Homework 1
Solutions

1. Conceptual Exercise 2.3.
   Let $X_n$ be the number of machines that are up at the beginning of day $n$, then at the beginning of day $n+1$, $k$ out of $X_n$ machines will remain up and $j$ out of $3-X_n$ machines will become up with probability
   \[ C_k^i p^k (1-p)^{i-k} C_j^{3-i} (1-q)^j q^{3-i-j} \]
   for $0 \leq k \leq i$, $0 \leq j \leq 3-i$. Since $X_{n+1}$ depends only on the current state and not on the past, \{\{X_n, n \geq 0\}\} is a DTMC on state space $S = \{0, 1, 2, 3\}$. Using the above expression, the transition probability matrix is given by
   \[
P = \begin{bmatrix}
   q^3 & 3q^2(1-q) & 3q(1-q)^2 & (1-q)^3 \\
   (1-p)q^2 & pq^2 + 2(1-p)q(1-q) & 2pq(1-q) + (1-p)(1-q)^2 & (1-q)^2p \\
   q(1-p)^2 & 2pq(1-p) + (1-q)(1-p)^2 & qp^2 + 2(1-q)p(1-p) & (1-q)p^2 \\
   (1-p)^3 & 3(1-p)^2p & 3(1-p)p^2 & p^3
   \end{bmatrix}.
   \]

   We have
   \[
   X_{n+1} = \begin{cases}
   1 & \text{w.p. } 1, \text{ if } X_n = 0 \\
   2 & \text{w.p. } 1, \text{ if } X_n = 1 \\
   2 & \text{w.p. } p, \text{ if } X_n = 2 \\
   0 & \text{w.p. } 1-p, \text{ if } X_n = 2.
   \end{cases}
   \]
   Since $X_{n+1}$ depends only on the current state and not on the past, \{\{X_n, n \geq 0\}\} is a DTMC on state space $S = \{0, 1, 2\}$. The transition probability matrix is given by
   \[
P = \begin{bmatrix}
   0 & 1 & 0 \\
   0 & 0 & 1 \\
   1-p & 0 & p
   \end{bmatrix}.
   \]

   We have
   \[
   X_{n+1} = \begin{cases}
   1 & \text{if the item that arrives to the shop during the } (n-1)^{\text{st}} \text{ minute is non-defective} \\
   0 & \text{otherwise}.
   \end{cases}
   \]
   Since $X_{n+1}$ does not depend on the past, \{\{X_n, n \geq 0\}\} is a DTMC on state space $S = \{0, 1\}$. The transition probability matrix is given by
   \[
P = \begin{bmatrix}
   1-p & p \\
   1-p & p
   \end{bmatrix}.
   \]
Let
\[
X_n = \begin{cases} 
0 & \text{if day } n \text{ and } n - 1 \text{ are both rainy} \\
1 & \text{if day } n \text{ is sunny and day } n - 1 \text{ is rainy} \\
2 & \text{if day } n \text{ is rainy and day } n - 1 \text{ is sunny} \\
3 & \text{if day } n \text{ and } n - 1 \text{ are both sunny.} 
\end{cases}
\]

Then, \{X_n, n \geq 0\} is a DTMC on state space \(S = \{0, 1, 2, 3\}\) with the transition probability matrix
\[
P = \begin{bmatrix} .4 & .6 & 0 & 0 \\ 0 & 0 & .2 & .8 \\ .3 & .7 & 0 & 0 \\ 0 & 0 & .1 & .9 \end{bmatrix}.
\]

We have
\[
X_{n+1} = \begin{cases} 
X_n + 1 & \text{w.p. } p \text{ if } 2 \leq X_n \leq N - 2 \\
X_n - 1 & \text{w.p. } 1 - p \text{ if } 2 \leq X_n \leq N - 2 \\
2 & \text{w.p. } p \text{ if } X_n = 1 \\
N & \text{w.p. } 1 - p \text{ if } X_n = 1 \\
1 & \text{w.p. } p \text{ if } X_n = N \\
N - 1 & \text{w.p. } 1 - p \text{ if } X_n = N.
\end{cases}
\]

Since \(X_{n+1}\) depends only on the current state and not on the past, \(\{X_n, n \geq 0\}\) is a DTMC on state space \(S = \{1, \ldots, N\}\). The transition probability matrix is given by
\[
P = \begin{bmatrix} 0 & p & 0 & \cdots & 0 & 1 - p \\ 1 - p & 0 & p & \cdots & 0 & 0 \\ 0 & 1 - p & 0 & \cdots & 0 & 0 \\ & \ddots & \ddots & \ddots & \ddots & \ddots \\ p & 0 & 0 & \cdots & 1 - p & 0 \end{bmatrix}.
\]