STOR 215 Final
4:00pm - 6:30pm, Friday, Dec 17, 2010.

1. Attempt all problems. Show all work.
2. The problem weights are as given in the parentheses.
3. The exam is open book, open notes.
4. Laptops, calculators, internet access are allowed.
5. No discussion about the exam is allowed with anybody except the proctors.

P1. Suppose we randomly draw four cards out of a standard deck of 52 cards.
(All the answers to the following questions must be given as decimal numbers, for example .35. Show all steps.)

(a) (4) Let $A$ be the event that all are face cards (there are sixteen face cards in a deck of 52 cards), and $B$ be the event that all four are red cards. Compute $P(A)$, $P(B)$ and $P(A|B)$. Are these two events independent? Why or why not?

Solution: Number of ways of picking four cards = $\binom{52}{4} = 270725$.
Number of ways of picking four face cards = $\binom{16}{4} = 1820$.
Number of ways of picking four red cards = $\binom{26}{4} = 14950$.
Number of ways of picking four red face cards = $\binom{8}{4} = 70$.

Hence $P(A) = 1820/270725 = .006722$, $P(B) = 14950/270725 = .05522$, $P(AB) = 70/270725 = .0002586$, $P(A|B) = 70/14950 = .0046822$.

Since $P(A|B) \neq P(A)$, events $A$ and $B$ are not independent.

(b) (4) Let $X$ be the number of black cards among the four cards drawn. What is the probability distribution of $X$?

Solution: $X$ can take values $k = 0, 1, 2, 3, 4$. $P(X = k) =$ Probability of getting exactly $k$ black cards among the four cards = $\binom{26}{k}\binom{26}{4-k}/\binom{52}{4}$. Computing this for all values of $k$ we get

$P(X = 0) = .0522$, $P(X = 1) = .2497$, $P(X = 2) = .3902$, $P(X = 3) = .2497$, $P(X = 4) = .0522$.

(c) (2) Compute the expected value of $X$.

Solution:

$$E(X) = \sum_{k=0}^{4} kP(X = k) = 2.$$
P2. Consider a complete graph $G = (V, E)$ with four nodes.

(a) (1) How many edges does it have?
   Solution: 6.

(b) (1) How many subgraphs of $G$ are there? (Not necessarily connected.)
   Solution: $2^6 = 64$.

(c) (1) How many of these subgraphs are distinct labeled spanning trees of $G$?
   Solution: $4^2 = 16$.

Now suppose we construct a random subgraph of $G$ by picking a random number of edges as follows: each edge is picked independently with probability $p$, where $0 \leq p \leq 1$ is a given number. Assume this randomization method in answering the following questions:

(a) (1) What is the probability that the randomly selected subgraph is a complete graph?
   Solution: $p^6$.

(b) (1) What is the probability that a given subgraph of three edges is picked?
   Solution: $p^3(1 - p)^3$.

(c) (1) What is the probability that the randomly picked subgraph is a spanning tree?
   Solution: There are 16 spanning trees. Each spanning tree has a probability of $p^3(1 - p)^3$ of being picked. Hence the required probability is $16p^3(1 - p)^3$.

P3. Consider a one way road network $G = (V, E)$ with $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $E = \{(1, 2), (1, 3), (1, 4), (2, 5), (2, 6), (3, 5), (3, 6), (3, 7), (4, 6), (4, 7), (5, 8), (6, 8), (7, 8)\}$.

The edge $(i, j)$ represents a one way road from node $i$ to node $j$. Let $w(i, j)$ represent the toll on the road $(i, j)$. We are given

\[
t(1, 2) = 1, t(1, 3) = 2, t(1, 4) = 3, t(2, 5) = 2, t(2, 6) = 2,
\]
\[
t(3, 5) = 1, t(3, 6) = 2, t(3, 7) = 3, t(4, 6) = 3, t(4, 7) = 3,
\]
\[
t(5, 8) = 3, t(6, 8) = 4, t(7, 8) = 3.
\]

(a) (1) Is this an acyclic network?
   Solution: Yes.
(b) (2) How many directed paths are there in this network from node 1 to node 8? List them.

**Solution:** 7. They are as listed below:

1 : 1 − 2 − 5 − 8,
2 : 1 − 2 − 6 − 8,
3 : 1 − 3 − 5 − 8,
4 : 1 − 3 − 6 − 8,
5 : 1 − 3 − 7 − 8,
6 : 1 − 4 − 6 − 8,
7 : 1 − 4 − 7 − 8.

(c) (3) Suppose the total toll paid in using a path is the sum of the tolls on the individual edges on that path. Use Dijkstra’s algorithm to find the least amount of total toll that needs to be paid to reach node 8 from node 1.

**Solution:** Least toll = 6.

(d) (1) List the least-total-toll paths from node 1 to 8.

**Solution:** Paths 1 and 3 listed above.

(e) (2) Suppose Alice commutes from node 1 to 8 every day. She is not interested in minimizing the toll paid, but chooses a path at random every day as follows. (Don’t worry about how she gets back from node 8 to node 1 every day!) At node one she chooses any of the three outgoing edges with equal probability. When she reaches the end of that edge, she chooses any of the outgoing edges from there with equal probability in an independent fashion, and so on until she reaches node 8. Suppose $L$ is the random path she chooses. Compute the probability distribution of $L$.

**Solution:** Consider the paths as listed in part 2.

\[ P(L = i) = \frac{1}{6}, \quad i = 1, 2, 6, 7, \]
\[ P(L = i) = \frac{1}{9}, \quad i = 3, 4, 5. \]

(f) (1) Compute the expected toll paid by Alice if she a chooses a path at random according to the mechanism described in the previous step.

**Solution:** Let $T(i)$ be the toll payed if Alice chooses path $i$. We have
\[ T(1) = 6, \quad T(2) = 7, \quad T(3) = 6, \quad T(4) = 8, \]
\[ T(5) = 8, \quad T(6) = 8, \quad T(7) = 9. \]

Hence the expected toll paid by Alice is
\[ \frac{1}{6} \times 6 + \frac{1}{6} \times 6 + \frac{1}{9} \times 7 + \frac{1}{9} \times 6 + \frac{1}{9} \times 8 + \frac{1}{9} \times 8 + \frac{1}{6} \times 8 + \frac{1}{6} \times 9 = 7.44. \]
P4. Consider a simplified list of activities involved in building a house. The following table gives the description of the activities, their durations (in days) and immediate predecessors.

<table>
<thead>
<tr>
<th>Activity</th>
<th>Description</th>
<th>Duration</th>
<th>Imm. Pred.</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>Build Foundation</td>
<td>5</td>
<td>None</td>
</tr>
<tr>
<td>b</td>
<td>Build walls and ceilings</td>
<td>8</td>
<td>a</td>
</tr>
<tr>
<td>c</td>
<td>Build roof</td>
<td>10</td>
<td>b</td>
</tr>
<tr>
<td>d</td>
<td>Do electrical wiring</td>
<td>5</td>
<td>b</td>
</tr>
<tr>
<td>e</td>
<td>Install windows</td>
<td>4</td>
<td>b</td>
</tr>
<tr>
<td>f</td>
<td>Install siding</td>
<td>6</td>
<td>e</td>
</tr>
<tr>
<td>g</td>
<td>Paint house</td>
<td>3</td>
<td>c,f</td>
</tr>
</tbody>
</table>

(a) (2) Draw the directed graph representing this project.

Solution:

(b) (3) Determine the minimum number of days it will take to complete building this house. Show the steps of the algorithm you followed in determining this.

Solution: The length of the longest path is 26. Hence the minimum number of days needed to finish the project is 26.

(c) (2) Which activities are critical? Why?

Solution: Every activity other than d is critical.

(d) (1) Suppose it is possible to speed up the installation of the siding by a day by hiring one more worker. Does it make sense to do this? Why or why not?

Solution: No. If activity f is speeded up, the length of the longest path still remains 26.

(e) (1) Suppose the electrical wiring takes 7 days instead of 5. Will it affect the project completion? Why or why not?

Solution: No. This is a slack activity, so speeding it up does not reduce the project duration.