We know that $Z = X_1 + X_2$ is a $P(\lambda_1 + \lambda_2)$ random variable. We have
\[
P(X_1 = i | Z = k) = \frac{P(X_1 = i; Z = k)}{P(Z = k)}
\]
\[
= \frac{P(X_1 = i; X_2 = k - i)}{P(Z = k)}
\]
\[
= \frac{P(X_1 = i)P(X_2 = k - i)}{P(Z = k)}
\]
(due to independence of $X_1$ and $X_2$)
\[
e^{-\lambda_1 \frac{i}{\lambda_1}} e^{-\lambda_2 \frac{k - i}{\lambda_2}}
\]
\[
e^{-\lambda_1 - \lambda_2 \frac{(\lambda_1 + \lambda_2)}{k}}
\]
\[
= \frac{k!}{i!(k-i)!} \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^i \left( \frac{\lambda_2}{\lambda_1 + \lambda_2} \right)^{k-i}.
\]
Thus, given $X_1 + X_2 = k$, $X_1$ is a $Bin(k, \frac{\lambda_1}{\lambda_1 + \lambda_2})$ random variable.

Suppose $T_i$ is the lifetime of $i$th light bulb ($i = 1, 2, ..., k$). Suppose these lifetimes are independent of each other and that the light bulbs are turned on at the same time. Let $X_i$ be the time interval during which exactly $i$ light bulbs are on. Then $X_i \sim exp(i\lambda)$, and
\[
\max\{T_1, T_2, ..., T_k\} = X_k + X_{k-1} + \cdots + X_2 + X_1.
\]
Hence
\[
E(\max\{T_1, T_2, ..., T_k\}) = E(X_k) + E(X_{k-1}) + \cdots + E(X_2) + E(X_1)
\]
\[
= \frac{1}{k\lambda} + \frac{1}{(k-1)\lambda} + \cdots + \frac{1}{2\lambda} + \frac{1}{\lambda}.
\]

3. Conceptual Problem 3.16.
Using the law of total probability, the complementary cdf of $Z$ is given by
\[
P(Z > z) = P(X_1 + \ldots + X_N > z)
\]
\[
= \sum_{k=1}^{\infty} P(X_1 + \ldots + X_N > z | N = k)(1-p)^{k-1}p
\]
\[
\begin{align*}
&= \sum_{k=1}^{\infty} \sum_{r=0}^{k-1} e^{-\lambda z} \frac{(\lambda z)^r}{r!} (1-p)^{k-1}p \\
&= \sum_{r=0}^{\infty} \sum_{k=r+1}^{\infty} e^{-\lambda z} \frac{(\lambda z)^r}{r!} (1-p)^{k-1}p \\
&= \sum_{r=0}^{\infty} e^{-\lambda z} \frac{(\lambda z)^r}{r!} p \left( \frac{1}{p} - \frac{1 - (1-p)^r}{p} \right) \\
&= \sum_{r=0}^{\infty} e^{-\lambda z} \frac{(\lambda z(1-p))^r}{r!} = e^{-\lambda z p}.
\end{align*}
\]

Thus, \( Z \sim \text{Exp}(\lambda p) \).


The number of miles until first failure is the minimum of four iid \( \text{Exp}(1/5000) \) random variables, the remaining number of miles until second failure is, by the strong memoryless property, the minimum of an \( \text{Exp}(1/1000) \) and three iid \( \text{Exp}(1/5000) \) random variables. Hence the expected total number of miles that can be covered without having to go to a tire shop is given by

\[
\frac{1}{5000} + \frac{1}{3 \cdot 5000} + \frac{1}{1000} = 1875 \text{ miles}.
\]

5. Computational Problem 3.10.

The remaining service times at the tellers are exponentially distributed random variables with mean 5 minutes, i.e., with parameter .2 \( \text{min}^{-1} \). Thus, the amount of time the customer has to wait is the minimum of three \( \text{Exp}(0.2) \) random variables, which is \( \text{Exp}(0.6) \). The mean is \( 1/0.6 = 1.667 \) minutes. The expected waiting time at the ATM is 2 minutes. Thus he should wait inside.