[2] and Peter Buczkowski[1]

[1] Department of Statistics and Operations Research
University of North Carolina at Chapel Hill

[2] American Airlines

Why Warranties Exist

- Legal
- Interests of consumers
- Interests of producers

Types of Warranties

- Renewable Free-Replacement Warranty Policy
- Non-Renewable Free-Replacement Warranty Policy
- Pro Rata Warranty Policy
- Non-Renewable Minimal-Repair Warranty Policy

Analysis of a Single sale: Notation

- $W =$ Warranty period.
- $F =$ cdf of the lifetime of the item under warranty.
- $D =$ cost of failure (repair/replacement/compensation)
- $\alpha > 0$ continuous discounting factor.
- $Z(W) =$ Expected total discounted cost to the producer of underwriting the warranty for a single item.

Analysis of a Single sale: Assumptions

- Consumers always file claims if they are eligible.
- Failure processing is instantaneous.
- Lifetimes, repair costs, etc are iid.

Renewable Free-Replacement Warranty Policy

Definition: If the consumer purchases an item at time 0 and it fails at time $x < W$, the producer has to provide a new identical replacement which comes with a new full warranty period W.

Notation:

$$\tilde{F}(\alpha, W) = \int_0^W e^{-\alpha x} dF(x). \tag{1}$$

Results:

$$E[Z(W)] = \frac{E[D]\tilde{F}(\alpha, W)}{1 - \tilde{F}(\alpha, W)}, \tag{2}$$

$$E[Z^2(W)] = \frac{E[D^2](1 - \tilde{F}(\alpha, W))\tilde{F}(2\alpha, W) + 2E^2[D]\tilde{F}(\alpha, W)\tilde{F}(2\alpha, W)}{(1 - \tilde{F}(\alpha, W))(1 - \tilde{F}(2\alpha, W))}. \tag{3}$$

Non-Renewable Free-Replacement Warranty Policy

Definition: If the consumer purchases an item at time 0 and it fails at time $x < W$, then the provider has to provide a free replacement which comes with the warranty period $W - x$.

- Let $M(t) =$ Renewal function associated with F.
- $m(t) = M'(t) =$ Renewal density.

Results:

$$E[Z(W)] = E[D] \int_0^W e^{-\alpha t} dM(t), \tag{4}$$

$$E[Z^2(W)] = 2E^2[D] \int_{v=0}^W \int_{u=v}^W e^{-\alpha(u+v)} dM(u-v) dM(v) + E[D^2] \int_0^W e^{-2\alpha u} dM(u). \tag{5}$$

Pro Rata Warranty Policy

Definition: If the consumer purchases an item at time 0, and it fails at time x and $x < W$, then the consumer buys a new item for price $\frac{x}{W}$ times the original price. The new item comes with a full warranty.

Notation: $d =$ the full purchase price.

Results:

$$E[Z(W)] = \frac{d \int_0^W (1 - \frac{x}{W}) e^{-\alpha x} dF(x)}{1 - \tilde{F}(\alpha, W)}, \tag{6}$$

$$E[Z^2(W)] = [d^2 (1 - \tilde{F}(\alpha, W)) \int_0^W (1 - \frac{x}{W})^2 e^{-2\alpha x} dF(x)$$

$$+ 2d^2 \int_0^W (1 - \frac{x}{W}) e^{-\alpha x} dF(x) \int_0^W (1 - \frac{x}{W}) e^{-2\alpha x} dF(x)] /$$

$$[(1 - \tilde{F}(\alpha, W))(1 - \tilde{F}(2\alpha, W))].$$

Non-Renewable Minimal-Repair Warranty Policy

Definition: Minimal Repair: The failure rate remains unchanged after repair.

Definition: The producer has to repair the failures happening during the warranty period $(0, W]$ and pay for the repair costs.

Notation: $r(t) = \frac{f(t)}{1-F(t)}$ = failure rate of the item.

Results:

$$E[Z(W)] = E[D] \int_0^W r(t) e^{-\alpha t} dt, \quad (7)$$

$$E[Z^2(W)] = E[D^2] \int_0^W r(t) e^{-2\alpha t} dt + E^2[D] (\int_0^W r(t) e^{-\alpha t} dt)^2. \quad (8)$$

Aggregate Sales

Notation:

- L = Product life cycle.
- $T(L)$ = the total discounted (back to time zero) warranty cost of all the sales during the life cycle L

Sales Processes:

- $S(t)$ = number of items sold during $[0, t]$.
- Examples:
  - Poisson process
  - Compound Poisson process
  - Non-homogenous Poisson Process
  - Process of independent increments
  - Batch renewal process
  - Markov modulated Poisson process, etc.

Characterizing Sales Processes

Assume that three functions $\mu(t), \nu(t), \rho(s,t)$ exist such that:

- $E(S(t+h) - S(t)) = \mu(t+h) - \mu(t) + o(h), \quad t, h \geq 0$
- $E((S(t+h) - S(t))^2) = \nu(t+h) - \nu(t) + o(h), \quad t, h \geq 0$
- $E((S(s+h_1) - S(s))(S(t+h_2) - S(t))) =$

  $\rho(s+h_1, t+h_2) - \rho(s, t+h_2) - \rho(s+h_1, t) + \rho(s,t) + o(h_1 h_2)$

  $t, s, h_1, h_2 \geq 0, \ \ [t, t+h_2] \cap [s, s+h_1] = \emptyset$

When μ, ν, ρ are sufficiently smooth, we can write

- $E(S(t+h) - S(t)) = \mu'(t)h + o(h), \quad t, h \geq 0$
- $E((S(t+h) - S(t))^2) = \nu'(t)h + o(h), \quad t, h \geq 0$
- $E((S(s+h_1) - S(s))(X(t+h_2) - X(t))) = \frac{\partial^2}{\partial s \partial t}\rho(s,t) h_1 h_2 + o(h_1 h_2)$,

  $t, s, h_1, h_2 \geq 0, \ \ [t, t+h_2] \cap [s, s+h_1] = \emptyset$

Characterizing Sales Processes: Examples

- **Independent Increments.** Here we have

$$\mu(t) = E(S(t)),$$
$$\nu(t) = Var(S(t)),$$
$$\rho(s,t) = \mu(s)\mu(t).$$

- **Batch Renewal Process.** Here sales occur in random batches of size R, and the inter-sales times are iid with cdf G, with the corresponding renewal function $M_G(t)$. We have

$$\mu(t) = E(R)M_G(t),$$
$$\nu(t) = E(R^2)M_G(t),$$
$$\partial_s\partial_t\rho(s,t) = E^2[R]dM_G(s)dM_G(t-s), \quad t > s.$$

Stochastic Sales Processes: Results

$$E[T(L)] = E[Z(W)] \int_0^L e^{-\alpha t} d\mu(t) \tag{9}$$

$$\begin{aligned} Var(T(L)) &= Var(Z(W)) \int_0^L e^{-2\alpha t} d\mu(t) + E^2[Z(W)] \int_0^L e^{-2\alpha t} d\nu(t) \\ &\quad +2E^2[Z(W)] \int_{u=0}^L \int_{v=u}^L e^{-\alpha(u+v)} d_u d_v \rho(u,v) \\ &\quad -(E[T(L)])^2. \end{aligned} \tag{10}$$

Funding a Warranty Reserve

Producers plan for warranty costs through a fund called the **warranty reserve**.

- The manufacturer contributes to the reserve in the following ways:
  1. Fund the reserve initially with seed money.
  2. Set aside a fraction of each sale for the reserve.
  3. Contribute (or deduct) from the reserve at regular intervals.
- If the reserve is too high, the excess funds could have been invested.
- If the reserve is too low, the company incurs administrative costs to locate the money needed for repair or replacement.

How should the manufacturer fund the warranty reserve?

Managing Warranty Reserves: Initial Funding Only

- Warranty reserves are funds set aside to cover the future warranty liabilities arising from all the sales during the product life cycle.

- $\alpha =$ continuous interest rate on the warranty reserves.

- $\psi(x) =$ the probability that the warranty reserve is never depleted given that the initial reserve is x dollars.

Results:

$$\psi(x) = Pr\{T(L) \leq x\}. \tag{11}$$

We want to compute x such that $\psi(x) \geq 1 - \beta$ where β is a given number. Approximating $T(L)$ by a Normal random variable with mean $E(T(L))$ and variance $\text{Var}(T(L))$, we get

$$\text{Warranty Reserve} \approx E(T(L)) + z_\beta \sqrt{\text{Var}(T(L))},$$

where z_β is the $(1 - \beta)$th percentile of a standard normal.

Numerical Example

Item Parameters:

- Warranty period 1 year.
- Non-renewable free-replacement warranty.
- Lifetimes are exponential with mean 1 year.
- Replacement costs are fixed = \$100.
- Product life cycle = 5 years.
- Discount factor = .068.

Sales Processes:

(a). Poisson with rate 200 per year.

(b). Compound Poisson with 365 batches sold per year, each batch is Poisson with mean 200/365.

(c). Non homogeneous Poisson with daily sales rate

$$\lambda(t) = \begin{cases} 10/31 & \text{for the first month in each year} \\ 10/28 & \text{for the second month in each year} \\ 14/31 & \text{for the third month in each year} \\ 14/30 & \text{for the fourth month in each year} \\ 14/31 & \text{for the fifth month in each year} \\ 25/30 & \text{for the sixth month in each year} \\ 25/31 & \text{for the seventh month in each year} \\ 25/31 & \text{for the eighth month in each year} \\ 25/30 & \text{for the ninth month in each year} \\ 14/31 & \text{for the tenth month in each year} \\ 14/30 & \text{for the eleventh month in each year} \\ 10/31 & \text{for the twelfth month in each year.} \end{cases} \quad (12)$$

(d). Batch Renewal sales process with batch sales of exactly one batch per day, with Poisson batch sizes with mean 200/365.

Numerical Example

Single Sale Analysis:

$$E(Z(W)) = 97,$$
$$\text{Var}(Z(W)) = 18696.$$

Warranty Reserve Analysis:

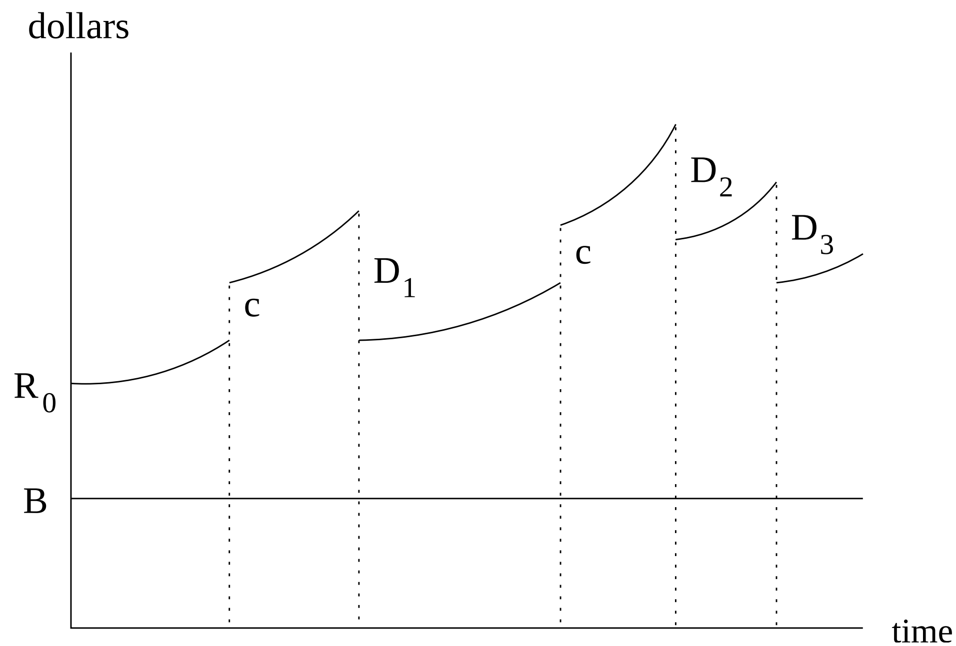
	Numerical Results		
Cases	$E[T(L)]$	Var$(T(L))$	Warranty Reserve
(a)	81955	13565138	87995
(b)	81955	17280899	88772
(c)	81820	13519202	87850
(d)	81947	13562611	87987

Table 1: Numerical results of the four cases.

Managing Warranty Reserves: Initial Funding and Contribution After Each Sale

At the beginning of each period, the manufacturer must decide on:

- c, the amount to contribute to the reserve after each sale, and
- R_0, the amount of money in the reserve account at the beginning of the period.



Model Assumptions

- Items are under warranty for a random amount of time. The warranty duration is a random variable with cdf $F(\cdot)$. While an item is under warranty, it fails at rate λ.

- The sales process is a nonhomogeneous Poisson Process with mean $\theta(t)$ sales per year.

- At time 0, the manufacturer observes the number of items under warranty, but not their remaining warranty durations.

- The reserve earns interest at rate $\alpha > 0$.

Notation

- $R(t)$ = reserve level at time t.
- $X(t)$ = number of items under warranty at time t.
- $S(t)$ = total number of sales in $[0,t]$. $\{S(t), t \geq 0\} \sim NPP(\theta(\cdot))$.
- $D(t)$ = total undiscounted costs of all claims up to time t. The repair costs have common mean $E[D]$ and second moment $E[D^2]$.

By convention, we use lower-case letters for the moments and a subscript of 2 for the second moments. (Ex: $E[X(t)] = x(t), E[X^2(t)] = x_2(t)$.)

We distinguish between the effects of the items sold before time 0 from the items sold after time 0, using a superscirpt of n for the portion of each process due to the new items and an o for the old items.

Differential Equation for $E[R(t)]$

Theorem 1. *Let* $r(t) = E[R(t)]$. *Then,*

$$\frac{dr(t)}{dt} = \alpha r(t) + c\theta(t) - \lambda E[D]x(t), \tag{13}$$

with initial condition $r(0) = R_0$.

In the derivation of the second moment, we will need expressions for the following additional quantities:

$E[R^n(t)]$, $E[R^o(t)X^o(t)]$, and $E[R(t)X(t)]$.

Differential Equation for $E[R^2(t)]$

Theorem 2. *Let* $r_2(t) = E[R^2(t)]$, $u(t) = E[R(t)X(t)]$, $r^n(t) = E[R^n(t)]$, *and* $v(t) = E[R^o(t)X^o(t)]$. *Then,*

$$\frac{dr_2(t)}{dt} = 2\alpha r_2(t) + c^2\theta(t) + \lambda E[D^2]x(t) + 2c\theta(t)r(t) - 2\lambda E[D]u(t), \quad (14)$$

where

$$\frac{du(t)}{dt} = (\alpha - h^n(t))\, u(t) + c\theta(t)\,(x(t) + 1) - \lambda E[D]x_2(t) + \theta(t)r(t),$$
$$+ (h^n(t) - h^o(t)) * (r^n(t)x^o(t) + v(t)) \,,$$
$$\frac{dr^n(t)}{dt} = \alpha r^n(t) + c\theta(t) - \lambda E[D]x^n(t),$$
$$\frac{dv(t)}{dt} = (\alpha - h^o(t))\, v(t) - \lambda E[D]x_2^o(t),$$

with initial conditions $r_2(0) = R_0^2, r^n(0) = 0, v(0) = u(0) = R_0X(0)$.

Making Decisions for c and R_0

The manufacturer wishes to remain above the target $B > 0$ with some prespecified confidence level β. We assume that the distribution of $R(t)$ is approximately Normal. Let T be the length of the fiscal period. We recommend that the manufacturer choose c and R_0 such that:

- $r(T) = R_0$, and

- $\min_{0 \le t \le T} \left[R(t) - z_\beta \sqrt{Var(R(t))} \right] = B$

Note: This will underestimate the true % below B.

Numerical Example

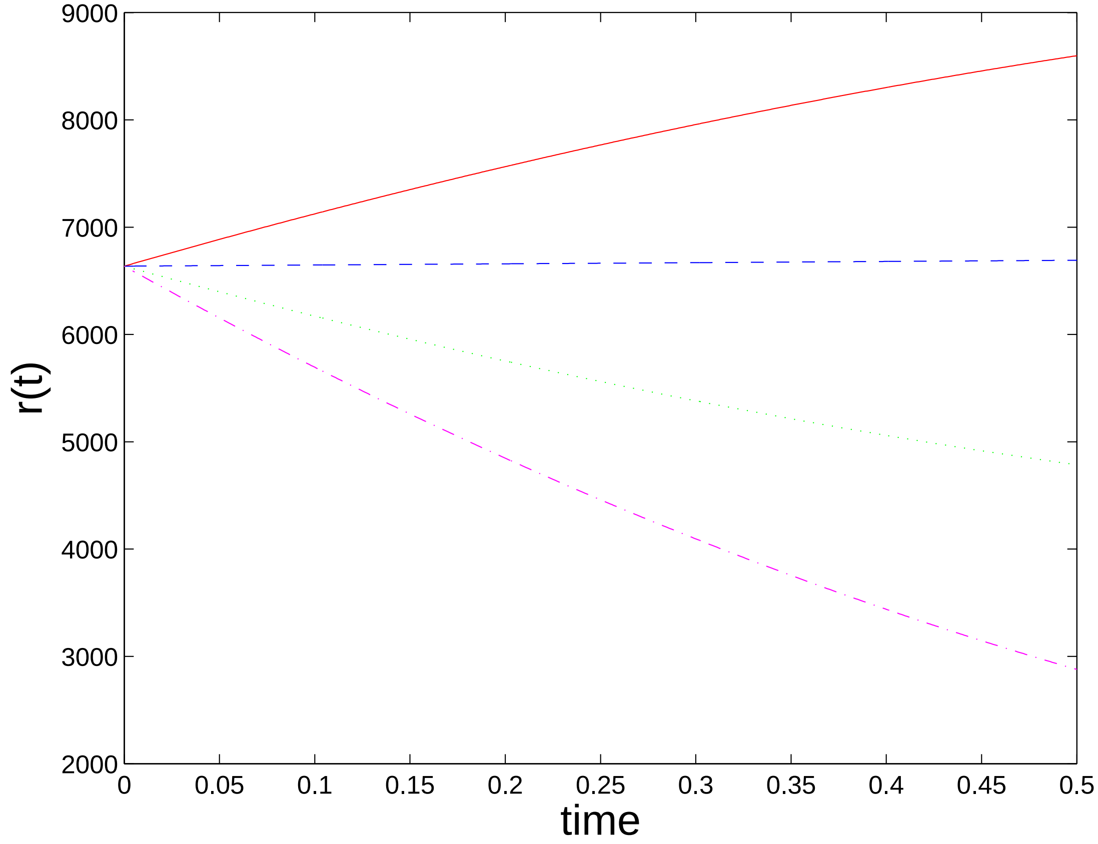
Consider the following numerical example, where items have a free replacement warranty with a fixed period of one year:

- $E[D] = 100$ and $E[D^2] = 10000$,
- $\theta(t)$=1000/year,
- α=.06,
- λ=0.1/year,
- $B = \$5000$.

Expected total discounted warranty cost for a single item $= \$9.71$.

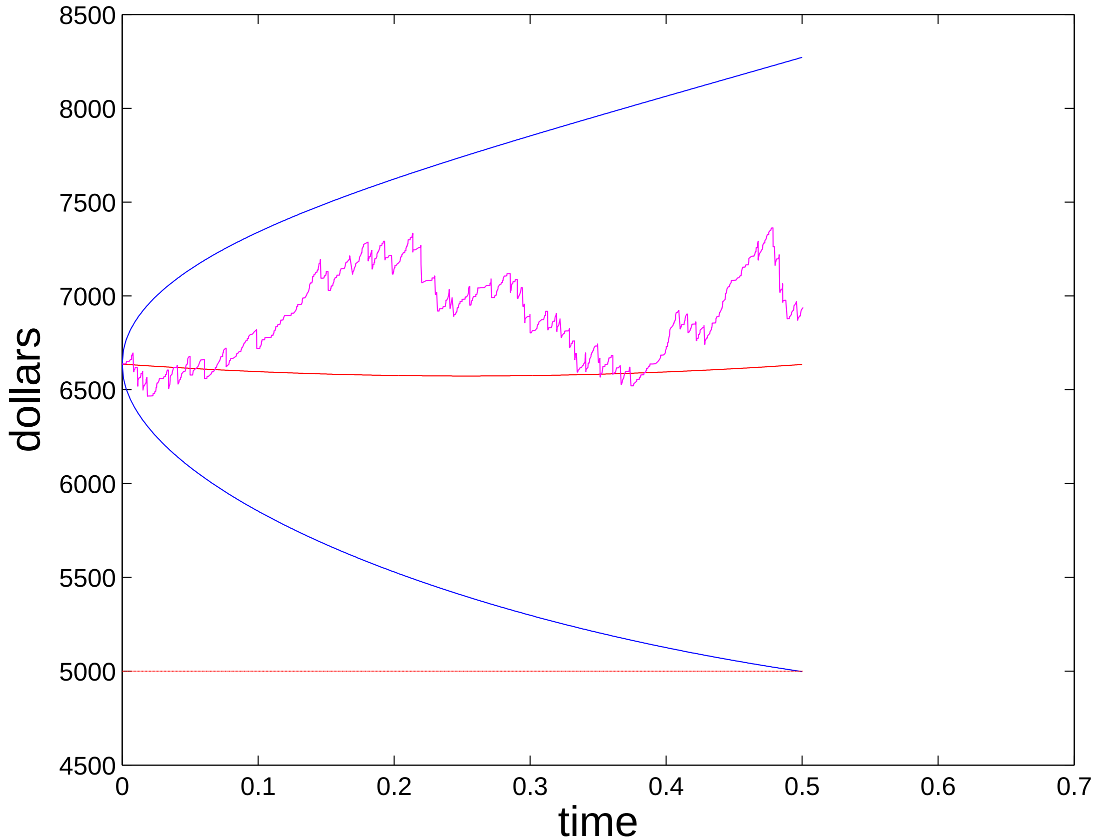
Effects of $X(0)$ on the Reserve

The figure below plots $r(t)$ for $X(0) = \{500, 1000, 1500, 2000\}$:

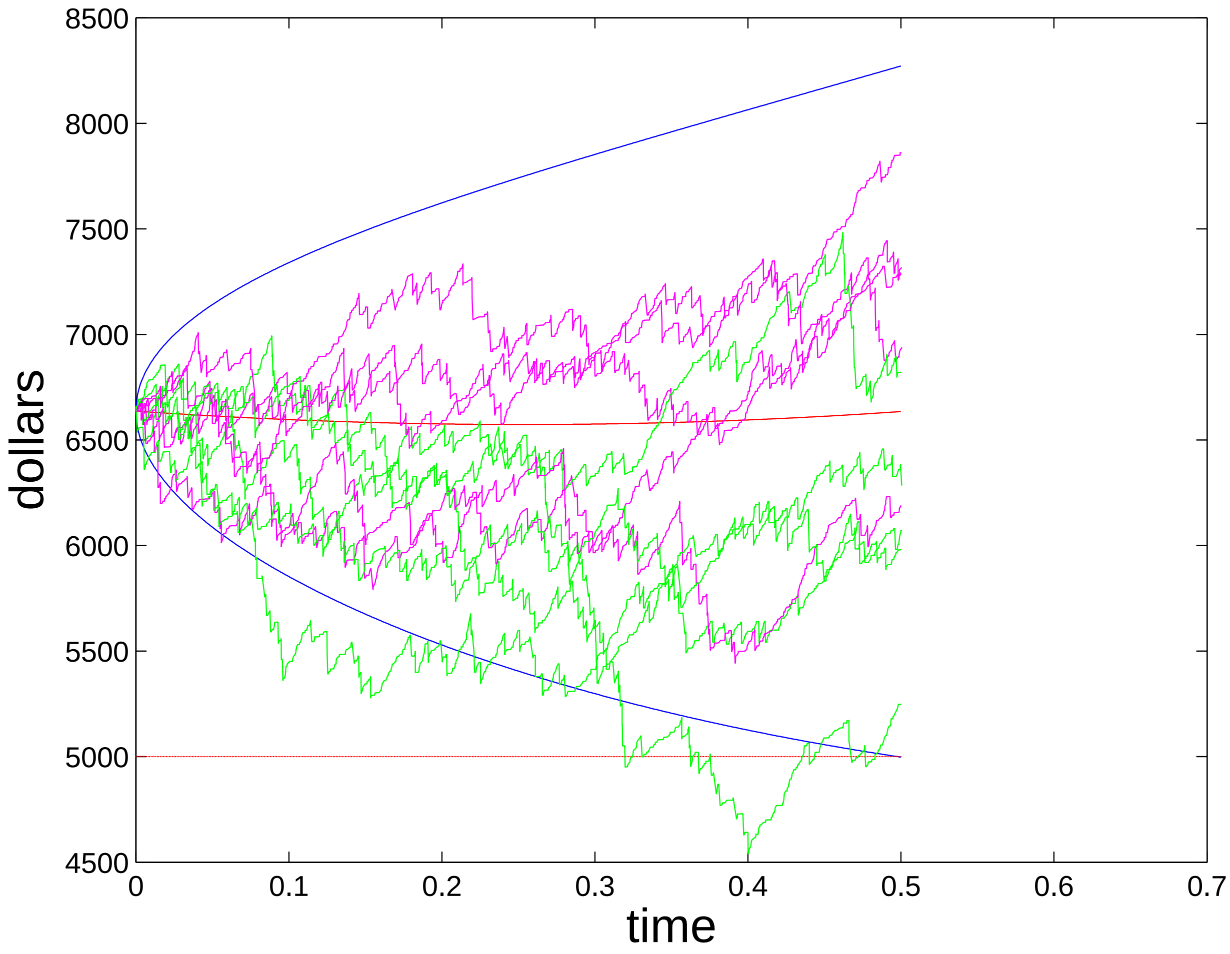


Simulation Examples

Consider the numbers of the previous example (Here, $X(0) = 1200$, $c = \$11.10$, and $R_0 = \$6636.85$).



Simulation Examples (cont.)



Simulation Results of Example 1

Time	Theor. Mean	Sim. Mean	Theor. SD	Sim SD	% below Mean-2*SD
0.0625	6609.3	6616.5	285.5668	289.3416	2.3%
0.125	6589.6	6603.6	404.3103	407.2525	2.9%
0.1875	6577.5	6587.2	493.1889	498.1227	3.3%
0.25	6573.3	6586.7	568.7701	577.0350	2.75%
0.3125	6576.8	6590.4	633.8545	645.2094	2.5%
0.375	6588.2	6601.9	693.2730	702.8973	2.2%
0.4375	6607.5	6612.7	746.7288	753.9137	2.3%
0.5	6634.7	6648.2	797.0033	798.1121	2.15%

Note: 4.9% fell below the target $B = \$5000$.

Future Work

- Obtain confidence bands for the entire process, rather than for each individual point.

- Compute the finite-time ruin probability of $R(t)$.

- Consider a mixture of warranty policies & different values of c for different warranty policies.

- Use simulation to look at different funding policies.

- Consider other objectives, such as minimizing a penalty function.