1. Attempt all problems. Show all work.

2. The problem weights are as given in the parentheses.

3. The exam is open book, open notes.

4. No discussion about the exam is allowed with anybody except me.

P1. Suppose you are in charge of distributing grant money to 5 applicants. You have a total of 100,000 dollars at your disposal. Each applicant must be given a non-negative integer multiple of thousand dollars.

(a) (2) How many possible distributions can you make if you want to distribute all the funds?

Solution: Let \( x_i \) be the award given to applicant \( i \). Then we want the number of integer solutions to

\[
x_1 + x_2 + x_3 + x_4 + x_5 = 100, \quad x_i \geq 0, i = 1, 2, 3, 4, 5.
\]

The answer is \( \binom{104}{4} = 4,598,126 \).

(b) (2) How many possible distributions can you make if you are allowed to distribute any amount of funds up to 100,000 dollars?

Solution: Assume there is a sixth applicant. Any amount awarded to this dummy sixth applicant is the amount that you don’t disburse. Then we are asked to compute the number of integer solutions to

\[
x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 100, \quad x_i \geq 0, i = 1, 2, 3, 4, 5, 6.
\]

The answer is \( \binom{105}{5} = 96,560,646 \).

P2. (4) Let \( A \) and \( B \) be two points on a regular graph paper. Point \( A \) is placed at coordinates (0,0) while point \( B \) is placed at coordinates \((m, n)\) where \( m > 0 \) and \( n > 0 \) are integers. How many ways are there to move from the point \( A \) to point \( B \) if the only moves allowed are (1) take one step to the right or (2) one step up at any time. Thus we can only move from point \((i, j)\) to point \((i+1, j)\) or point \((i, j+1)\). We cannot go left or down.

Example: If \( m = 1, n = 1 \), there are two ways to go from \((0,0)\) to \((1,1)\):

1. \((0,0) \rightarrow (0,1) \rightarrow (1,1)\).
2. \((0,0) \rightarrow (1,0) \rightarrow (1,1)\).
Solution: We need to take \(m\) right steps and \(n\) up-steps in any order in order to reach \((m, n)\) from \((0, 0)\). Thus the total number of ways is
\[
\binom{m+n}{m}.
\]

P3. (2) What is the coefficient of \(x^9y^8\) in the binomial expansion of \((x^3 + y^2)^7\)?

Solution:
\[
(x^3 + y^2)^7 = \sum_{k=0}^{7} \binom{7}{k}(x^3)^k(y^2)^{7-k}.
\]

For \(k = 3\) we get term \((x^3)^3(y^2)^4 = x^9y^8\). Hence the required coefficient is
\[
\binom{7}{3} = 35.
\]

P4. A class consists of six boys and eight girls. The teacher wants to form a committee of five students by choosing three girls and two boys. There is one boy named Bob and one boy named Charles and one girl named Alice in the class.

(a) (2) How many ways are there to pick a committee at random?

Solution: We need to choose 2 boys from 6 and 3 girls from 8. Hence the total number of ways is
\[
\binom{6}{2}\binom{8}{3} = 840.
\]

(b) (2) What is the probability that both Bob and Alice are on the committee?

Solution: We first pick Bob and Alice. Then we pick one more boy from the remaining five and two more girls from the remaining seven. Thus the total number of ways of picking a committee with Bob and Alice on it are
\[
\binom{5}{1}\binom{7}{2} = 105.
\]

Hence the required probability is
\[
\frac{105}{840} = 0.125.
\]

(c) (2) What is the probability that Bob is on the committee given that Alice is on the committee?

Solution: Let \(B\) be the event that Bob is on the committee, and \(A\) be the event that Alice is on the committee. Number of committees with Alice on it is
\[
\binom{6}{2}\binom{7}{2} = 315.
\]
Hence
\[ P(A) = \frac{315}{840} = \frac{3}{8}. \]

We have computed
\[ P(AB) = \frac{105}{840} = \frac{1}{8}. \]

Hence
\[ P(B|A) = \frac{P(AB)}{P(A)} = \frac{(1/8)/(3/8)}{1/3} = 1/3. \]

(d) Is the event “Bob is on the committee” independent of the event “Alice is on the committee”?

**Solution:** Number of committees with Bob on it is
\[ \binom{5}{1} \binom{8}{3} = 105. \]

Hence
\[ P(B) = \frac{280}{840} = 1/3. \]

We have computed
\[ P(A) = \frac{3}{8}, \quad P(AB) = 1/8. \]

Hence
\[ P(A)P(B) = (1/3)(3/8) = 1/8 = P(AB). \]

Hence \( A \) and \( B \) are independent.

(e) Is the event “Bob is on the committee” independent of the event “Charles is on the committee”?

**Solution:** Let \( C \) be the event that Charles is on the committee. We have
\[ P(C) = P(B) = 1/3, \]

and
\[ P(BC) = 1/15. \]

Hence
\[ P(BC) \neq P(B)P(C). \]

Hence \( B \) and \( C \) are not independent.

P5. An oil company uses a high-tech method called the seismic reflection survey to predict whether drilling an oil well at a particular off-shore site will actually produce oil. If the off-shore site actually has oil, the survey will detect it with probability .9, and if there is no oil at the site, the survey will falsely detect it with probability .05. From prior experience in this area, the oil company knows that roughly 30 percent of the oil wells in this region produce oil, others come up dry. Suppose the survey detects
oil at this site. What is the probability that an oil well drilled at this site will actually produce oil?

**Solution:** Let $E$ be the event that the oil well has oil, and $E^c$ be its complement. Let $D$ be the event that the survey detects oil, and $D^c$ be its complement. Then we are given

$$P(E) = .3, \quad P(E^c) = 1 - P(E) = .7$$

$$P(D|E) = .9, \quad P(D|E^c) = .05.$$  

We are asked to compute $P(E|D)$. By using Bayes’ rule we get

$$P(E|D) = \frac{P(D|E)P(E)}{P(D|E)P(E) + P(D|E^c)P(E^c)}$$

$$= \frac{.9 * .3}{.9 * .3 + .05 * .7} = \frac{27}{30.5} = .8852.$$