BOUNDS FOR FLUID MODELS DRIVEN BY SEMI-MARKOV INPUTS

N. Gautam, V. G. Kulkarni, Z. Palmowski, and T. Rolski.

Mailing Address for proofs :
N. Gautam
Assistant Professor,
Department of Industrial and Manufacturing Engineering,
The Pennsylvania State University,
207 Hammond Building,
University Park, PA 16802

Lead Author :
N. Gautam
Phone : 814-865-1239
Fax : 814-863-4745

Abstract

We consider an infinite buffer fluid model whose input is driven by independent semi-Markov processes. The output capacity of the buffer is a constant. We derive upper and lower bounds for the limiting distribution of the stationary buffer content process. We discuss examples and applications in telecommunication networks.

Short Title : Bounds for fluid models with semi-Markov inputs

BOUNDS FOR FLUID MODELS DRIVEN BY SEMI-MARKOV INPUTS

N. Gautam [1], V.G. Kulkarni [2,4],

Z. Palmowski [3,5] and T. Rolski [3,6]

Abstract

In this paper we consider an infinite buffer fluid model whose input is driven by independent semi-Markov processes. The output capacity of the buffer is a constant. We derive upper and lower bounds for the limiting distribution of the stationary buffer content process. We discuss examples and applications where the results can be used to determine bounds on the loss probability in telecommunication networks.

Keywords: queueing fluid model, exponential bound, semi-Markov flow.

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[1]Dept. of Industrial Engineering, Penn State Univ, University Park, PA 16802
[2]Dept. of Operations Research, University of North Carolina, Chapel Hill, NC 27599
[3]Mathematical Institute, Wrocław University, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
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1 Introduction

In high-speed telecommunication networks (mainly asynchronous transfer mode (ATM)), the optimal design and admission control problems frequently require computing a certain Quality of Service (QoS) or Grade of Service that the network users need to be assured. This QoS is based on loss probability, delay, delay-jitter, etc. We mainly focus on the loss probability aspect where loss occurs whenever a buffer overflows. Therefore it becomes important to study the buffer content processes.

The high speed (e.g. 155-622 Mbits/sec) of the ATM network implies that it can transmit millions of cells (53-byte ATM packets) per second. This makes fluid-flow models useful in describing the flow of cells. We analyze the packetized traffic by approximating it by fluids, following the large literature using fluid-flow models for communication systems (see Anick et al [1], Elwalid and Mitra [9], etc). Chen and Yao [5] and [6], Ott and Shanthikumar [28], Harrison [16], Chen and Mandelbaum [4], etc, demonstrate how to convert any discrete arrival system into a fluid-flow system and apply the fluid-model results. Therefore our results can be applied to a wide variety of networks, not just high-speed ATM networks.

The most popular method to analyze the buffer content process is using the effective bandwidth approximation. The effective-bandwidth methodology, although simple to use, is based on an exponential approximation to the tail of the distribution of the buffer content in steady state. This approximation holds when the buffer sizes are very large, and the tail probabilities are small. Several researchers have attempted to redress these shortcomings. For example, Elwalid et al [11] and [12] modify the effective-bandwidth methodology and develop the Chernoff Dominant Eigenvalue (CDE) approximation for single-class traffic. To avoid approximations, other approaches have been developed. They include deriving upper and lower bounds for the tail of buffer content process in steady state with a Markov additive input by discretizing time and using extensions of Kingman's exponential bounds for waiting times in the stationary regime in a $G/G/1$ queue (see Kingman [18], Ross [31], Artiges and Nain [2], and Liu et al [24]). Artiges and Nain [2] obtain exponential bounds for multiplexing multiclass Markovian on-off sources, where the upper bounds are similar to those in Palmowski and Rolski [29].

Liu et al [24] obtain exponential bounds for a large class of single resource systems fed by multiplexing Markov Arrival Processes in discrete time. In this paper using the exponential change of measure, we generalize the results in Liu et al [24] to the continuous time case and a more general input process. Obviously, the parameters in the exponents of the lower and upper bounds obtained in both papers using a Markov additive input model are the same. The technique of exponential change of measure are presented in Palmowski and Rolski [29], who develop bounds for the distribution of the buffer-content process whose input traffic is modulated by a Continuous Time Markov Chain (CTMC). In this paper we generalize their results to the semi-Markov modulated traffic.

The paper is organized as follows. In Section 2 we describe the model of a single buffer with traffic from K independent sources modulated by semi-Markov processes. In Section 3 we explain the notations used for semi-Markov processes for the case when there is only a single source ($K = 1$). In Section 4 we derive bounds for the buffer content distribution for the single source ($K = 1$) case. In Section 5 we generalize the results in Section 4 to K sources. In Section 6 we demonstrate how to compute the bounds and in Section 7 we illustrate the results using several examples.

2 Single Buffer Fluid Model

Consider a single buffer that admits single-class traffic from K independent sources, and the k^{th} source driven by a random environment process $\{Z^k(t), t \geq 0\}$, $k = 1, 2, \ldots, K$ (see Figure 1). Note that $Z^k(t)$ can be thought of as the state of the k^{th} input source at time t. When source k is in state $Z^k(t)$, it generates fluid at rate $r^k_{Z^k(t)}$ into the buffer. Let $X(t)$ be the amount of fluid in the buffer at time

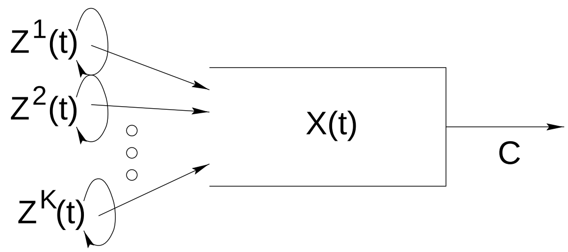

Figure 1: Single Buffer Fluid Model

t. The buffer has infinite capacity and is serviced by a channel of constant capacity c. The dynamics of the buffer-content process $\{X(t), t \geq 0\}$ is described by

$$\frac{dX(t)}{dt} = \begin{cases} \sum_{k=1}^{K} r^k_{Z^k(t)} - c & \text{if } X(t) > 0 \\ \{\sum_{k=1}^{K} r^k_{Z^k(t)} - c\}^+ & \text{if } X(t) = 0. \end{cases} \tag{1}$$

where $\{x\}^+ = \max(x, 0)$. The solution is given by (see Kulkarni and Rolski [23])

$$X(t) = \sup_{0 \leq u \leq t} \left(Y(t), \int_u^t \left(\sum_{k=1}^{K} r^k_{Z^k(s)} - c \right) ds \right),$$

where

$$Y(t) = X(0) + \int_0^t \left(\sum_{k=1}^{K} r^k_{Z^k(s)} - c \right) ds.$$

It has been shown in Kulkarni and Rolski [23] that the buffer-content process $\{X(t), t \geq 0\}$ is stable if

$$\sum_{k=1}^{K} E\{r^k_{Z^k(\infty)}\} < c, \tag{2}$$

in which case $X(t) \to X$ in distribution with

$$X = \sup_{u \leq 0} \int_u^0 \left(\sum_{k=1}^{K} r^k_{Z^k(s)} - c \right) ds. \tag{3}$$

One of the main aims of this paper is to analyze the limiting behavior of the buffer-content process $\{X(t), t \geq 0\}$ namely,

$$P\{X > x\} = \lim_{t \to \infty} P\{X(t) > x\}. \tag{4}$$

We obtain upper and lower bounds for this limiting distribution $P\{X > x\}$. We first consider a single source ($K = 1$) and then extend the analysis to multiple sources.

3 Single Source Model : Notation

Consider the fluid model in Section 2 (see Figure 1) with a single source ($K = 1$) modulated by a Semi-Markov Process (SMP) $\{Z(t), t \geq 0\}$ on state space $\{1, 2, ..., \ell\}$. Fluid is generated at rate r_i at time t when the SMP is in state $Z(t) = i$. We first introduce the relevant notation for the SMP $\{Z(t), t \geq 0\}$.

Let S_n denote the time of the nth jump epoch in the SMP with $S_0 = 0$. Define Z_n as the state of the SMP immediately after the nth jump, i.e.,

$$Z_n = Z(S_n+).$$

Let

$$G_{ij}(x) = P\{S_1 \le x; Z_1 = j | Z_0 = i\}. \tag{5}$$

The kernel of the SMP is

$$G(x) = [G_{ij}(x)]_{i,j=1,\ldots,\ell}\ .$$

Note that $\{Z_n, n \ge 0\}$ is a discrete time Markov chain (DTMC) with transition probability matrix

$$P = G(\infty).$$

We assume that this DTMC is irreducible and recurrent. Let

$$G_i(x) = P\{S_1 \le x | Z_0 = i\} = \sum_{j=1}^{\ell} G_{ij}(x)$$

and the expected time the SMP spends in state i be

$$\tau_i = E(S_1 | Z_0 = i).$$

Let

$$\pi_i = \lim_{n\to\infty} P\{Z_n = i\}$$

be the stationary distribution of the DTMC $\{Z_n, n \ge 0\}$. It is given by the unique non-negative solution to

$$\pi = \pi P \text{ and } \sum_i \pi_i = 1. \tag{6}$$

The stationary distribution of the SMP is given by

$$p_i = \lim_{t\to\infty} P\{Z(t) = i\} = \frac{\pi_i \tau_i}{\sum_{m=1}^{\ell} \pi_m \tau_m}. \tag{7}$$

Since

$$E\{r_{Z(\infty)}\} = \sum_{m=1}^{\ell} p_m r_m,$$

for stability we assume that (see Equation (2))

$$\sum_{m=1}^{\ell} p_m r_m < c. \tag{8}$$

We also assume that

$$c < \max_{1\le m\le \ell} r_m, \tag{9}$$

otherwise the buffer will always be empty in steady-state.

Let $L(t)$, be the remaining sojourn time of the SMP in the current state at time t. It is known that (see Kulkarni [19])

$$\lim_{t\to\infty} P\{Z(t) = m, L(t) > x\} = p_m \int_x^{\infty} \frac{1 - G_m(s)}{\tau_m}\, ds.$$

$\{L(t), t \ge 0\}$ is called the supplementary age process of $\{Z(t), t \ge 0\}$.

4 Single Source Model : Results

The following theorem gives the bounds on the limiting distribution of $X(t)$. First we need the following definitions. Let

$$\phi_{ij}(\delta) = E(e^{\delta(r_i-c)S_1}; Z_1 = j | Z_0 = i), \tag{10}$$

and

$$\Phi(\delta) = [\phi_{ij}(\delta)]_{i,j\in 1,\ldots,\ell} \ . \tag{11}$$

Note that $\Phi(\delta)$ is a matrix with non-negative elements. Let $e(\Phi(\delta))$ be the Perron-Frobenius eigenvalue of $\Phi(\delta)$. Let η be the smallest real-positive value satisfying

$$e(\Phi(\eta)) = 1. \tag{12}$$

Let $h = [h_1, h_2, \ldots, h_\ell]$ be the corresponding left eigenvector, i.e.,

$$h = h\ \Phi(\eta). \tag{13}$$

Define

$$H = \sum_{i=1}^{\ell} \frac{h_i}{\eta(r_i - c)} \left(\sum_{j=1}^{\ell} (\phi_{ij}(\eta)) - 1 \right), \tag{14}$$

$$\Psi_{max}(i,j) = \sup_x \left\{ \frac{h_i e^{-\eta(r_i-c)x} \int_x^\infty e^{\eta(r_i-c)y} dG_{ij}(y)}{\frac{p_i}{\tau_i} \int_x^\infty dG_{ij}(y)} \right\} \tag{15}$$

and

$$\Psi_{min}(i,j) = \inf_x \left\{ \frac{h_i e^{-\eta(r_i-c)x} \int_x^\infty e^{\eta(r_i-c)y} dG_{ij}(y)}{\frac{p_i}{\tau_i} \int_x^\infty dG_{ij}(y)} \right\}. \tag{16}$$

Theorem 1 *Suppose (8) and (9) hold and there exists solution of the equation (12) (which is unique) and*

$$\Psi_{min}(i,j) > 0 \quad \textit{and} \quad \Psi_{max}(i,j) < \infty \qquad \textit{for all } i,j \in \{1,\ldots,\ell\}\ . \tag{17}$$

Then $X(t) \to X$ in distribution and

$$C_* e^{-\eta x} \le P\{X > x\} \le C^* e^{-\eta x}, \textit{ for all } x \ge 0 \quad , \tag{18}$$

where η is from Equation (12) and

$$\begin{aligned} C^* &= \frac{H}{\min_{i:r_i>c,j:p_{ij}>0} \{\Psi_{min}(i,j)\}} \\ C_* &= \frac{H}{\max_{i:r_i>c,j:p_{ij}>0} \{\Psi_{max}(i,j)\}} \ . \end{aligned}$$

We now describe the main steps in the proof of this theorem below. Each step is discussed in detail in the following subsections.

1. The reversed SMP
   From Equation (3) we have

$$\begin{aligned} P\{X > x\} &= P\{\sup_{u \le 0} \int_u^0 (r_{Z(s)} - c) ds > x\} \\ &= P\{\sup_{t \ge 0} \int_0^t (r_{Z(-s)} - c) ds > x\}. \end{aligned}$$

Now define

$$Y(s) = Z(-s).$$

Then $\{Y(s), -\infty < s < \infty\}$ is the time reversed version of $\{Z(s), -\infty < s < \infty\}$. Therefore

$$P\{X > x\} = P\{\sup_{t \geq 0} \int_0^t (r_{Y(s)} - c) ds > x\}.$$

Thus we need to study the reversed process $\{Y(t), -\infty < t < \infty\}$. In general $\{Y(t), -\infty < t < \infty\}$ is not an SMP. A necessary and sufficient condition for it to be an SMP is given in in Subsection 4.1.

2. Markov process and its generator
Even if $\{Y(t), -\infty < t < \infty\}$ is an SMP, it is not a Markov process in general. Hence we construct a canonical Markov process $\{w(t), t \geq 0\}$ defined by $w(t) = (Y(t), S(t))$, where $S(t)$ is the supplementary age process of $\{Y(t), -\infty < t < \infty\}$ (as defined in Section 3). Let Q be its generator. We shall show in Subsection 4.2 that

$$[Q(x)]_{ij} = \begin{cases} \frac{\partial}{\partial x} - \rho_i(x) + \rho_i(x) a_{ii} \mathbf{D} & i = j, \\ \rho_i(x) a_{ij} \mathbf{D} & i \neq j. \end{cases}$$

Hence the process defined by

$$M(t) = \frac{v(w(t))}{v(w(0))} e^{-\int_0^t \frac{Qv}{v}(w(s)) ds}$$

is a martingale (see Ethier and Kurtz [13]), for any function $v(\cdot, \cdot) \in \mathcal{D}(Q)$ (the domain of the generator Q) with strictly positive infimum.

3. Choice of $v(\cdot, \cdot)$
We next show (see Subsection 4.3) that it is possible to choose a $v(\cdot, \cdot) \in \mathcal{D}(Q)$ such that

$$M(t) = \frac{v(w(t))}{v(w(0))} e^{-\int_0^t \frac{Qv}{v}(w(s)) ds} = \frac{v(w(t))}{v(w(0))} e^{\zeta \int_0^t (r_{Y(s)} - c) ds},$$

for some positive real number ζ. Note that $\{M(t), t \geq 0\}$ is a mean one martingale with respect to the natural filtration of $\{w(t), t \geq 0\}$ process, i.e., $E\{M(t)\} = 1$ for all t.

4. Martingale $M(t)$
Let

$$\tau(x) = \inf\{t > 0 : \int_0^t (r_{Y(s)} - c)\, ds > x\}. \tag{19}$$

From the above definition, we have,

$$\left\{ \sup_{t \geq 0} \int_0^t (r_{Y(s)} - c) ds > x \right\} \equiv \{\tau(x) < \infty\}.$$

Hence,

$$P\{\sup_{t \geq 0} \int_0^t (r_{Y(s)} - c) ds > x\} = P\{\tau(x) < \infty\}.$$

Now define a modified measure $\tilde{P}_t$ on $\mathcal{F}_t = \sigma\{w(u) : 0 \leq u \leq t\}$ (see Subsection 4.4) by

$$\frac{d\tilde{P}_t}{dP_t} = M(t).$$

Then,

$$\begin{aligned} P\{\tau(x) < \infty\} &= \int_{\tau(x)<\infty} dP \\ &= \int_{\tau(x)<\infty} [M(\tau(x))]^{-1} d\tilde{P}. \end{aligned}$$

From Lemma 4

$$\tilde{P}(\tau(x) < \infty) = 1 \ . \tag{20}$$

Therefore

$$\begin{aligned} P\{\tau(x) < \infty\} &= \int_{\tau(x)<\infty} \frac{v(w(0))}{v(w(\tau(x)))} e^{-\zeta x} d\tilde{P} \\ &= e^{-\zeta x} \tilde{E}\left\{\frac{v(w(0))}{v(w(\tau(x)))}; \tau(x) < \infty\right\} && (21) \\ &= e^{-\zeta x} \sum_{j=1}^{\ell} \int_0^\infty \tilde{E}_{(j,y)} \left\{\frac{v(j,y)}{v(w(\tau(x)))}\right\} \pi_j(y) dy, && (22) \end{aligned}$$

where $\tilde{E}$ is the expectation with respect to $\tilde{P}$ and we use (20) to justify equality between (21) and (22).

5. The bounds
Clearly, at time $\tau(x)$, $w(\tau(x))$ can only be in a state (i, s) such that $r_i > c$. Hence the lower bound on $\tilde{E}_{(j,y)} \left\{\frac{1}{v(w(\tau(x)))}\right\}$ is $1/\{\max_{i:r_i>c} \sup_x v(i, x)\}$ and the upper bound on $\tilde{E}_{(j,y)} \left\{\frac{1}{v(w(\tau(x)))}\right\}$ is $1/\{\min_{i:r_i>c} \inf_x v(i, x)\}$. These yield the bounds (see Subsection 4.5 for the expressions) on $P\{X > x\}$ in terms of the parameters of the reversed process $Y(t)$.

6. Original process
We convert the bounds on $P\{X > x\}$ in terms of that of the $\{Z(t), -\infty < t < \infty\}$ process. This is done in Subsection 4.6.

4.1 Time-reversed Semi-Markov Processes

As mentioned in step 1 in Section 4, we shall use reversed SMPs in the proof of Theorem 1. Hence, we collect the relevant results about reversed SMPs here.
Definition : An SMP with kernel $G(\cdot)$ is called a *non-anticipative SMP* if

$$[G_{ij}(x)] = [G_i(x)p_{ij}].$$

We call such an SMP "non-anticipative" since the sojourn time in the current state does not depend upon the following state. In other words, given the current state, the sojourn time and the next state, are independent of each other (a property that is exhibited by CTMCs). In case of a general SMP, the sojourn time depends on both the current state and the following state.

We first restate a theorem proved in Chari [3].

Theorem 2 *Let $\{Z(t), -\infty < t < \infty\}$ be a semi-Markov process with kernel $G(x) = [G_{ij}(x)]$. Then the time-reversed process $\{Y(t) = Z(-t), -\infty < t < \infty\}$ is an SMP if and only if $\{Z(t), -\infty < t < \infty\}$ is a non-anticipative SMP. If $\{Z(t), -\infty < t < \infty\}$ is a non-anticipative SMP, then the reversed SMP is also a non-anticipative SMP with kernel*

$$F(x) = [F_{ij}(x)] = [F_i(x)a_{ij}], \tag{23}$$

where

$$F_i(x) = G_i(x), \tag{24}$$

$$A = [a_{ij}] = \left[p_{ji}\frac{\pi_j}{\pi_i}\right] \tag{25}$$

and π *is given by Equation (6).*

Note that the stationary distribution of the reversed SMP is the same as that of the original SMP.

We will use the following notations. Let S_n denote the time of the nth jump epoch in the SMP $\{Y(t), t \geq 0\}$, with $S_0 = 0$. Define Y_n as the state of the SMP immediately after the nth jump, i.e.,

$$Y_n = Y(S_n+).$$

Define

$$\chi(\delta) = [\chi_{ij}(\delta)] = [E(e^{\delta(r_i - c)S_1}; Y_1 = j | Y_0 = i)]. \tag{26}$$

Note that $\chi(\delta)$ is a matrix with non-negative elements. Let $e(\chi(\delta))$ be the Perron-Frobenius eigenvalue of $\chi(\delta)$. Let ζ be the smallest real positive value satisfying

$$e(\chi(\zeta)) = 1. \tag{27}$$

Let

$$, (x) = [, _{ij}(x)] = [E(e^{\zeta(r_i - c)(S_1 - x)}; Y_1 = j | Y_0 = i, S_1 > x)]. \tag{28}$$

4.2 Reversed Markov Process and Its Generator

In this Subsection we explain in detail the results shown in step 2 of Section 4. We assume that $\{Z(t), t \geq 0\}$ is a non-anticipative SMP. We consider its reversed (non-anticipative) semi-Markov process $\{Y(t), t \geq 0\}$ with kernel $F(x)$, together with its supplementary age process $\{S(t)\}$. The process $\{w(t), t \geq 0\}$ defined by

$$w(t) = (Y(t), S(t)) \tag{29}$$

is Markovian and moreover it is a piecewise-deterministic Markov process according to the terminology of Davis [7, 8].

The generator

$$Q = [q_{ij}]_{i,j=1,\ldots,\ell} \tag{30}$$

and its domain $\mathcal{D}(Q)$ are defined as follows. We use column vector function $f = (f_1, \ldots, f_\ell)^T$, where $f_i : R_+ \to R$ are measurable functions, and adopt a notation that $f(i, x) = f_i(x)$ (we use them interchangeably). Consider the Markov process $w(t)$ on the probability space $(\Omega, \mathcal{F}, P_{(i,y)})$, where $P_{(i,y)}$ is the underlying probability measure for which $Y(0) = i$, $S(0) = y$. For measurable functions $f, f^* : R_+ \to R^\ell$ we denote

$$M_{f,f^*}(w(t)) = f_{Y(t)}(S(t)) - f_i(y) - \int_0^t f^*(Y(s), S(s))\, ds\ , \qquad t \geq 0\ .$$

We look for all measurable vector functions f, f^* such that $\{M_{f,f^*}(w(t)), t \geq 0\}$ is a $P_{(i,y)}$ martingale for all (i, y). We denote this family of f as $\mathcal{D}(Q)$ and we call the mapping $Q : f \to f^*$ as the (full) generator. The results of Theorem 26.14 from Davis [8] are adapted to the process $w(t)$ as follows: The family $\mathcal{D}(Q)$ consists of all measurable functions $f(x) = (f_1(x), \ldots, f_\ell(x))$, such that $f_i(x)$ is absolutely continuous on $[0, s_i)$, and $E|f(i, T_i)| < \infty$, where $s_i = \inf\{t : F_i(t) = 1\}$ and T_i is a random variable with distribution function $F_i(x)$. Also, $C_{\rm b}(R_+)$ is the set of all continuous and bounded functions $f : R_+ \to R$. Then, the family of functions $f(x) = (f_1(x), \ldots, f_\ell(x)) \in C_{\rm b}^\ell(R_+)$, which are absolutely continuous on $[0, s_i)$ are included in $\mathcal{D}(Q)$. Since condition $E|f(i,T_i)| < \infty$ is computationally very difficult to check we only consider family of functions $C_{\rm b}^{\ell}(R_+)$.

Define the hazard rate function

$$\rho_i(x) = \frac{f_i(x)}{\bar{F}_i(x)},$$

where $f_i(x) = \frac{dF_i(x)}{dx}$, and $\bar{F}_i(x) = 1 - F_i(x)$.

Following Davis [7, 8], the full generator of $\{w(t)\}$ is as follows :

Lemma 1 *The full generator of the process $w(t)$ is $(Qf)(i,x) = (Q(x)f(x))_i$, where*

$$[Q(x)]_{ij} = \begin{cases} \frac{\partial}{\partial x} - \rho_i(x) + \rho_i(x)a_{ii}\mathbf{D} & i = j \\ \rho_i(x)a_{ij}\mathbf{D} & i \neq j \end{cases} \tag{31}$$

where $i,j = 1,\dots,\ell$ and $(\mathbf{D}d)(x) = d(0)$ for a function $d \in \mathcal{C}_{\rm b}(R_+)$. Moreover $\mathcal{D}(Q)$ is its domain.

4.3 The v Function

We now continue with step 3 (of Section 4) and show how to choose $v(x) = (v_1(x),\dots,v_\ell(x)) \in \mathcal{D}(Q)$. Following the idea from Palmowski and Rolski [30], we look for the smallest $\beta < 0$ fulfilling

$$Q(x)v(x) = \beta\Delta v(x) \tag{32}$$

where $\Delta = \mathrm{diag}([r_i - c])$

$$v \in \mathcal{C}_{\rm b}^{\ell}(R_+) \text{ and} \tag{33}$$

$$\inf_{i,x} v_i(x) > 0. \tag{34}$$

The i-th row of Equation (32) is the first order nonhomogeneous differential equation

$$\frac{\partial}{\partial x}v_i(x) + \rho_i(x)\left(\sum_{j=1}^{\ell} a_{ij}v_j(0) - v_i(x)\right) = \beta(r_i - c)v_i(x) \tag{35}$$

whose general solution is:

$$v_i(x) = (\bar{F}_i(x))^{-1}e^{\beta(r_i-c)x}\left[v_i(0) - \int_0^x e^{-\beta(r_i-c)z}f_i(z)\,dz\sum_{j=1}^{\ell} a_{ij}v_j(0)\right].$$

A necessary condition for (34) is

$$v_i(0) - \int_0^\infty e^{-\beta(r_i-c)z}f_i(z)\,dz\sum_{j=1}^{\ell} a_{ij}v_j(0) \geq 0$$

for all $i = 1,\dots\ell$. This is equivalent to $u \geq \chi(-\beta)u$ where $u = (v_1(0),\dots,v_\ell(0))^T$ and $v_i(0) > 0$.

Lemma 2 *The smallest possible $\beta < 0$ fulfilling*

$$Q(x)v(x) = \beta\Delta v(x)$$

is $-\zeta$. Then,

$$v(x) = \,,\, (x)u\;, \tag{36}$$

where $\chi(\zeta)u = u$.

Proof. We look for the smallest β and vector $u > 0$ such that $\chi(-\beta)u \leq u$. If there exist $-\beta_0 > \zeta$ and $u > 0$ fulfilling the above inequality, then multiplying it by the left Perron-Frobenius vector $\nu_{-\beta_0} > 0$, we would have $e(\chi(-\beta_0))\nu_{-\beta_0} u \leq \nu_{-\beta_0} u$ so $e(\chi(-\beta_0)) \leq 1$. But this is a contradiction as $e(\chi(\alpha))$ is a convex function and $e(\chi(\zeta)) = 1$. Thus $-\beta = \zeta$ and $(v_1(0), \ldots, v_\ell(0)) = (u_1, \ldots, u_\ell)$, where $\chi(\zeta)u = u$. In that case:

$$v_i(x) = (\bar{F}_i(x))^{-1} e^{-\zeta(r_i - c)x} \sum_{j=1}^{\ell} a_{ij} u_j \int_x^\infty e^{\zeta(r_i-c)z} f_i(z)\, dz \tag{37}$$

$$= \sum_{j=1}^{\ell} ,\, _{ij}(x) u_j \ , \tag{38}$$

that is $v(x) = \, , (x)u$. ∎

4.4 The Martingale $M(t)$

As mentioned in step 4 of Section 4 we develop the mean one martingale $M(t)$ as follows. By Proposition 3.2 in Ethier and Kurtz [13], we have that

$$M(t) = \frac{v(w(t))}{v(i,x)} e^{-\int_0^t \frac{Qv}{v}(w(t))\, ds} = \frac{v(w(t))}{v(i,x)} e^{\zeta \int_0^t (r_{Y(s)} - c)\, ds}$$

is a martingale on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P_{(i,x)})$, where we may suppose that $\Omega = D[0,\infty)$, $\mathcal{F}_t$ is the history of $w(t)$ up to time t and $\mathcal{F}$ is a smallest σ-field generated by all subsets from $\mathcal{F}_t$.

Following Palmowski and Rolski [30], we define a new probability measure $\tilde{P}_{(i,x)}$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t^w\})$ by

$$\frac{d\tilde{P}_{(i,x),t}}{dP_{(i,x),t}} = M(t)$$

on which a canonical Markov process $\{w(t)\}$ is Markovian with full generator

$$\tilde{Q}g = \frac{Q(vg)}{v} - g\frac{Qv}{v} = \frac{Q(vg)}{v} + \zeta g \Delta.$$

Thus the i-th component is

$$\left[\tilde{Q}g\right](i,x) = \frac{\partial}{\partial x} g_i(x) + \frac{\rho_i(x)}{v_i(x)} \sum_{j=1}^{\ell} a_{ij} u_j \left[\sum_{j=1}^{\ell} \frac{a_{ij} u_j}{\sum_k a_{ik} u_k} g_j(0) - g_i(x) \right]. \tag{39}$$

Lemma 3 *On the new probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \tilde{P}_{(i,x)})$ the process $\{Y(t), -\infty < t < \infty\}$ is again a semi-Markov process specified by $([\tilde{a}_{ij}], [\tilde{\rho}_i(x)], [r_i])$, where*

$$\tilde{a}_{ij} = \frac{a_{ij} u_j}{\sum_{k=1}^{\ell} a_{ik} u_k} = \frac{\chi_{ij}(\zeta) u_j}{u_i} \ , \tag{40}$$

$$\tilde{\rho}_i(x) = \frac{\rho_i(x) \sum_{j=1}^{\ell} a_{ij} u_j}{v_i(x)} = \frac{f_i(x) e^{\zeta(r_i - c)x} \sum_{j=1}^{\ell} a_{ij} u_j}{\sum_{j=1}^{\ell} \int_x^\infty e^{\zeta(r_i - c)z} a_{ij} f_i(z) u_j\, dz} \ , \tag{41}$$

$$\tilde{f}_i(x) = \frac{f_i(x) e^{\zeta(r_i - c)x} \sum_{j=1}^{\ell} a_{ij} u_j}{u_i} \tag{42}$$

and

$$\tilde{\tau}_i = \frac{(\chi'(\zeta)u)_i}{u_i(r_i - c)} \ . \tag{43}$$

Proof. The first parts of Equations (40) and (41) are simple consequences of the definition of generator Q and the function $v_i(x)$. Moreover, the second part of Equation (40) follows from

$$\begin{aligned} \frac{a_{ij}u_j}{\sum_{k=1}^{\ell} a_{ik}u_k} &= \frac{a_{ij}\int_0^\infty e^{\zeta(r_i-c)x} f_i(x)\,dx\,u_j}{\sum_{k=1}^{\ell} a_{ik}\int_0^\infty e^{\zeta(r_i-c)x} f_i(x)\,dx\,u_k} \\ &= \frac{\chi_{ij}(\zeta)u_j}{\sum_{k=1}^{\ell} \chi(\zeta)_{ik}u_k} \\ &= \frac{\chi_{ij}(\zeta)u_j}{u_i} \ . \end{aligned}$$

From Equation (41), we get,

$$\tilde{f}_i(x) = \frac{f_i(x)e^{\zeta(r_i-c)x}\sum_{j=1}^{\ell} a_{ij}u_j}{d_i} \ , \tag{44}$$

where d_i is a normalizing constant. Hence

$$d_i = \int_0^\infty f_i(z)e^{\zeta(r_i-c)z}\sum_{j=1}^{\ell} a_{ij}u_j\,dz = \sum_{j=1}^{\ell}\chi(\zeta)_{ij}u_j = u_i \tag{45}$$

and we obtain Equation (42). Using Equation (42), Equation (43) is fulfilled by

$$\begin{aligned} (r_i - c)\tilde{\tau}_i &= (r_i - c)\int_0^\infty x\tilde{f}_i(x)\,dx \\ &= \frac{\sum_{k=1}^{\ell}\chi'_{ik}(\zeta)u_k}{u_i} \\ &= \frac{(\chi'(\zeta)u)_i}{u_i} \ . \end{aligned}$$

This completes the proof of this Lemma. ■

Let $\tilde{\pi}\tilde{P} = \tilde{\pi}$ be the stationary distribution of the Markov chain $\{Y_n\}$ under the new probability measure. In matrix notation, Equation (40) reads as

$$\tilde{P} = [\tilde{a}_{ij}] = \text{diag}(u^{-1})\chi(\zeta)\text{diag}(u) \ . \tag{46}$$

Therefore from Equation (46)

$$\tilde{\pi} = \tilde{\pi}\text{diag}(u^{-1})\chi(\zeta)\text{diag}(u)$$

and hence,

$$\tilde{\pi} = \nu\,\text{diag}(u) \tag{47}$$

where ν is the left (row) eigenvector of $\chi(\zeta)$ corresponding to eigenvalue 1 such that $\nu u = 1$, i.e.,

$$\nu = \nu\chi(\zeta).$$

The drift $\tilde{d} = \tilde{E}Y(t) - c$ is

$$\tilde{d} = \sum_{i=1}^{\ell} \frac{r_i \tilde{\pi}_i \tilde{\tau}_i}{\sum_{j=1}^{\ell} \tilde{\pi}_j \tilde{\tau}_j} - c.$$

Hence $\tilde{d} > 0$ if and only if

$$\sum_{i=1}^{\ell} \tilde{\pi}_i \tilde{\tau}_i (r_i - c) > 0$$

which is, by Equations (43) and (47), equivalent to

$$\nu \chi'(\zeta) u > 0 \ . \tag{48}$$

Consider the eigenvalue problem

$$\begin{cases} \chi(\alpha) h_\alpha &= \kappa(\alpha) h_\alpha \\ \nu_\alpha \chi(\alpha) &= \kappa(\alpha) \nu_\alpha \\ \nu_\alpha h_\alpha = 1 & \\ \alpha \geq 0 & \end{cases} \tag{49}$$

Function $\kappa(\alpha)$ is convex (by Kingman-Miller theorem; see Miller [25] and the beautiful proof of this theorem by Kingman [17]).

Lemma 4 $\tilde{d} > 0$

Proof. Differentiating the first line in (49) we obtain

$$\chi'(\alpha) h_\alpha + \chi(\alpha) h_\alpha^{'} = \kappa'(\alpha) h_\alpha + \kappa(\alpha) h_\alpha^{'}.$$

Multiplying both the sides from the right by ν_α and rearranging we arrive at

$$\nu_\alpha \chi'(\alpha) h_\alpha = \kappa'(\alpha) + (\kappa(\alpha) - 1) \nu_\alpha h_\alpha^{'}. \tag{50}$$

Since $\chi(0) = P$, we have,

$$h_0 = Pe = e, \qquad \nu_0 = \pi P = \pi, \qquad \kappa(0) = 1, \tag{51}$$

where $e = [1, \ldots, 1]^T$ is the (column) vector of one's. The stability condition $d = \sum_i p_i r_i - c < 0$ and Equation (50) yield

$$\begin{aligned} \kappa'(0) &= \pi \chi'(0) e \\ &= \sum_{i=1}^{\ell} a_i \sum_{j=1}^{\ell} \frac{\partial}{\partial \alpha} \chi_{ij}(\alpha(r_i - c)) \mid_{\alpha=0} \\ &= \sum_{i=1}^{\ell} a_i \tau_i (r_i - c) < 0. \end{aligned}$$

Since $\kappa(0) = 1$, $\kappa'(0) < 0$ and $\kappa(\alpha)$ is a convex function, there is $\kappa'(\zeta) > 0$ for $\kappa(\zeta) = 1$. Substituting $\alpha = \zeta$ in (50) and bearing in mind the Inequality (48), $\tilde{d} > 0$ is equivalent to

$$\nu \chi'(\zeta) u = \kappa'(\zeta) > 0$$

The proof is completed. ■

4.5 The Bounds For The Reversed SMP

Continuing with the analysis in step 5 (of Section 4), consider a reversed non-anticipative SMP $\{Y(t), \infty < t < \infty\}$ with kernel $F(x) = [F_{ij}(x)] = [F_i(x)a_{ij}]$. Define

$$\pi_m(x) = a_m \frac{1 - F_m(x)}{\tau_m},$$

where a_m is the stationary probability of the reversed SMP $\{Y(t), -\infty < t < \infty\}$. Let

$$b_m = \frac{a_m}{\tau_m \zeta(r_m - c)}. \tag{52}$$

Theorem 3 *If conditions (33) and (34) are fulfilled, then the steady-state distribution of the buffer-content process is bounded as*

$$C_* e^{-\zeta x} \le P\{X > x\} \le C^* e^{-\zeta x}, \tag{53}$$

where

$$\begin{aligned} C^* &= \frac{b(I - A)u}{\min\limits_{m: r_m > c} \inf\limits_{x \ge 0} v_m(x)} \\ C_* &= \frac{b(I - A)u}{\max\limits_{m: r_m > c} \sup\limits_{x \ge 0} v_m(x)} \end{aligned} .$$

Proof. Let $\tau(x)$ be as in Equation (19). Continuing from Equation (22), we have,

$$P\{\tau(x) < \infty\} = e^{-\zeta x} \sum_{j=1}^{\ell} \int_0^\infty \tilde{E}_{(j,y)} \left\{ \frac{v_j(y)}{v(w(\tau(x)))} \right\} \pi_j(y)\, dy$$

which can be bounded above by

$$\le e^{-\zeta x} \frac{1}{\min\limits_{m: r_m > c} \inf\limits_{x \ge 0} v_m(x)} \sum_{j=1}^{\ell} \int_0^\infty v_j(y) \pi_j(y)\, dy. \tag{54}$$

However

$$\begin{aligned} \sum_{j=1}^{\ell} \int_0^\infty v_j(y) \pi_j(y)\, dy &= \sum_{j=1}^{\ell} \frac{a_j}{\tau_j} \int_0^\infty v_j(y)\, \bar{F}_j(y)\, dy \\ &= \sum_{j=1}^{\ell} \frac{a_j}{\tau_j} \int_0^\infty \sum_{k=1}^{\ell} a_{jk} u_k \int_y^\infty e^{\zeta(r_j - c)z} f_j(z)\, dz e^{-\zeta(r_j - c)y}\, dy \end{aligned}$$

and integrating by parts,

$$\begin{aligned} &= \sum_{j=1}^{\ell} \left[\frac{a_j}{\tau_j \zeta(r_j - c)} \sum_{k=1}^{\ell} a_{jk} u_k \left[\tilde{F}_j(-\zeta(r_j - c)) - 1 \right] \right] \\ &= \sum_{j=1}^{\ell} b_j \sum_{k=1}^{\ell} a_{jk} u_k \tilde{F}_j(-\zeta(r_j - c)) - \sum_{j=1}^{\ell} b_j \sum_{k=1}^{\ell} a_{jk} u_k \\ &= \sum_{j=1}^{\ell} b_j (\chi(\zeta) u)_j - \sum_{j=1}^{\ell} b_j (Au)_j = b(I - A)u, \end{aligned}$$

where $\tilde{F}(\alpha)$ is a Laplace transform. Thus by Equation (54) the proof of the upper bound in Equation (53) is completed. In a similar way we obtain the lower bounds in Equation (53). ∎

4.6 Bounds for Non-anticipative SMP

In the previous subsection we derived bounds for the buffer-content process in terms of the parameters of the reversed process $\{Y(t), -\infty < t < \infty\}$, as illustrated in Theorem 3. In this subsection, following step 6 of Section 4, we convert the bounds in Theorem 3 in terms of the original (non-anticipative) SMP $\{Z(t), -\infty < t < \infty\}$. We assume that $\{Z(t), -\infty < t < \infty\}$ is an ℓ-state non-anticipative semi-Markov process with kernel $[G_{ij}(x)] = [G_i(x)p_{ij}]$. All other parameters of the SMP (viz. π_j, p_j, τ_j, $\phi_{ij}(\delta)$, η, h_j) are as described in Sections 3 and 4.

From Equations (25) and (24) in Theorem 2 we have

$$a_{ij} = p_{ji}\frac{\pi_j}{\pi_i} \tag{55}$$

and

$$F_i(x) = G_i(x).$$

From the definition of $\chi_{ij}(\delta)$ in Equation (26) and Equation (10), we obtain

$$\begin{aligned}\chi_{ij}(\delta) &= \tilde{F}_{ij}(-\delta(r_i - c)) \\ &= a_{ij}\tilde{F}_i(-\delta(r_i - c)) \\ &= \tilde{G}_i(-\delta(r_i - c))p_{ji}\frac{\pi_j}{\pi_i},\end{aligned} \tag{56}$$

and

$$\begin{aligned}\phi_{ij}(\delta) &= \tilde{G}_{ij}(-\delta(r_i - c)) \\ &= \tilde{G}_i(-\delta(r_i - c))p_{ij}.\end{aligned} \tag{57}$$

Lemma 5 *For a given a real positive number δ, the matrices $\chi(\delta)$ and $\Phi(\delta)$ have identical eigenvalues. Therefore*

$$e(\Phi(\delta)) = e(\chi(\delta))$$

and $\eta = \zeta$.

Proof. Let $\mathcal{G}$ and Δ be diagonal matrices such that

$$\mathcal{G} = \text{diag}\{\tilde{G}_1(-\delta(r_1 - c)), \tilde{G}_2(-\delta(r_2 - c)), \ldots, \tilde{G}_\ell(-\delta(r_\ell - c))\}$$

and

$$\Delta = \text{diag}\{\pi_1, \pi_2, \ldots, \pi_\ell\}.$$

Clearly,

$$\Phi(\delta) = \mathcal{G}P \text{ and}$$

$$\chi(\delta) = \mathcal{G}\Delta^{-1}P'\Delta.$$

Let λ be an eigenvalue of $\Phi(\delta)$. Therefore if $|D|$ denotes the determinant of the square matrix D and if D' denotes the transpose of a matrix D, we have

$$\begin{aligned}& |\mathcal{G}P - \lambda I| = 0, \\ \Rightarrow\ & |\mathcal{G}|\ |P - \lambda\mathcal{G}^{-1}| = 0, \\ \Rightarrow\ & |\mathcal{G}|\ |P' - \lambda\mathcal{G}^{-1}| = 0, \\ \Rightarrow\ & |\mathcal{G}|\ |\Delta^{-1}|\ |\Delta|\ |\Delta^{-1}P'\Delta - \lambda\mathcal{G}^{-1}| = 0, \\ \Rightarrow\ & |\mathcal{G}\Delta^{-1}P'\Delta - \lambda I| = 0.\end{aligned}$$

Thus λ is also an eigenvalue of $\chi(\delta)$. For a given a real positive number δ, the matrices $\chi(\delta)$ and $\Phi(\delta)$ have identical eigenvalues. Specifically, the largest real positive eigenvalues of $\chi(\delta)$ and $\Phi(\delta)$ are identical. Also, the smallest δ that satisfies the largest real positive eigenvalues of $\chi(\delta)$ and $\Phi(\delta)$ to be identical and equal to one, is unique. Therefore

$$e(\Phi(\delta)) = e(\chi(\delta))$$

and $\eta = \zeta$. ■

Since $\eta = \zeta$, the eigenvectors h and u are given by

$$h\Phi(\eta) = h \qquad \text{and} \qquad \chi(\eta)u = u.$$

Lemma 6 *The eigenvector u, the vector $v(x)$ and the matrix , (x) are related to the corresponding terms of the original non-anticipative SMP $\{Z(t), -\infty < t < \infty\}$ as given below :*

$$u_j = \frac{\tilde{G}_j(-\eta(r_j - c))}{\pi_j} h_j, \tag{58}$$

$$,\ _{ij}(x) = \frac{\int_x^\infty e^{\eta(r_i - c)y} dG_i(y)}{\int_x^\infty e^{\eta(r_i - c)x} dG_i(y)} p_{ji} \frac{\pi_j}{\pi_i}, \tag{59}$$

and

$$v_i(x) = \frac{\int_x^\infty e^{\eta(r_i - c)y} dG_i(y)}{\int_x^\infty e^{\eta(r_i - c)x} dG_i(y)} \frac{h_i}{\pi_i}. \tag{60}$$

Proof. Since $\chi(\eta)u = u$, from Equation (56) for all i, we get

$$\begin{aligned} u_i &= \sum_{j=1}^{\ell} \chi_{ij}(\eta) u_j \\ &= \sum_{j=1}^{\ell} \tilde{G}_i(-\eta(r_i - c)) p_{ji} \frac{\pi_j}{\pi_i} u_j. \end{aligned} \tag{61}$$

Note that

$$u_j = \frac{\tilde{G}_j(-\eta(r_j - c))}{\pi_j} h_j$$

satisfies Equation (61) since

$$\begin{aligned} \sum_{j=1}^{\ell} \tilde{G}_i(-\eta(r_i - c)) p_{ji} \frac{\pi_j}{\pi_i} u_j &= \frac{\tilde{G}_i(-\eta(r_i - c))}{\pi_i} \sum_{j=1}^{\ell} p_{ji} \pi_j u_j, \\ &= \frac{\tilde{G}_i(-\eta(r_i - c))}{\pi_i} \sum_{j=1}^{\ell} \tilde{G}_j(-\eta(r_j - c)) p_{ji} h_j \\ &= \frac{\tilde{G}_i(-\eta(r_i - c))}{\pi_i} \sum_{j=1}^{\ell} \phi_{ji}(\eta) h_j \\ &= \frac{\tilde{G}_i(-\delta(r_i - c))}{\pi_i} h_i, \\ &= u_i \end{aligned}$$

using Equation (57) and the relation $h\Phi(\eta) = h$. From Equation (28),

$$
\begin{aligned}
, _{ij}(x) &= \frac{\int_x^\infty e^{\zeta(r_i-c)y} dF_i(y)}{\int_x^\infty e^{\zeta(r_i-c)x} dF_i(y)} a_{ij} \\
&= \frac{\int_x^\infty e^{\eta(r_i-c)y} dG_i(y)}{\int_x^\infty e^{\eta(r_i-c)x} dG_i(y)} p_{ji} \frac{\pi_j}{\pi_i}. \qquad (62)
\end{aligned}
$$

Note that from Equation (36),

$$
v(x) = \, , (x) \, u. \qquad (63)
$$

Using Equation (62) and

$$
h_i = \sum_{j=1}^{\ell} \tilde{G}_j(-\eta(r_j - c)) p_{ji} h_j,
$$

we have

$$
\begin{aligned}
v_i(x) &= \sum_{j=1}^{\ell} \frac{\int_x^\infty e^{\eta(r_i-c)y} dG_i(y)}{\int_x^\infty e^{\eta(r_i-c)x} dG_i(y)} p_{ji} \tilde{G}_j(-\eta(r_j - c)) \frac{h_j}{\pi_i} \\
&= \frac{\int_x^\infty e^{\eta(r_i-c)y} dG_i(y)}{\int_x^\infty e^{\eta(r_i-c)x} dG_i(y)} \frac{h_i}{\pi_i}.
\end{aligned}
$$

■

From the definition of b_i in Equation (52), it follows that

$$
b_i = \frac{p_i}{\tau_i \eta(r_i - c)}. \qquad (64)
$$

The following theorem describes the bounds of Theorem 3 in terms of the original non-anticipative SMP $\{Z(t), -\infty < t < \infty\}$.

Theorem 4 *The steady-state distribution of the buffer-content process is bounded as*

$$
C_* e^{-\eta x} \le P(X > x) \le C^* e^{-\eta x},
$$

where

$$
C^* = \frac{\sum_{i=1}^{\ell} \frac{h_i}{\eta(r_i - c)} (\tilde{G}_i(-\eta(r_i - c)) - 1)}{\min_{i: r_i > c} \inf_x \left\{ \frac{h_i \tau_i}{p_i} \frac{\int_x^\infty e^{\eta(r_i-c)y} dG_i(y)}{\int_x^\infty e^{\eta(r_i-c)x} dG_i(y)} \right\}} \qquad (65)
$$

and

$$
C_* = \frac{\sum_{i=1}^{\ell} \frac{h_i}{\eta(r_i - c)} (\tilde{G}_i(-\eta(r_i - c)) - 1)}{\max_{i: r_i > c} \sup_x \left\{ \frac{h_i \tau_i}{p_i} \frac{\int_x^\infty e^{\eta(r_i-c)y} dG_i(y)}{\int_x^\infty e^{\eta(r_i-c)x} dG_i(y)} \right\}} \qquad (66)
$$

provided that

$$
C_* > 0 \quad \textit{and} \quad C^* < \infty \,. \qquad (67)
$$

Proof. Using the time reversed process $\{Y(t), -\infty < t < \infty\}$, we get from Theorem 3

$$C^* = \frac{b(I-A)u}{\min_{i:r_i>c} \inf_x v_i(x)} \tag{68}$$

$$C_* = \frac{b(I-A)u}{\max_{i:r_i>c} \sup_x v_i(x)}. \tag{69}$$

Now we rewrite b, A, u and $v_i(x)$ in Equations (68) and (69) in terms of the parameters of the $\{Z(t), -\infty < t < \infty\}$ process using Equations (64), (55), (58) and (60). After some algebra, we obtain equations (65) and (66). Let us note that conditions (33) and (34) are equivalent assumptions that

$$C_* > 0 \quad \text{and} \quad C^* < \infty \ . \tag{70}$$

Hence the proof is complete. ■

4.7 General SMP Proof

Theorem 4 of the previous subsection holds only for the special case when the SMP driving the input to a buffer can be modeled as a non-anticipative SMP. There are several applications where the non-anticipative SMP models fail. In this subsection, we prove Theorem 1 (which is stated for a general SMP) using Theorem 4 for the non-anticipative SMP. We first state and prove a theorem that explains how to convert a general SMP into a non-anticipative SMP.

Let $\{Z(t), t \geq 0\}$ be an ℓ-state (not necessarily non-anticipative) SMP with kernel $[G_{ij}(x)]$, transition probabilities $p_{ij} = G_{ij}(\infty)$, and stationary probabilities π_j (from $\pi = \pi G(\infty)$). Also, S_n and Y_n denote respectively the nth transition epoch and the state of the SMP immediately after the nth transition, i.e., $Y_n = Z(S_n+)$. Let $N(t)$ be the number of transitions by the SMP until time t. Define

$$\overline{Z}(t) = (Z(S_{N(t)}+), Z(S_{N(t)+1}+)). \tag{71}$$

Let $\overline{S}_n$ be the nth transition epoch of the $\{\overline{Z}(t), t \geq 0\}$ process and $\overline{Y}_n$ be the state of the $\{\overline{Z}(t), t \geq 0\}$ process immediately after the nth transition, i.e.,

$$\overline{Y}_n = \overline{Z}(\overline{S}_n).$$

Observe that $\overline{S}_n = S_n$ and $\overline{Y}_n = (Y_n, Y_{n+1})$.

Theorem 5 *The ℓ^2-state process $\{\overline{Z}(t), t \geq 0\}$ is a non-anticipative SMP with state space $\{(i,j) : 1 \leq i, j \leq \ell\}$, kernel*

$$\overline{G}_{(i,j),(k,l)}(x) = \begin{cases} p_{kl} G_{ij}(x)/G_{ij}(\infty) & \text{if } j = k \\ 0 & \text{if } j \neq k, \end{cases} \tag{72}$$

transition probabilities

$$\overline{p}_{(i,j),(k,l)} = \begin{cases} p_{kl} & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases} \tag{73}$$

and stationary probabilities

$$\overline{\pi}_{(i,j)} = \pi_i p_{ij}. \tag{74}$$

Proof. Let

$$\begin{aligned} \overline{G}_{(i,j),(k,l)}(x) &= P\{\overline{Y}_{n+1} = (k,l), \overline{S}_{n+1} \leq x | \overline{Y}_n = (i,j), \overline{S}_n, \overline{Y}_{n-1}, \ldots\} \\ &= P\{Y_{n+2} = l, Y_{n+1} = k, S_{n+1} \leq x | Y_{n+1} = j, Y_n = i, S_n, Y_{n-1}, \ldots\} \\ &= P\{Y_2 = l, Y_1 = k, S_1 \leq x | Y_1 = j, Y_0 = i\}, \end{aligned}$$

since $\{Z(t), t \geq 0\}$ is an SMP. Also, given Y_1, Y_2 is conditionally independent of S_1. Hence we can write

$$\begin{aligned}
\overline{G}_{(i,j),(k,l)}(x) &= P\{Y_2 = l, Y_1 = k | Y_1 = j, Y_0 = i\}\ P\{S_1 \leq x | Y_1 = j, Y_0 = i\} \\
&= \begin{cases} p_{kl} \cdot P\{S_1 \leq x | Y_1 = j, Y_0 = i\} & \text{if } j = k \\ 0 \cdot P\{S_1 \leq x | Y_1 = j, Y_0 = i\} & \text{if } j \neq k. \end{cases} \\
&= \overline{p}_{(i,j),(k,l)}\ \frac{P\{S_1 \leq x, Y_1 = j | Y_0 = i\}}{P\{Y_1 = j | Y_0 = i\}} \\
&= \overline{p}_{(i,j),(k,l)}\ \frac{G_{ij}(x)}{G_{ij}(\infty)} \\
&= \overline{p}_{(i,j),(k,l)}\ \overline{G}_{(i,j)}(x),
\end{aligned}$$

which is of the form (like the non-anticipative SMP)

$$[G_{ab}(x)] = [G_a(x) p_{ab}].$$

Hence we get Equations (72) and (73). It is easy to verify that $\overline{\pi}_{(i,j)} = \pi_i p_{ij}$ satisfies

$$\overline{\pi}\,\overline{G}(\infty) = \overline{\pi} \text{ and } \sum_{i=1}^{\ell}\sum_{j=1}^{\ell} \overline{\pi}_{(i,j)} = 1.$$

Hence we get Equation (74). ∎

Let $\{Z(t), t \geq 0\}$ be a general (not necessarily non-anticipative) SMP. Construct a non-anticipative SMP $\{\overline{Z}(t), t \geq 0\}$ as described in Equation (71). Now we use Theorem 4 for the ℓ^2-state non-anticipative SMP $\{\overline{Z}(t), t \geq 0\}$. Similar to the Equations (11), and (5) in Section 4, we now define $\overline{\Phi}(v)$, $\overline{r}_{(i,j)}$, and $\tilde{\overline{G}}_{(i,j)}(\cdot)$.

From the definition of the non-anticipative SMP $\{\overline{Z}(t), t \geq 0\}$ and Equations (73), (72) and (74), it is straightforward to show that

$$\overline{r}_{(i,j)} = \begin{cases} r_i & \text{if } p_{ij} > 0 \\ 0 & \text{if } p_{ij} = 0, \end{cases} \tag{75}$$

$$\begin{aligned}
\tilde{\overline{G}}_{(i,j)}(v(\overline{r}_{(i,j)} - c)) &= \frac{\tilde{G}_{ij}(v(r_i - c))}{G_{ij}(\infty)} \\
&= \frac{\phi_{ij}(v)}{p_{ij}}.
\end{aligned} \tag{76}$$

We have,

$$\begin{aligned}
\overline{\phi}_{(i,j),(k,l)}(v) &= E(e^{v(\overline{r}_{(i,j)} - c)\overline{S}_1}; \overline{Y}_1 = (k,l) | \overline{Y}_0 = (i,j)) \\
&= \begin{cases} 0 & \text{if } k \neq j \\ \tilde{\overline{G}}_{(i,j)}(-v(\overline{r}_{(i,j)} - c))\overline{p}_{(i,j),(j,l)} & \text{if } k = j \end{cases} \\
&= \begin{cases} 0 & \text{if } k \neq j \\ \frac{\phi_{ij}(v)}{p_{ij}} p_{kl} & \text{if } k = j. \end{cases}
\end{aligned} \tag{77}$$

Lemma 7 *The smallest real positive value of δ that satisfies*

$$e(\Phi(\delta)) = 1$$

is η if and only if η is the smallest real positive value that satisfies

$$e(\overline{\Phi}(\delta)) = 1.$$

Proof. For a given v, let λ_i be such that

$$w^i \Phi(v) = \lambda_i w^i,$$

where λ_i is the ith eigenvalue of $\Phi(v)$ and w^i is the corresponding eigenvector. Therefore λ_1, λ_2, ..., λ_ℓ are the ℓ eigenvalues of $\Phi(v)$ and w^1, w^2, ..., w^ℓ the corresponding eigenvectors. Without loss of generality we can assume that

$$Re(\lambda_1) \geq Re(\lambda_2) \geq \ldots \geq Re(\lambda_\ell),$$

where $Re(x)$ denotes the real part of the complex number x.

From the expression of $\overline{\Phi}(v)$ in Equation (77), it follows that there are at most ℓ linearly independent rows for the matrix $\overline{\Phi}(v)$. Hence at least $\ell^2 - \ell$ eigenvalues are zero. Let $\overline{\lambda}_1$, $\overline{\lambda}_2$, ..., $\overline{\lambda}_\ell$ be the ℓ possibly non-zero eigenvalues of $\overline{\Phi}(v)$ and $\overline{w}^1$, $\overline{w}^2$, ..., $\overline{w}^\ell$ the corresponding eigenvectors. Without loss of generality we can assume that

$$Re(\overline{\lambda}_1) \geq Re(\overline{\lambda}_2) \geq \ldots \geq Re(\overline{\lambda}_\ell).$$

From Perron-Frobenius theorem, $\overline{\lambda}_1$ is real and positive and hence $Re(\overline{\lambda}_1) \geq 0$. Note that we can choose each eigenvector $\overline{w}^n$ such that

$$\overline{w}^n_{(i,j)} = p_{ij} w^n_i.$$

Then also for a given v, $\overline{\lambda}_n = \lambda_n$ for all $n \in 1, 2, \ldots, \ell$. In fact,

$$\begin{aligned}
[\overline{w}^n \overline{\Phi}(v)]_{(k,l)} &= \sum_{j=1}^{\ell} \sum_{i=1}^{\ell} \overline{w}^n_{(i,j)} \overline{\phi}_{(i,j),(k,l)}(v) \\
&= \sum_{i=1}^{\ell} \overline{w}^n_{(i,k)} \overline{\phi}_{(i,k),(k,l)}(v) \\
&= \sum_{i=1}^{\ell} \overline{w}^n_{(i,k)} \frac{\phi_{ik}(v)}{p_{ik}} p_{kl} \\
&= \sum_{i=1}^{\ell} p_{ik} w^n_i \frac{\phi_{ik}(v)}{p_{ik}} p_{kl} \\
&= \lambda_n p_{kl} w^n_k \\
&= \lambda_n \overline{w}^n_{(k,l)} \\
&= \overline{\lambda}_n \overline{w}^n_{(k,l)}.
\end{aligned}$$

When $v = \eta$, the largest real-positive eigenvalue for both $\overline{\Phi}(\eta)$ and $\Phi(\eta)$ are equal to one. Hence the proof. ∎

In the next lemma we derive the relationship between the left eigenvectors h and $\overline{h}$.

Lemma 8 *Let $\overline{h}\,\overline{\Phi}(\eta) = \overline{h}$ and $h\,\Phi(\eta) = h$, then*

$$\overline{h}_{(i,j)} = h_i p_{ij}. \tag{78}$$

Proof. From the definition of $\overline{h}$, we have,

$$\begin{aligned}
\overline{h}_{(k,l)} &= \sum_{j=1}^{\ell} \sum_{i=1}^{\ell} \overline{h}_{(i,j)} \overline{\phi}_{(i,j),(k,l)}(\eta) \\
&= \sum_{i=1}^{\ell} \overline{h}_{(i,k)} \overline{\phi}_{(i,k),(k,l)}(\eta) \\
&= \sum_{i=1}^{\ell} \overline{h}_{(i,k)} \frac{\phi_{ik}(\eta)}{p_{ik}} p_{kl}.
\end{aligned}$$

Since $h_j = \sum_{i=1}^{\ell} \phi_{ij}(\eta) h_i$, verify that

$$\overline{h}_{(i,j)} = h_i p_{ij} \tag{79}$$

solves $\overline{h}\,\overline{\Phi}(\eta) = \overline{h}$. ■

Define $\overline{\tau}_{(i,j)}$ as the expected time the non-anticipative SMP $\{\overline{Z}(t), t \geq 0\}$ spends in state (i,j). Therefore,

$$\begin{aligned}
\overline{\tau}_{(i,j)} &= E\{\overline{S}_1 | \overline{Z}_0 = (i,j)\} \\
&= \int_0^\infty (1 - \overline{G}_{(i,j)}(x)) dx \\
&= \int_0^\infty (1 - G_{ij}(x)/G_{ij}(\infty)) \, dx.
\end{aligned}$$

Let $\overline{p}_{(i,j)}$ be the probability that in steady-state, the non-anticipative SMP $\{\overline{Z}(t), t \geq 0\}$ is state (i,j) and is given by

$$\overline{p}_{(i,j)} = \frac{\overline{\tau}_{(i,j)} \overline{\pi}_{(i,j)}}{\sum_{k=1}^{\ell} \sum_{l=1}^{\ell} \overline{\tau}_{(k,l)} \overline{\pi}_{(k,l)}}.$$

Subsequently we need an expression for $\overline{\tau}_{(i,j)} p_{ij} / \overline{p}_{(i,j)}$. This is shown in the following lemma.

Lemma 9

$$\frac{\overline{\tau}_{(i,j)} p_{ij}}{\overline{p}_{(i,j)}} = \frac{\tau_i}{p_i}. \tag{80}$$

Proof.

$$\begin{aligned}
\frac{\overline{\tau}_{(i,j)} p_{ij}}{\overline{p}_{(i,j)}} &= \frac{\overline{\tau}_{(i,j)} \pi_i p_{ij}}{\overline{p}_{(i,j)} \pi_i} \\
&= \frac{\overline{\tau}_{(i,j)} \overline{\pi}_{(i,j)}}{\overline{p}_{(i,j)} \pi_i} \\
&= \frac{1}{\pi_i} \sum_{k=1}^{\ell} \sum_{l=1}^{\ell} \overline{\tau}_{(k,l)} \overline{\pi}_{(k,l)} \\
&= \frac{1}{\pi_i} \sum_{k=1}^{\ell} \sum_{l=1}^{\ell} \pi_k p_{kl} \int_0^\infty (1 - G_{kl}(x)/G_{kl}(\infty)) \, dx \\
&= \frac{1}{\pi_i} \sum_{k=1}^{\ell} \pi_k \int_0^\infty (1 - G_k(x)) dx \\
&= \frac{1}{\pi_i} \sum_{k=1}^{\ell} \pi_k \tau_k \\
&= \frac{\tau_i}{p_i}.
\end{aligned}$$

■

Next we give the

Proof of Theorem 1 :

From Equation (65) for the non-anticipative SMP $\{\overline{Z}(t), t \geq 0\}$, we have

$$\overline{C}^* = \frac{\sum_{i=1}^{\ell} \sum_{j=1}^{\ell} \frac{\overline{h}_{(i,j)}}{\eta(\overline{r}_{(i,j)} - c)} (\tilde{\overline{G}}_{(i,j)}(-\eta(\overline{r}_{(i,j)} - c)) - 1)}{\min_{i,j:\overline{r}_{(i,j)} > c} \inf_x \left\{ \frac{\overline{h}_{(i,j)} \overline{\tau}_{(i,j)}}{\overline{p}_{(i,j)}} \frac{\int_x^\infty e^{\eta(\overline{r}_{(i,j)} - c)y} d\overline{G}_{(i,j)}(y)}{\int_x^\infty e^{\eta(\overline{r}_{(i,j)} - c)x} d\overline{G}_{(i,j)}(y)} \right\}}. \tag{81}$$

We substitute the expressions $\overline{h}_{(i,j)}$, $\overline{r}_{(i,j)}$, $\tilde{\overline{G}}_{(i,j)}(\cdot)$, $\overline{G}_{(i,j)}(y)$ and $\overline{\tau}_{(i,j)}p_{ij}/\overline{p}_{(i,j)}$ from Equations (79), (75), (76), (72) and (80) respectively into Equation (81) to get

$$\begin{aligned}
\overline{C}^* &= \frac{\sum_{i=1}^{\ell}\sum_{j=1}^{\ell}\frac{h_i p_{ij}}{\eta(r_i-c)}(\tilde{G}_{ij}(-\eta(r_i-c))/p_{ij}-1)}{\min\limits_{i,j:r_i>c,p_{ij}>0}\inf\limits_{x}\left\{\frac{h_i\tau_i}{p_i}\frac{\int_x^\infty e^{\eta(r_i-c)y}dG_{ij}(y)}{\int_x^\infty e^{\eta(r_i-c)x}dG_{ij}(y)}\right\}} \\
&= \frac{\sum_{i=1}^{\ell}\frac{h_i}{\eta(r_i-c)}\left(\sum_{j=1}^{\ell}(\phi_{ij}(\eta))-1\right),}{\min\limits_{i,j:r_i>c,p_{ij}>0}\inf\limits_{x}\left\{\frac{h_i e^{-\eta(r_i-c)x}\int_x^\infty e^{\eta(r_i-c)y}dG_{ij}(y)}{\frac{p_i}{\tau_i}\int_x^\infty dG_{ij}(y)}\right\}} \\
&= C^*.
\end{aligned}$$

Note that assumptions (17) yields (67). Using a similar analysis for $\overline{C}_*$, Theorem 1 is proved. ■

In Section 6 we demonstrate how to compute the bounds and in Section 7 we illustrate the results using several examples.

5 Multiplexing Of Independent Sources

In this section we consider the model in Section 2. As illustrated in Figure 1, there are K independent sources, each modulated by an SMP $\{Z^k(t), t \geq 0\}$ on state space $\{1, 2, \ldots, \ell_k\}$. Fluid is generated from source k at rate r_i^k at time t when its modulating SMP is in state i. The kernel of the SMP modulating the kth input source is $G^k(x) = [G_{ij}^k(x)]$. The expected time the kth SMP spends in state i is τ_i^k. The stationary distributions of the kth SMP $\{Z^k(t), t \geq 0\}$ and its underlying DTMC are respectively p^k and π^k. We assume for stability and non-triviality that

$$\sum_{k=1}^{K}\sum_{m=1}^{\ell_k} p_m^k r_m^k < c < \sup_{k,m} r_m^k.$$

Let $A_k(t)$ be the total amount of fluid input from the kth source to the buffer over time $(0, t]$, where

$$A_k(t) = \int_0^t r_{Z^k(u)}^k du. \tag{82}$$

Then the effective bandwidth of the kth source, $eb^k(\cdot)$, is given by

$$eb^k(v) = \lim_{t\to\infty}\frac{1}{v\ t}\log E\{\exp(vA_k(t))\}. \tag{83}$$

Let η be the smallest positive solution to

$$\sum_{k=1}^{K} eb^k(\eta) = c. \tag{84}$$

As a first step we analyze a single source model where source k acts as the only input source to an infinite-capacity buffer with output capacity

$$c^k(\eta) = eb^k(\eta). \tag{85}$$

See Figure 2. For this single source model, following the analysis in Section 4, for each source k, we compute $\Phi^k(\eta)$ and h^k from Equations (10) and (13) as

$$\begin{aligned}
\phi_{ij}^k(\eta) &= E(e^{\eta(r_i^k - c^k(\eta))S_1^k}; Z_1^k = j | Z_0^k = i), &(86)\\
h^k &= h^k\ \Phi^k(\eta). &(87)
\end{aligned}$$

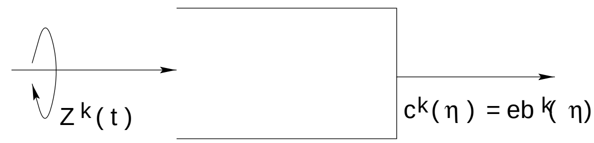

Figure 2: Single Source Modification

Define

$$P^k(i,j) = P\{Z_1^k = j | Z_0^k = i\}. \tag{88}$$

Continuing with the single source model, we also define

$$H^k = \sum_{i=1}^{\ell_k} \frac{h_i^k}{\eta(r_i^k - c^k(\eta))} \left(\sum_{j=1}^{\ell_k} (\phi_{ij}^k(\eta)) - 1 \right), \tag{89}$$

$$\Psi_{min}^k(i,j) = \inf_x \left\{ \frac{h_i^k e^{-\eta(r_i^k - c^k(\eta))x} \int_x^\infty e^{\eta(r_i^k - c^k(\eta))y} dG_{ij}^k(y)}{\frac{p_i^k}{\tau_i^k} \int_x^\infty dG_{ij}^k(y)} \right\}, \tag{90}$$

and

$$\Psi_{max}^k(i,j) = \sup_x \left\{ \frac{h_i^k e^{-\eta(r_i^k - c^k(\eta))x} \int_x^\infty e^{\eta(r_i^k - c^k(\eta))y} dG_{ij}^k(y)}{\frac{p_i^k}{\tau_i^k} \int_x^\infty dG_{ij}^k(y)} \right\}. \tag{91}$$

Using these quantities, the bounds on the multiplexed traffic are given in the following theorem.

Theorem 6 *If $\Psi_{\min}^k(i,j) > 0$ and $\Psi_{\max}^k(i,j) < \infty$ for all $i,j \in \{1,\ldots,\ell_k\}$, $k \in \{1,\ldots,K\}$, then the bounds on the limiting distribution of the buffer-content process $\{X(t), t \geq 0\}$ driven by K independent SMP sources are given by*

$$C_* e^{-\eta x} \leq P(X > x) \leq C^* e^{-\eta x},$$

where η is given by the solution to Equation (84),

$$C^* = \frac{\prod_{k=1}^K H^k}{\min\limits_{\mathcal{A}} \prod_{k=1}^K \Psi_{min}^k(i_k, j_k)},$$

$$C_* = \frac{\prod_{k=1}^K H^k}{\max\limits_{\mathcal{A}} \prod_{k=1}^K \Psi_{max}^k(i_k, j_k)}$$

and

$$\mathcal{A} = \left\{ (i_1,j_1),(i_2,j_2),\ldots,(i_K,j_K) : \sum_{k=1}^K r_{i_k}^k > c \text{ and } \forall k,\ P^k(i_k,j_k) > 0 \right\}. \tag{92}$$

Proof. We describe only a brief outline of the proof of the theorem by illustrating the main steps of the proof since most of the analysis is identical to the proof of Theorem 1.

1. The non-anticipative SMPs

   For each $k \in [1,2,\ldots,K]$, we first construct an ℓ_k^2 state non-anticipative SMP $\{\overline{Z}^k(t), t \geq 0\}$ from the SMP $\{Z^k(t), t \geq 0\}$ following the technique in Subsection 4.7.

2. <u>The reversed SMP</u>
Next we construct the reversed SMP of $\{\overline{Z}^k(t), t \geq 0\}$ as

$$Y^k(s) = \overline{Z}^k(-s).$$

Therefore

$$P\{X > x\} = P\{\sup_{t\geq 0} \int_0^t (\sum_{k=1}^K r^k_{Y^k(s)} - c)ds > x\}.$$

3. <u>Markov process and its generator</u>
We construct a Markov process $\{W(t), t \geq 0\}$ defined by

$$W(t) = (Y^1(t), S^1(t), Y^2(t), S^2(t), \ldots, Y^K(t), S^K(t)),$$

where $S^k(t)$ is the supplementary age process of $\{Y^k(t), -\infty < t < \infty\}$ (as defined in Section 3). Let Q^k $(1 \leq k \leq K)$ be the full generator of $\{w^k(t)\} = \{(Y^k(t), S^k(t))\}$ as in Equation (29).

4. <u>Martingale $M(t)$</u>
Let η satisfy Equation (84) and for this η let $v^k(i,x)$ satisfy

$$v^k(i,x) = [, ^k(x)u^k]_i$$

where

$$, ^k_{ij}(x) = E(e^{\eta(r^k_i - c^k(\eta))(S^k_1 - x)}; Y^k_1 = j | Y^k_0 = i, S^k_1 > x)$$

(see Equation (28)) and u^k is the right eigenvector satisfying

$$\chi^k(\eta)u^k = u^k$$

with $\chi^k_{ij}(\eta) = E(e^{\eta(r^k_i - c^k(\eta))S^k_1}; Y^k_1 = j | Y^k_0 = i)$ (see Equation (26)). Assume that

$$v^k \in \mathcal{C}_{\mathrm{b}}(R_+) \quad \text{and} \quad \inf_x v^k(i,x) > 0 \ . \tag{93}$$

Then we have $(Q^k v^k)(i,x) = -\eta(r^k_i - c)v^k(i,x)$ and following Proposition 3.2 from Ethier and Kurtz [13] we define exponential martingales by

$$M^k(t) = \frac{v^k(w(t))}{v^k(i,x)} e^{-\eta \int_0^t (Y^k(s) - c^k(\eta))\, ds}, k = 1, \ldots, K \ .$$

Since sources are independent, the process

$$\begin{aligned} M(t) = \prod_{k=1}^K M^k(t) &= \frac{v^1(w^1(t)) \ldots v^K(w^K(t))}{v^1(i^1,x^1) \ldots v(i^K,x^K)} e^{-\eta \int_0^t \sum_{k=1}^K (Y^k(s) - c^k(\eta))\, ds} \\ &= \frac{v(W(t))}{v(i,x)} e^{-\eta \int_0^t \sum_{k=1}^K (Y^k(s) - c^k(\eta))\, ds} \\ &= \frac{v(W(t))}{v(i,x)} e^{-\eta \int_0^t (\sum_{k=1}^K Y^k(s) - c)\, ds} \end{aligned}$$

is a martingale, where $v(i,x) = v(i^1,x^1,\ldots,i^K,x^K) = \prod_{k=1}^K v^k(i^k,x^k)$ and $i = (i^1,\ldots,i^K)$, $i^k \in \{1,\ldots,\ell_k\}$ and $x = (x^1,\ldots,x^K) \in R^K_+$.

Then we also construct a new probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \tilde{P})$ by

$$\frac{d\tilde{P}_t}{dP_t} = M(t) \ . \tag{94}$$

5. <u>First Passage Time $\tau(x)$</u>
Let

$$\tau(x) = \inf\{t > 0 : \int_0^t (\sum_{k=1}^K r_{Y^k(s)}^k - c)\, ds > x\}. \tag{95}$$

From the above definition, we have,

$$\left\{ \sup_{t \geq 0} \int_0^t (\sum_{k=1}^K r_{Y^k(s)}^k - c) ds > x \right\} \equiv \{\tau(x) < \infty\}.$$

Hence,

$$P\{\sup_{t \geq 0} \int_0^t (\sum_{k=1}^K r_{Y^k(s)}^k - c) ds > x\} = P\{\tau(x) < \infty\}.$$

Then, by Equation (94)

$$\begin{aligned} P\{\tau(x) < \infty\} &= \int_{\tau(x)<\infty} dP \\ &= \int_{\tau(x)<\infty} [M(\tau(x))]^{-1} d\tilde{P}. \end{aligned}$$

Lemma 4 yields that $\tilde{E} r_{Z^k(\infty)}^k > c^k(\eta)$. Hence $\tilde{P}(\tau(x) < \infty) = 1$ and then

$$\begin{aligned} P\{\tau(x) < \infty\} &= \int_{\tau(x)<\infty} \frac{v(W(0))}{v(W(\tau(x)))} e^{-\eta x} d\tilde{P} \\ &= e^{-\eta x} \tilde{E} \left\{ \frac{v(W(0))}{v(W(\tau(x)))} \right\}, \end{aligned} \tag{96}$$

where $\tilde{E}$ is the expectation with respect to $\tilde{P}$. Define

$$\begin{aligned} \pi_j^k &= \lim_{n\to\infty} P\{Y_n^k = j\} \\ A_{ij}^k &= P\{Y_1^k = j | P\{Y_0^k = i\} \\ p_i^k &= \lim_{t\to\infty} P\{Y^k(t) = i\} \\ \tau_i^k &= E\{S_1^k | Y_0^k = i\} \\ b_i^k &= \frac{p_i^k}{\tau_i^k \eta (r_i^k - c^k)}. \end{aligned}$$

Next we bound

$$\begin{aligned} &\tilde{E} \left\{ \frac{v(W(0))}{v(W(\tau(x)))} \right\} \\ &= v(W(0)) \tilde{E} \left\{ \frac{1}{v(W(\tau(x)))} \right\} \\ &= \prod_{k=1}^K \sum_{j_k=1}^{\ell_k^2} \int_0^\infty v^k(j^k, y^k) \pi_{j_k}^k (y_k) dy_k \tilde{E} \left\{ \frac{1}{v(W(\tau(x)))} \right\} \\ &= \prod_{k=1}^k b^k (I - A^k) u^k \tilde{E} \left\{ \frac{1}{v(W(\tau(x)))} \right\}. \end{aligned}$$

At time $\tau(x)$, the $W(t)$ process can only be in states $(i_1, s_1, i_2, s_2 \ldots, i_K, s_K)$ that satisfy $\sum_{k=1}^{K} r_{i_k}^k > c$. Hence we have

$$\tilde{E}\left\{\frac{1}{v(W(\tau(x)))}\right\} \leq \frac{1}{\displaystyle\min_{i_1,i_2,\ldots,i_K:\sum_{k=1}^{K} r_{i_k}^k > c} \prod_{k=1}^{k} \left\{\inf_x v_k(i^k, x)\right\}}$$

and

$$\tilde{E}\left\{\frac{1}{v(W(\tau(x)))}\right\} \geq \frac{1}{\displaystyle\max_{i_1,i_2,\ldots,i_K:\sum_{k=1}^{K} r_{i_k}^k > c} \prod_{k=1}^{k} \left\{\sup_x v_k(i^k, x)\right\}}.$$

Therefore the bounds on Equation (96) are

$$L_* e^{-\eta x} \leq P\{\tau(x) < \infty\} \leq L^* e^{-\eta x} \ , \tag{97}$$

where

$$L^* = \frac{\displaystyle\prod_{k=1}^{K} b^k (I - A^k) u^k}{\displaystyle\min_{i_1,i_2,\ldots,i_K:\sum_{k=1}^{K} r_{i_k}^k > c} \prod_{k=1}^{K} \left\{\inf_x v_k(i^k, x)\right\}}$$

$$L_* = \frac{\displaystyle\prod_{k=1}^{K} b^k (I - A^k) u^k}{\displaystyle\max_{i_1,i_2,\ldots,i_K:\sum_{k=1}^{K} r_{i_k}^k > c} \prod_{k=1}^{K} \left\{\sup_x v_k(i^k, x)\right\}}.$$

6. Original process

   Next we rewrite L^* and L_* in terms of the ℓ_k^2 state SMPs $\{\overline{Z}^k(t), t \geq 0\}$ (for all $k \in [1, 2, \ldots, K]$). Finally, we convert the expressions to those of the original ℓ_k state SMPs $\{Z^k(t), t \geq 0\}$ and obtain C^*, C_* and Equation (92) of Theorem 6. Note that condition (93) put on the functions $v^k(i, x)$ is equivalent to the condition that $L_* > 0$ and $L^* < \infty$, that is that $C_* > 0$ and $C^* < \infty$, which follows from the assumptions of the Theorem 6.

■

6 Computation of C^* and C_*

In this section we address two concerns while using Theorem 1 in Section 4 :

- how to compute η in Equation (12) and hence h in Equation (13)?
- how to compute $\Psi_{max}(i, j)$ and $\Psi_{min}(i, j)$ in Equations (15) and (16) respectively?

6.1 Computation of η and h

To compute η, we have to solve Equation (12). In terms of the effective bandwidth of the source, it is the same as solving for η in

$$eb(\eta) = c \tag{98}$$

whenever $e(\Phi(v)) = 1$ has a solution (see Gautam [14]). Also, condition (17) is satisfied whenever $e(\Phi(v)) = 1$ has a solution. Therefore the easiest way to calculate η is to use Equation (98). Since a semi-Markov process is a special case of a Markov regenerative process (MRGP), to compute the effective bandwidth of a source modulated by an SMP we use the results from Kulkarni [21].

The eigenvector h is obtained by solving

$$h\ \Phi(\eta) = h.$$

Note : An interesting scenario that arises in practical situations is the case when $e(\Phi(v)) = 1$ has no solutions. This is also the case when condition (17) is not satisfied. We do not address this issue in this paper. However, we will publish a technique to get around this scenario in a forthcoming paper.

6.2 Computation of Ψ_{max} and Ψ_{min}

Consider a nonnegative random variable Y with distribution $G_{ij}(x)/G_{ij}(\infty)$ and density

$$g_{ij}(x) = \frac{dG_{ij}(x)}{dx}\frac{1}{G_{ij}(\infty)}.$$

The failure rate function of Y is defined by

$$\lambda_{ij}(x) = \frac{g_{ij}(x)}{1 - \frac{G_{ij}(x)}{G_{ij}(\infty)}}. \tag{99}$$

Y is said to be an increasing failure rate (IFR) random variable if

$$\lambda_{ij}(x) \uparrow x$$

and Y is said to be a decreasing failure rate (DFR) random variable if

$$\lambda_{ij}(x) \downarrow x.$$

It is possible to obtain closed form algebraic expressions for $\Psi_{max}(i,j)$ and $\Psi_{min}(i,j)$ in Equations (15) and (16) respectively if a random variable Y with distribution $G_{ij}(x)/G_{ij}(\infty)$ is an IFR or DFR random variable. The following theorem describes how to compute $\Psi_{max}(i,j)$ and $\Psi_{min}(i,j)$ in those cases. Let x^* and x_* be such that

$$x^* = \arg\sup_x \left\{ \frac{h_i \int_x^\infty e^{\eta(r_i-c)y} dG_{ij}(y)}{\frac{p_i}{\tau_i} e^{\eta(r_i-c)x} \int_x^\infty dG_{ij}(y)} \right\} \tag{100}$$

and

$$x_* = \arg\inf_x \left\{ \frac{h_i \int_x^\infty e^{\eta(r_i-c)y} dG_{ij}(y)}{\frac{p_i}{\tau_i} e^{\eta(r_i-c)x} \int_x^\infty dG_{ij}(y)} \right\}. \tag{101}$$

Theorem 7 *If Y is IFR or DFR, then $\Psi_{max}(i,j)$ and $\Psi_{min}(i,j)$ in Equations (15) and (16) respectively occur at x values given by the following table*

	IFR		DFR	
	$r_i > c$	$r_i \le c$	$r_i > c$	$r_i \le c$
x^*	0	∞	∞	0
$\Psi_{max}(i,j)$	$\frac{\phi_{ij}(-\eta(r_i-c))\tau_i h_i}{p_{ij}p_i}$	$\frac{\tau_i h_i \lambda_{ij}(\infty)}{p_i(\lambda_{ij}(\infty)-\eta(r_i-c))}$	$\frac{\tau_i h_i \lambda_{ij}(\infty)}{p_i(\lambda_{ij}(\infty)-\eta(r_i-c))}$	$\frac{\phi_{ij}(-\eta(r_i-c))\tau_i h_i}{p_{ij}p_i}$
x_*	∞	0	0	∞
$\Psi_{min}(i,j)$	$\frac{\tau_i h_i \lambda_{ij}(\infty)}{p_i(\lambda_{ij}(\infty)-\eta(r_i-c))}$	$\frac{\phi_{ij}(-\eta(r_i-c))\tau_i h_i}{p_{ij}p_i}$	$\frac{\phi_{ij}(-\eta(r_i-c))\tau_i h_i}{p_{ij}p_i}$	$\frac{\tau_i h_i \lambda_{ij}(\infty)}{p_i(\lambda_{ij}(\infty)-\eta(r_i-c))}$

where

$$\lambda_{ij}(\infty) = \lim_{x\to\infty} \lambda_{ij}(x).$$

Proof. Let Y^t be the remaining life associated with the random variable Y. Then Y^t will have distribution $G^t_{ij}(a)$ given by

$$\begin{aligned} 1 - G^t_{ij}(a) &= P\{Y^t > a\} \\ &= P\{Y - t > a | Y > t\}. \end{aligned}$$

It is known that (see Ross [32]) if Y is IFR (DFR), then Y^t is stochastically decreasing (increasing) in t and hence $E[f(Y^t)]$ is decreasing (increasing) in t for all increasing functions f. Therefore if Y is IFR (DFR) and $r_i > c$, then

$$E(e^{\eta(r_i-c)[Y-x]}|Y > x) = \frac{e^{-\eta(r_i-c)x}\int_x^\infty e^{\eta(r_i-c)y} dG_{ij}(y)}{\int_x^\infty dG_{ij}(y)}$$

is decreasing (increasing) in x. Similarly, if Y is IFR (DFR), then Y^t is stochastically decreasing (increasing) in t and hence $E[f(Y^t)]$ is increasing (decreasing) in t for all decreasing functions f. Therefore if Y is IFR (DFR) and $r_i \le c$, then

$$E(e^{\eta(r_i-c)[Y-x]}|Y > x) = \frac{e^{-\eta(r_i-c)x}\int_x^\infty e^{\eta(r_i-c)y} dG_{ij}(y)}{\int_x^\infty dG_{ij}(y)}$$

is increasing (decreasing) in x. Note that

$$\lim_{x\to 0}\left\{\frac{h_i\int_x^\infty e^{\eta(r_i-c)y} dG_{ij}(y)}{\frac{p_i}{\tau_i} e^{\eta(r_i-c)x}\int_x^\infty dG_{ij}(y)}\right\} = \frac{\tilde{\phi}_{ij}(-\eta(r_i-c))\tau_i h_i}{p_{ij}p_i}$$

and using L'Hospital's Rule we show that

$$\begin{aligned} \lim_{x\to\infty}\left\{\frac{h_i\int_x^\infty e^{\eta(r_i-c)y} dG_{ij}(y)}{\frac{p_i}{\tau_i} e^{\eta(r_i-c)x}\int_x^\infty dG_{ij}(y)}\right\} &= \lim_{x\to\infty}\left\{\frac{-e^{\eta(r_i-c)x}\frac{dG_{ij}(x)}{dx}\frac{h_i\tau_i}{p_i}}{-e^{\eta(r_i-c)x}\frac{dG_{ij}(x)}{dx} + \eta(r_i-c)e^{\eta(r_i-c)x}(G_{ij}(\infty)-G_{ij}(x))}\right\} \\ &= \frac{h_i\tau_i}{p_i}\lim_{x\to\infty}\left\{\frac{p_{ij}g_{ij}(x)}{p_{ij}g_{ij}(x) - \eta(r_i-c)(p_{ij}-G_{ij}(x))}\right\} \\ &= \frac{\tau_i h_i \lambda_{ij}(\infty)}{p_i(\lambda_{ij}(\infty) - \eta(r_i-c))}. \end{aligned}$$

Hence the result follows. ■

If Y is not an IFR or DFR random variable, then x^*, x_*, $\Psi_{max}(i,j)$ and $\Psi_{min}(i,j)$ can be obtained only numerically.

7 Examples

7.1 General On-Off Source

Consider a source modulated by a two-state (on and off) process that alternates between on and off states. The random amount of time the process spends in the on state (called *on-times*) has cdf $U(\cdot)$ with mean τ_U and the corresponding *off-time* cdf is $D(\cdot)$ with mean τ_D. The successive on and off-times are independent and on-times are independent of off-times. Fluid is generated continuously at rate r during the on state and at rate 0 during the off state. A source modulated by such a 2-state (on-off) process is called a *general on-off source with on-time distribution* $U(\cdot)$, *off-time distribution* $D(\cdot)$ *and (peak) rate* r.

Consider a general on-off source with on-time distribution $U(\cdot)$, off-time distribution $D(\cdot)$ and rate r that inputs traffic into an infinite-capacity buffer. The output capacity of the buffer is a constant c. Assume

$$\frac{r\tau_U}{\tau_U+\tau_D} < c < r.$$

Equation (11) reduces to

$$\Phi(v) = \begin{bmatrix} 0 & \tilde{D}(vc) \\ \tilde{U}(-v(r-c)) & 0 \end{bmatrix},$$

where $\tilde{U}(\cdot)$ and $\tilde{D}(\cdot)$ are the Laplace Stieltjes transforms (LSTs) of $U(t)$ and $D(t)$ respectively. In this subsection, we assume that $e(\Phi(\eta)) = 1$ has a solution and it implies that

$$e(\Phi(\eta)) = \sqrt{\tilde{U}(-\eta(r-c))\ \tilde{D}(\eta c)} = 1$$

can be solved. Hence η is the smallest real-positive solution to

$$\tilde{U}(-\eta(r-c))\ \tilde{D}(\eta c)\ =\ 1.$$

Also, Equation (13) reduces to

$$h = [1\ \ \tilde{D}(\eta c)].$$

Furthermore, from Equations (14), (16) and (15) we get

$$H = \frac{(1-\tilde{D}(\eta c))r}{c(r-c)},$$

$$\Psi_{min} = \begin{bmatrix} 0 & \inf_x \left\{ \frac{(\tau_U+\tau_D)\int_x^\infty e^{-\eta c(y-x)}dD(y)}{1-D(x)} \right\} \\ \inf_x \left\{ \tilde{D}(\eta c)\frac{(\tau_U+\tau_D)\int_x^\infty e^{\eta(r-c)(y-x)}dU(y)}{1-U(x)} \right\} & 0 \end{bmatrix}$$

and

$$\Psi_{max} = \begin{bmatrix} 0 & \sup_x \left\{ \frac{(\tau_U+\tau_D)\int_x^\infty e^{-\eta c(y-x)}dD(y)}{1-D(x)} \right\} \\ \sup_x \left\{ \tilde{D}(\eta c)\frac{(\tau_U+\tau_D)\int_x^\infty e^{\eta(r-c)(y-x)}dU(y)}{1-U(x)} \right\} & 0 \end{bmatrix}.$$

From Theorem 1, we can derive that

$$C_* e^{-\eta x} \le P\{X > x\} \le C^* e^{-\eta x}, \tag{102}$$

where

$$C^* = \frac{\tilde{U}(-\eta(r-c))-1}{\tau_U+\tau_D} \frac{r}{c(r-c)\eta \inf_x \left\{ \frac{\int_x^\infty e^{\eta(r-c)(y-x)}dU(y)}{1-U(x)} \right\}} \tag{103}$$

and

$$C_* = \frac{\tilde{U}(-\eta(r-c))-1}{\tau_U+\tau_D} \frac{r}{c(r-c)\eta \sup_x \left\{ \frac{\int_x^\infty e^{\eta(r-c)(y-x)}dU(y)}{1-U(x)} \right\}}. \tag{104}$$

If $U(\cdot)$ is IFR/DFR, the supremums and the infimums in the above equations can be obtained from Theorem 7 in Section 6.2.

7.2 Erlang On-Off Source

Consider the special case of the general on-off source in Subsection 7.1 with *Erlang*(N_U, α) on-time distribution, *Erlang*(N_D, β) off-time distribution and rate r. Note that, $\tau_U = N_U/\alpha$ and $\tau_D = N_D/\beta$. Equation (11) reduces to

$$\Phi(v) = \begin{bmatrix} 0 & \tilde{D}(vc) \\ \tilde{U}(-v(r-c)) & 0 \end{bmatrix} = \begin{bmatrix} 0 & \left(\frac{\beta}{\beta+vc}\right)^{N_D} \\ \left(\frac{\alpha}{\alpha-v(r-c)}\right)^{N_U} & 0 \end{bmatrix}.$$

It is possible to show that $e(\Phi(v)) = 1$ always has a solution. Hence η is obtained by solving

$$e(\Phi(\eta)) = \sqrt{\tilde{U}(-\eta(r-c))\ \tilde{D}(\eta c)} = 1$$

and

$$h = \begin{bmatrix} 1 & \left(\frac{\beta}{\beta+\eta c}\right)^{N_D} \end{bmatrix}.$$

Using the fact that the *Erlang* random variable has an increasing hazard rate function, from Theorem 7 in Section 6.2, we see that

$$\Psi_{min} = \begin{bmatrix} 0 & (N_U/\alpha + N_D/\beta)\left(\frac{\beta}{\beta+\eta c}\right)^{N_D} \\ \left(\frac{\beta}{\beta+\eta c}\right)^{N_D}(N_U/\alpha + N_D/\beta)\frac{\alpha}{\alpha-\eta(r-c)} & 0 \end{bmatrix} \text{ and}$$

$$\Psi_{max} = \begin{bmatrix} 0 & (N_U/\alpha + N_D/\beta)\frac{\beta}{\beta+\eta c} \\ \left(\frac{\beta}{\beta+\eta c}\right)^{N_D}(N_U/\alpha + N_D/\beta)\left(\frac{\alpha}{\alpha-\eta(r-c)}\right)^{N_U} & 0 \end{bmatrix}.$$

Using Theorem 1, we have

$$C_* e^{-\eta x} \leq P\{X > x\} \leq C^* e^{-\eta x}, \tag{105}$$

where

$$C^* = \frac{\left(\frac{\alpha}{\alpha-\eta(r-c)}\right)^{N_U} - 1}{\tau_U + \tau_D} \frac{r}{c(r-c)\eta\left\{\frac{\alpha}{\alpha-\eta(r-c)}\right\}} \tag{106}$$

and

$$C_* = \frac{\left(\frac{\alpha}{\alpha-\eta(r-c)}\right)^{N_U} - 1}{\tau_U + \tau_D} \frac{r}{c(r-c)\eta\left\{\left(\frac{\alpha}{\alpha-\eta(r-c)}\right)^{N_U}\right\}}. \tag{107}$$

Consider a numerical example of an Erlang on-off source with on-time distribution $Erlang(N_U, \alpha)$ and off-time distribution $Erlang(N_D, \beta)$. Let $r = 15$, $c = 10$, $\tau_U = 1/70$ and $\tau_D = 1/30$. We keep the means constant (i.e. τ_U and τ_D are held constant) but decrease the variances by increasing N_U and N_D. In Figure 3 we illustrate for four pairs of (N_U, N_D), (namely, $(1,1), (4,3), (9,8)$, and $(16,14)$), the logarithm of the upper and lower bounds on the limiting distribution of the buffer-content process.

From the figure we notice that as the variance decreases, the bounds move further apart. Also note that, C^* increases with decrease in variance and C_* decreases with decrease in variance. Since η increases with decrease in variance, the tail of the limiting distribution rapidly goes to zero.

Remark : Let $N_U \to \infty$, $\alpha \to \infty$ such that $N_U/\alpha = \tau_U$, a finite positive number and $N_D \to \infty$, $\beta \to \infty$ such that $N_D/\beta = \tau_D$, a finite positive number. Such a source is a deterministic on-off source with on-times τ_U and off-times τ_D. The upper and lower bounds for this limiting case of an Erlang on-off source is illustrated in Figure 4.

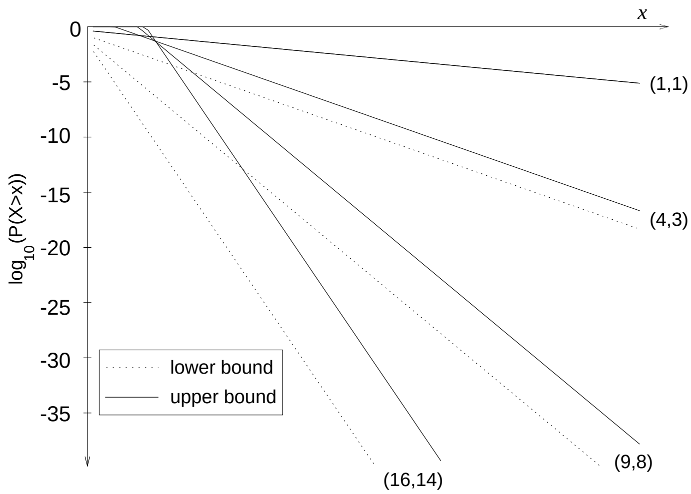

Figure 3: Logarithm of the upper and lower bounds as a function of x

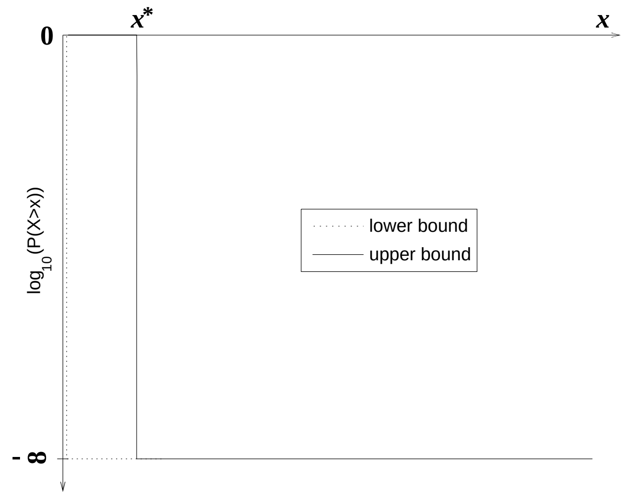

Figure 4: Logarithm of the bounds as a function of x for the limiting case

In Figure 5, we demonstrate the exact probabilities $P(X > x)$ for different c values increasing from $r\tau_U/(\tau_U + \tau_D)$ to r. We compare these exact probabilities with the bounds obtained by limiting case of the Erlang distribution. The x^* values in Figures 4 and 5 are identical. Hence we can conclude that the limiting case of the Erlang distribution does produce bounds that make sense for the deterministic on-off source.

7.3 Tandem Buffers - Single Source

In this subsection, an exponential on-off source with on-time parameter α, off-time parameter β and rate r inputs traffic to an infinite-capacity buffer with output capacity c_1. The output from the buffer acts as an input to another infinite-capacity buffer whose output capacity is c_2.

The effective bandwidth of the exponential on-off source is

$$eb(v) = \frac{rv - \alpha - \beta + \sqrt{(rv - \alpha - \beta)^2 + 4\beta rv}}{2v}. \tag{108}$$

Assume,

$$r\beta/(\alpha + \beta) < c_2 < c_1 < r.$$

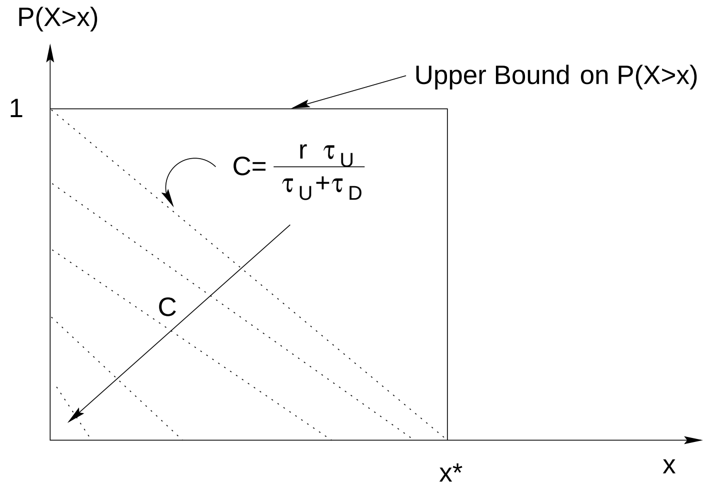

Figure 5: The deterministic on-off source probabilities and bounds

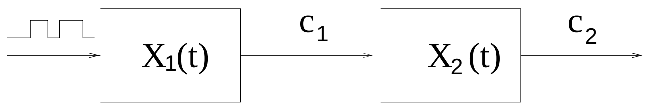

Figure 6: Exponential ON-OFF input to buffers in tandem

We study the buffer-content processes of the respective buffers $\{X_1(t), t \geq 0\}$ and $\{X_2(t), t \geq 0\}$. See Figure 6 for an illustration of the model.

For the exponential on-off source we have,

$$P\{X_1 > x\} = \frac{r\beta}{c_1(\alpha+\beta)} e^{-\eta_1 x}, \tag{109}$$

where

$$\eta_1 = \frac{c_1\alpha + c_1\beta - \beta r}{c_1(r-c_1)}.$$

Using the analysis in Narayanan [26], we model the output process from the first buffer as an alternating renewal process. Thus the input source to the second buffer can be modeled as a general on-off source with and on-time distribution $U(t)$ (with mean $r/(c_1(\alpha+\beta)-r\beta)$), off-time distribution $D(t)$ (with mean $1/\beta$) and rate c_1, where

$$U(t) = \left(\frac{a_2^2 a_3}{2a_1}\right) \sum_{k=0}^{\infty} \left[\left(\frac{a_2}{2a_1}\right)^{2k} \frac{(2k)!}{k!(k+1)!} \right] \left[1 - \sum_{n=0}^{2k} \left(\frac{e^{-a_1 t/r}(a_1 t/r)^n}{n!} \right) \right], \tag{110}$$

with $a_1 = \beta r - \beta c_1 + \alpha c_1$, $a_2 = \sqrt{4\alpha\beta c_1(r-c_1)}$, $a_3 = 1/(2\beta(r-c_1))$, and

$$D(t) = 1 - e^{-\beta t}.$$

The LST of the distribution $U(\cdot)$ is

$$\tilde{U}(w) = \begin{cases} \frac{w+\beta+c_1 s_0(w)}{\beta} & \text{if } w \geq w^* \\ \infty & \text{otherwise,} \end{cases}$$

where $w^* = (2\sqrt{c_1\alpha\beta(r-c_1)} - r\beta - c_1\alpha - c_1\beta)/r$, $s_0(w) = \frac{-b-\sqrt{b^2+4w(w+\alpha+\beta)c_1(r-c_1)}}{2c_1(r-c_1)}$ and $b = (r - 2c_1)w + (r-c_1)\beta - c_1\alpha$. The LST of the distribution $D(\cdot)$ is

$$\tilde{D}(w) = \begin{cases} \frac{\beta}{\beta+w} & \text{if } w > -\beta \\ \infty & \text{otherwise.} \end{cases}$$

From Kulkarni and Gautam [22] we have the effective bandwidth of this general on-off source, $eb_2(v)$, given by

$$eb_2(v) = \begin{cases} eb_1(v) & \text{if } 0 \le v \le v^* \\ (eb_1(v^*) - c_1)\frac{v^*}{v} + c_1 & \text{if } v > v^*, \end{cases}$$

where

$$v^* = \frac{\beta}{r}\left(\sqrt{\frac{c_1\alpha}{\beta(r-c_1)}} - 1\right) + \frac{\alpha}{r}\left(1 - \sqrt{\frac{\beta(r-c_1)}{c_1\alpha}}\right) \tag{111}$$

and $eb_1(v)$ is from Equation (108). Note that η_2 is obtained by solving

$$eb_2(\eta_2) = c_2.$$

If $\eta_2 \le v^*$, we have,

$$\Phi(\eta_2) = \begin{bmatrix} 0 & \tilde{D}(\eta_2 c_2) \\ \tilde{U}(-\eta_2(c_1 - c_2)) & 0 \end{bmatrix},$$

and $e(\Phi(\eta_2)) = 1$ (note that if $\eta_2 > v^*$, then $e(\Phi(\eta_2)) = 1$ has no solutions). Then we obtain h_2 by solving

$$[1 \;\; h_2]\,\Phi(\eta_2) = [1 \;\; h_2]$$

as $h_2 = \tilde{D}(\eta_2 c_2)$.

Intuitively, a random variable with the distribution $U(t)$ (of Equation (110)) is a Decreasing Failure Rate random variable (since $U(t)$ represents the busy period distribution). The intuition can be verified (after a lot of algebra) using the expression for $U(t)$ in Equation (110).

Using Theorem 1, the steady-state distribution of the buffer-content process $\{X_2(t), t \ge 0\}$ is bounded as

$$C_{2*}e^{-\eta_2 x} \le P(X_2 > x) \le C_2^* e^{-\eta_2 x},$$

where

$$C_2^* = \frac{\frac{\tilde{D}(\eta_2 c_2) - 1}{-\eta_2 c_2} + \frac{\tilde{U}(-\eta_2(c_1-c_2)) - 1}{\eta_2(c_1 - c_2)} h_2}{\frac{h_2 c_1(\alpha+\beta)}{\beta(c_1(\alpha+\beta) - r\beta)} \lim_{x\to\infty}\left\{\frac{\int_x^\infty e^{\eta_2(c_1-c_2)y}\,dU(y)}{\int_x^\infty e^{\eta_2(c_1-c_2)x}\,dU(y)}\right\}}, \tag{112}$$

$$C_{2*} = \frac{\frac{\tilde{D}(\eta_2 c_2) - 1}{-\eta_2 c_2} + \frac{\tilde{U}(-\eta_2(c_1-c_2)) - 1}{\eta_2(c_1 - c_2)} h_2}{\frac{h_2 c_1(\alpha+\beta)}{\beta(c_1(\alpha+\beta) - r\beta)}\tilde{U}(-\eta_2(c_1 - c_2))}. \tag{113}$$

7.4 Tandem Buffers - Markov Modulated On-Off Sources

Consider the tandem buffers model in Figure 7. Input to the first buffer is from N independent and identical exponential on-off sources with on-time parameter α, off-time parameter β and rate r. The output from buffer 1 is directly fed into buffer 2. The output capacities of buffer 1 and 2 are c_1 and c_2 respectively. We study the limiting distributions of the contents of the two buffers.

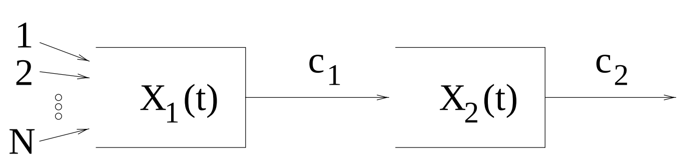

Figure 7: Tandem buffers model with multiple sources

Buffer 1: Let $Z_1(t)$ be the number of sources that are in the on state at time t. Clearly $\{Z_1(t), t \geq 0\}$ is an SMP (more specifically, a CTMC). Assume,

$$\frac{Nr\beta}{\alpha+\beta} < c_1 < Nr.$$

We can show that (see Gautam [14]) $\Phi(\delta)$ is given by

$$\phi_{ij}(\delta) = \begin{cases} \frac{i\alpha}{i\alpha+(N-i)\beta-(ir-c_1)\delta} & \text{if } j = i-1 \\ \frac{(N-i)\beta}{i\alpha+(N-i)\beta-(ir-c_1)\delta} & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases},$$

and $e(\Phi(\delta)) = 1$ always has solutions. Using the expression for $eb(v)$ in Equation (108) and solving for η_1 in $N\ eb(\eta_1) = c_1$ we get

$$\eta_1 = \frac{N(c_1\alpha + c_1\beta - N\beta r)}{c_1(Nr - c_1)}.$$

The eigenvectors are obtained by solving

$$h = h\Phi(\eta_1).$$

The limiting distribution of the buffer-content process $\{X_1(t) t \geq 0\}$ is given by

$$C_{1*}e^{-\eta_1 x} \leq P\{X > x\} \leq C_1^* e^{-\eta_1 x},$$

where

$$C_1^* = \frac{\sum_{i=0}^N \frac{h_i}{\eta_1(ir-c_1)}(\sum_{j=0}^N(\phi_{ij}(\eta_1)) - 1)}{\min_{i:ir>c_1} \frac{h_i}{p_i} \frac{1}{i\alpha + (N-i)\beta - \eta_1(ir-c_1)}},$$

$$C_{1*} = \frac{\sum_{i=0}^N \frac{h_i}{\eta_1(ir-c_1)}(\sum_{j=0}^N(\phi_{ij}(\eta_1)) - 1)}{\max_{i:ir>c_1} \frac{h_i}{p_i} \frac{1}{i\alpha + (N-i)\beta - \eta_1(ir-c_1)}},$$

and

$$p_i = \frac{a_i\tau_i}{\sum_{m=0}^N a_m\tau_m} = \frac{N!}{i!(N-i)!}\frac{\alpha^{N-i}\beta^i}{(\alpha+\beta)^N}.$$

Buffer 2: Let $M = \lceil \frac{c_1}{r} \rceil$. Define

$$Z_2(t) = \begin{cases} Z_1(t) & \text{if } X_1(t) = 0 \\ M & \text{if } X_1(t) > 0, \end{cases} \tag{114}$$

where $Z_1(t)$ is the number of sources on at time t. Let $R_1(t)$ be the output rate from the first buffer at time t. We assume that

$$\frac{Nr\beta}{\alpha+\beta} < c_2 < c_1.$$

We can see that the $\{Z_2(t), t \geq 0\}$ process is an SMP on state space $\{0, 1, \ldots, M\}$ with kernel

$$G(t) = [G_{ij}(t)]$$

derived below. For $i = 0, 1, \ldots, M-1$ and $j = 0, 1, \ldots, M$, let

$$G_{ij}(t) = \begin{cases} \frac{i\alpha}{i\alpha+(N-i)\beta}\left(1 - \exp\{-(i\alpha + (N-i)\beta)t\}\right) & \text{if } j = i-1 \\ \frac{(N-i)\beta}{i\alpha+(N-i)\beta}\left(1 - \exp\{-(i\alpha + (N-i)\beta)t\}\right) & \text{if } j = i+1 \\ 0 & \text{otherwise .} \end{cases}$$

To describe $G_{Mj}(t)$, we need to define the first passage time in $\{X_1(t), t \geq 0\}$ process as follows :

$$T = \min\{t > 0 : X_1(t) = 0\}.$$

Then for $j = 0, 1, \ldots, M-1$, we have

$$G_{Mj}(t) = P\{T \leq t, Z_1(T) = j | X_1(0) = 0, Z_1(0) = M\}.$$

Note that $G_{MM}(t) = 0$. The Laplace Stieltjes transform (LST) of $G_{Mj}(t)$ can be computed using the analysis in Narayanan and Kulkarni [27]. See Gautam [14] for a detailed derivation.

From Kulkarni and Gautam [22] we have the effective bandwidth of the output from Buffer 1, $eb_2(v)$, given by

$$eb_2(v) = \begin{cases} N\ eb_1(v) & \text{if } 0 \leq v \leq v^* \\ (N\ eb_1(v^*) - c)\frac{v^*}{v} + c & \text{if } v > v^*, \end{cases}$$

where $eb_1(v)$ is from Equation (108) and

$$v^* = \frac{\beta}{r}\left(\sqrt{\frac{c_1\alpha}{\beta(Nr - c_1)}} - 1\right) + \frac{\alpha}{r}\left(1 - \sqrt{\frac{\beta(Nr - c_1)}{c_1\alpha}}\right). \tag{115}$$

Hence solving

$$eb_2(\eta_2) = c_2,$$

we get

$$\eta_2 = \min\left\{\frac{N(c_2\alpha + c_2\beta - N\beta r)}{c_2(Nr - c_2)}, \frac{h(v^*) - c_1 v^*}{c_2 - c_1}\right\},$$

where

$$h(v^*) = \frac{\left(rv^* - \alpha - \beta + \sqrt{(rv^* - \alpha - \beta)^2 + 4\beta r v^*}\right)N}{2}. \tag{116}$$

If $\eta_2 \leq v^*$, then

$$\phi_{ij}(\eta_2) = \begin{cases} \tilde{G}_{ij}(-\eta_2(ir - c_2)) & \text{if } 0 \leq i \leq M-1, \\ \tilde{G}_{ij}(-\eta_2(c_1 - c_2)) & \text{if } i = M \end{cases}$$

and $e(\Phi(\eta_2)) = 1$. We obtain h by solving

$$h\Phi(\eta_2) = h.$$

It can be shown that the random variables associated with the distribution $G_{Mj}(x)/G_{Mj}(\infty)$ have a decreasing failure rate. Hence from Theorem 7, $\Psi_{min}(M, j)$ and $\Psi_{max}(M, j)$ occur at $x = \infty$ and $x = 0$ respectively. Thus using Theorem 1, we can find bounds for the steady-state distribution of the buffer-content process $\{X_2(t), t \geq 0\}$.

In Figure 8 we illustrate the upper and lower bounds on the limiting distribution of the buffer-content process

$$\lim_{t\to\infty} P\{X_2(t) > x\} = P\{X_2 > x\}$$

for a numerical example with $\alpha = 1$, $\beta = 0.3$, $r = 1$, $c_1 = 13.22$, $c_2 = 10.71$ and $N = 16$.

In a forthcoming paper, we will discuss the case $\eta_2 > v^*$ (when $e(\Phi(v)) = 1$ has no solutions) and use the results to solve Quality of Service problems in multipriority networks.

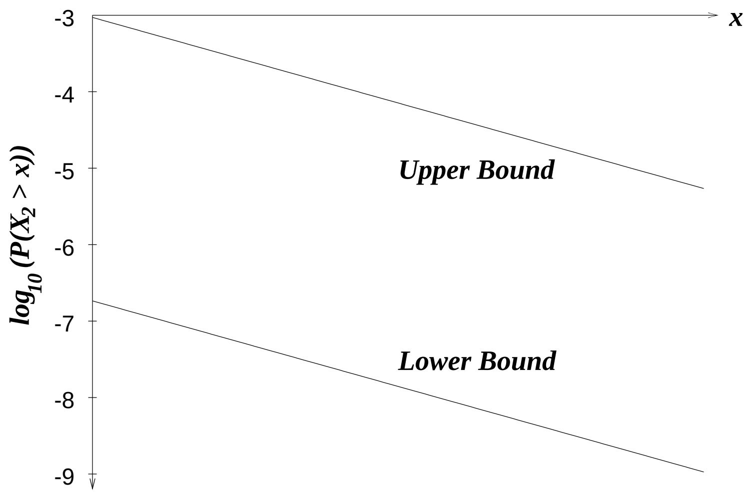

Figure 8: The upper and lower bounds as a function of x

8 Conclusions And Extensions

In this paper we have derived upper and lower bounds on the steady-state distribution of the buffer-content process (with constant output capacity) whose input is modulated by a single SMP. Unlike the effective-bandwidth approximation where the results are valid only in the asymptotic case of large buffers and small tail probabilities, the SMP bounds are valid for all buffer sizes and tail probabilities. We also illustrated how to compute the SMP bounds and obtain closed form algebraic expressions for certain special cases. We also reported the results (upper and lower bounds) using several examples.

We have also derived the bounds for the buffer-content process when the input to the buffer is generated from several sources, each modulated by an independent SMP. Since the result is very similar to the single SMP source, we do not go into the details of the proof of the theorem. However we conclude that the only significant difference is the minimization or the maximization being done over a slightly different set. We consider examples and applications of Theorem 6 in detail in a few other papers dealing with multiplexing multi-class traffic in high-speed telecommunication networks.

It may be possible to extend the SMP bounds result to more general sources like MRGP sources and general Markovian sources, and with time-varying output capacities.

We also believe that $P(X > x) \sim Ce^{-\eta x}$ for some constant $C > 0$ (perhaps under the mild conditions), however this problem for more than one stream seems to be very difficult. In fact, we think that $W(\tau(x)) \stackrel{\mathcal{D}}{\to} W^0$ for some variable W^0. Then from (96) and fact that function v is bounded and continuous we would obtain the needed assertion.

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