Homework 10

Solutions

1. Computational Exercise 5.2.
Let $L \sim \text{Erl}(3, 1)$ be the lifetime of a battery. Then, the expected inter replacement time is given by

$$E_T = E \min(3, L)$$

$$= \int_0^3 xf_L(x) \, dx + \int_3^\infty 3f_L(x) \, dx$$

$$= \int_0^3 xe^{-x} \frac{x^2}{2} + 3PL > 3$$

$$= 3 - 13.5e^{-3} \text{(using table of integrals)} = 2.3279.$$  

Thus, the long run replacement rate is given by

$$\lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{E_T} = 0.4296.$$  

2. Computational Exercise 5.6.
Let $\{X_n, n \geq 0\}$ be the DTMC of Example 2.4. Suppose an order has just been delivered at time 0, and $X_0 = 5$. Let $T$ be the time when the next order is placed. Let $m_i = E_T | X_0 = i$. A first step analysis shows that

$$m_5 = 1 + 0.498 m_5 + 0.149 m_4 + 0.224 m_3 + 0.224 m_2,$$

$$m_4 = 1 + 0.498 m_4 + 0.149 m_3 + 0.224 m_2,$$

$$m_3 = 1 + 0.498 m_3 + 0.149 m_2, \quad m_2 = 1 + 0.498 m_2.$$  

Solving, the expected time between two consecutive orders is given as $E_T = m_5 = 1.822$. From Example 5.3, it is clear that the number of orders placed up to time $t$ is a renewal process. Hence, the number of orders placed per week in the long run is given by

$$\frac{1}{E_T} = 0.5488.$$  

3. Computational Exercise 5.10.
Use results of Example 5.16. We have

$$E(U_1) = \frac{k}{\lambda} = 2/2 = 10 \text{ days, } \ E(D_1) = \frac{1}{\mu} = 1 \text{ days.}$$

The revenue rate is $A = $200 per day, and the repair cost rate is $B = $10 per day. Hence, from Example 5.16, the long run net revenue per day is given by

$$E(-BD_1 + AU_1) = \frac{-10 \times 24 + 200 \times 10}{10 + 1} = \frac{1760}{11} = 160$$

dollars per day.
In this case, the expected cost over a cycle is given by

\[ EC_1 = 90 - \int_1^4 80(4 - t)\frac{t^2}{63} dt = 64.2857. \]

The expected length of the cycle is 3.036 as computed in the solution to Computational Problem 7.12. and the long run cost rate is given by

\[ \lim_{t \to \infty} \frac{C(t)}{t} = \frac{EC_1}{ET_1} = \frac{64.2857}{3.036} = 21.1745 \text{ dollars/year}. \]

5. Computational Exercise 5.16.
We say that a new cycle starts whenever the pump starts filling an empty tank. Let \( U \), \( D \), and \( R \) be the up, down, and repair times of the oil pump, respectively. Then the cycle length is \( U + D \). When the pump is up, the tank is filling at a rate of 400 gallons/day, and at the end of the up time there are 400\( U \) gallons of oil. This oil gets consumed at a rate 100 gallons/day, hence the tank will be empty after 4\( U \) days. Hence the down time is given by \( D = \max(R, 4U) \). The cdf of \( D \) is given by \( \mathbb{P}(D < x) = \mathbb{P}(\max(R, 4U) < x = \mathbb{P}(R < x \mathbb{P}(4U < x = (1 - e^{-x})(1 - e^{-0.05x}), \)

and the expected value is given by

\[ ED = \int_0^\infty \mathbb{P}(D > x) dx = \int_0^\infty (e^{-x} + e^{-0.05x} - e^{-1.05x}) dx = 20.048. \]

We have \( ET_1 = EU + D = 5 + 20.048 = 25.048 \) days. The oil in the tank increases from 0 to 400\( U \) over \([0, U]\) and then decreases linearly from 400\( U \) to zero over the interval \([U, 5U]\). Thus the average oil stock over \([0, 5U]\) is 200\( U \). Hence the storage cost over the cycle is

\[ EC_1 = 0.05E(200U)(5U) = 50EU^2 = 2500 \text{ dollars}. \]

Hence, the long run cost rate is given by

\[ \lim_{t \to \infty} \frac{C(t)}{t} = \frac{EC_1}{ET_1} = \frac{2500}{25.048} = 99.81 \text{ dollars/day}. \]