Computation of Warranty Reserves for Non-Stationary Sales Processes

Vidyadhar G. Kulkarni
and
Shau-Shiang Ja
Department of Operations Research
University of North Carolina
Chapel Hill, N.C. 27599-3180
kulkarni@or.unc.edu and ja@or.unc.edu

August 1, 1997

Abstract

Warranty reserve analysis is very important for the producers because during the life cycle of the product they have to set aside a fixed sum of money in the bank for paying all the replacement or repair costs of the products which fail during the warranty period. In this paper, we first derive the first and second moments of the producer's total discounted warranty cost of single sale for single-component items under four different kinds of warranty policies: renewable free-replacement, non-renewable free-replacement, pro rata, and non-renewable minimal-repair warranty policy. By using these results we also compute the mean and variance of the producer's total discounted warranty cost of the total sales over the product life cycle. Finally, we use these quantities to derive the level of warranty reserve by using the Normal approximation. We illustrate with several simple examples.

1 Introduction

In everyday life, we frequently purchase goods which include a warranty. A warranty could be offered for goods as expensive as automobiles or as cheap as light bulbs, even though we usually don't pay much attention to the warranty on cheap items.

A warranty is a contract between the producer and the consumer to guarantee a satisfactory performance of the item during a fixed period of time. It doesn't just define the right of the consumer, but also defines the responsibility and liability of the producer. Since the producer has to replace or repair the failed items during the warranty period, a warranty costs money. Even though sometimes the warranty cost is not very large, it does affect the producer's revenue. From the consumer's point of view, the question is whether it is worth paying for the warranty, if there is a choice. ¿From the above description, we can see that the warranty cost analysis is important for both the producer and the consumer.

Warranty reserve are the funds which the producer sets aside at the beginning of the sales period for the warranty cost of all future sales. The funds in the reserve account can also earn interest. The amount of funds in the warranty reserve fluctuates with time, increasing due to accrued interest, and decreasing whenever a warranty claim arises. The aim of the producer is to start the production cycle with enough warranty reserve to cover the future liability costs over the product life cycle with a given probability.

Several authors have considered the problem of warranty reserve. Menke[9] estimates the total warranty reserve required for a fixed lot size of units which have exponential failure time distribution. Amato and Anderson[1] extend Menke's model to allow for discounting and price adjustments. Patankar and Worms[14] derive prediction intervals for warranty reserves and cash flow. Tapiero and Posner[15] present an alternative approach to modeling of warranty reserves. Thomas[16] finds the expected warranty reserve cost per unit with discounting for failure distributions that included the uniform, gamma, and Weibull. Eliashberg, Singpurwalla, and Wilson[5] consider the problem of assessing the size of a reserve needed by the manufacturer to meet future claims for a two-dimensional warranty.

In this paper, we develop a more accurate method to estimate the warranty reserve. We compute the first and seconds moments of the total discounted warranty cost of single sale for four different warranty policies in section 2. In section 3, we assume that the sales over $[0, L]$ occur according a nonhomogeneous Poisson process. Each item sold is covered by a warranty over a period of length W starting from the time of sales. The actual warranty policy can be any one of the four analyzed in section 2. Using the results of section 2, we compute the mean and variance of the total discounted warranty cost of all the items sold in $[0, L]$ in section 3. The cash flow is analyzed through the total discounted warranty cost of the aggregate sales in section 4. By using the Normal approximation, we get the warranty reserve if the risk probability which the producer want to take is given.

We first define some terminology we use in this paper. Warranty period W is the fixed period of time during which the producer has the responsibility to replace or repair the failed items. Life cycle L is the period over which the producer continues to supply the item. In this paper we consider single-component items, that is, once the component fails, the whole item fails. The lifetimes of the item are assumed to be i.i.d. with common cdf $F(\cdot)$, pdf $f(\cdot)$, and failure rate $r(t) = f(t)/[1 - F(t)]$. Let $M(\cdot)$ be the renewal function associated with $F(\cdot)$, and $m(t) = dM(t)/dt$ be the renewal density. Suppose an item is sold at time $t \in [0, L]$. Then it is covered under warranty for the period $[t, t + W]$. However, the precise costs to the producer differ under different warranty policies. We assume that the item can be replaced or repaired back to an operating condition after a failure and the repair time is zero because in comparison to the warranty period and life cycle of the item it's relatively very small. We also assume that the consumers keep claiming the warranties until the warranties expire. The repair or replacement cost of the ith failure, D_i, is a random variable. All D_is are i.i.d., and are independent of the failure times. Lastly, let $\alpha > 0$ be the continuous discount factor, i.e. the present value of cost $\$C$ incurred at time t is $\$Ce^{-\alpha t}$.

2 Analysis: Single Sale

In this section, we compute the first and second moments of the producer's total discounted warranty cost of single sale. These quantities are needed in the analysis of the warranty costs for the aggregate sales. We analyze four different warranty policies in the next four subsections. Let $Z(W)$ be the total discounted warranty cost of single sale with a warranty period W.

2.1 Renewable Free-Replacement Warranty Policy

Under the free-replacement warranty policy, the producer has to supply as many replacements as needed during the warranty period and pay all the cost. In this subsection we deal with the case that the warranty period is renewed after each replacement. Thus if the consumer purchases the item at time 0 and the item fails at time $x < W$, the producer has to provide a replacement which comes with a new full warranty period W. We shall use the following notation:

$$\tilde{F}(\alpha, W) = \int_0^W e^{-\alpha x} dF(x). \tag{1}$$

Theorem 1 : *Under renewable free-replacement warranty policy,*

$$E[Z(W)] = \frac{E[D]\tilde{F}(\alpha, W)}{1 - \tilde{F}(\alpha, W)}, \tag{2}$$

and

$$E[Z^2(W)] = \frac{E[D^2](1 - \tilde{F}(\alpha, W))\tilde{F}(2\alpha, W) + 2E^2[D]\tilde{F}(\alpha, W)\tilde{F}(2\alpha, W)}{(1 - \tilde{F}(\alpha, W))(1 - \tilde{F}(2\alpha, W))}. \tag{3}$$

Proof. : By conditioning on the time of the first failure, X_1, and first replacement cost, D_1, we have

$$E[Z(W) \mid X_1 = x, D_1 = d] = \begin{cases} de^{-\alpha x} + e^{-\alpha x} E[Z(W)] & \text{if } 0 < x \leq W \\ 0 & \text{if } x > W \end{cases} \tag{4}$$

which yields

$$\begin{aligned} E[Z(W)] &= E[D] \int_0^W e^{-\alpha x} dF(x) + \int_0^W e^{-\alpha x} E[Z(W)] dF(x). \\ &= E[D]\tilde{F}(\alpha, W) + E[Z(W)]\tilde{F}(\alpha, W). \end{aligned} \tag{5}$$

Rearranging, we get equation (2). The second moment can be derived by the same kind of renewal argument. We have

$$E[Z^2(W) \mid X_1 = x, D_1 = d] = \begin{cases} E[(de^{-\alpha x} + e^{-\alpha x}Z(W))^2] & \text{if } 0 < x \le W \\ 0 & \text{if } x > W \end{cases} \tag{6}$$

$$= \begin{cases} e^{-2\alpha x}E[d^2 + 2dZ(W) + Z^2(W)] & \text{if } 0 < x \le W \\ 0 & \text{if } x > W \end{cases} \tag{7}$$

which yields

$$\begin{aligned} E[Z^2(W)] &= E[D^2]\int_0^W e^{-2\alpha x}dF(x) + 2E[D]E[Z(W)]\int_0^W e^{-2\alpha x}dF(x) \\ &\quad + \int_0^W e^{-2\alpha x}E[Z^2(W)]dF(x) \qquad (8) \\ &= E[D^2]\tilde{F}(2\alpha, W) + 2E[D]E[Z(W)]\tilde{F}(2\alpha, W) \\ &\quad + E[Z^2(W)]\tilde{F}(2\alpha, W) \qquad (9) \end{aligned}$$

Therefore,

$$E[Z^2(W)] = \frac{E[D^2]\tilde{F}(2\alpha, W) + 2E[D]E[Z(W)]\tilde{F}(2\alpha, W)}{1 - \tilde{F}(2\alpha, W)}. \tag{10}$$

We substitute $E[Z(W)]$ from equation (2) into the above equation and get equation (3).

■

We illustrate the above theorem with a simple example.

Example 1 : Suppose the lifetime of the product is an exponential random variable with mean $\frac{1}{\theta}$, i.e.

$$F(t) = 1 - e^{-\theta t}.$$

Using equation (1), we get

$$\tilde{F}(\alpha, W) = \frac{\theta}{\alpha + \theta}(1 - e^{-(\alpha+\theta)W}).$$

¿From Theorem 1, we get

$$\begin{aligned} E[Z(W)] &= \frac{E[D]\frac{\theta}{\alpha+\theta}(1 - e^{-(\alpha+\theta)W})}{1 - \frac{\theta}{\alpha+\theta}(1 - e^{-(\alpha+\theta)W})} \\ &= \frac{E[D]\theta(1 - e^{-(\alpha+\theta)W})}{\alpha + \theta e^{-(\alpha+\theta)W}}. \qquad (11) \end{aligned}$$

$$\begin{aligned} E[Z^2(W)] &= \frac{E[D^2][1 - \frac{\theta}{\alpha+\theta}(1 - e^{(\alpha+\theta)W})]\frac{\theta}{2\alpha+\theta}(1 - e^{(2\alpha+\theta)W}) + 2E^2[D]\frac{\theta}{\alpha+\theta}(1 - e^{(\alpha+\theta)W})\frac{\theta}{2\alpha+\theta}(1 - e^{(2\alpha+\theta)W})}{[1 - \frac{\theta}{\alpha+\theta}(1 - e^{(\alpha+\theta)W})][1 - \frac{\theta}{2\alpha+\theta}(1 - e^{(2\alpha+\theta)W})]} \\ &= \frac{E[D^2][(\alpha+\theta) - \theta(1 - e^{(\alpha+\theta)W})](1 - e^{(2\alpha+\theta)W}) + 2E^2[D]\theta^2(1 - e^{(\alpha+\theta)W})(1 - e^{(2\alpha+\theta)W})}{[(\alpha+\theta) - \theta(1 - e^{(\alpha+\theta)W})][(2\alpha+\theta) - \theta(1 - e^{(2\alpha+\theta)W})]} \qquad (12) \end{aligned}$$

■

2.2 Non-Renewable Free-Replacement Warranty Policy

Now we consider a free-replacement warranty that is not renewed after each replacement. That is, if the consumer purchases the item at time 0 and the item fails at time $x < W$ then the provider has to provide a free replacement which comes with the warranty period $W - x$. We compute $E[Z(W)]$ and $E[Z^2(W)]$ in the Theorem 2. We need the following two lemmas in the proof of that theorem.

Lemma 1 *:*

$$\int_0^W e^{-\alpha x} dF(x) + \int_{t=0}^W \int_{x=0}^{W-t} e^{-\alpha(t+x)} dF(x) dM(t) = \int_0^W e^{-\alpha t} dM(t). \tag{13}$$

Proof. :

$$\begin{aligned}
\int_{t=0}^W \int_{x=0}^{W-t} e^{-\alpha(t+x)} dF(x) dM(t) &= \int_{t=0}^W \int_{u=t}^W e^{-\alpha u} dF(u-t) dM(t) & (14)\\
&= \int_{u=0}^W e^{-\alpha u} \int_{t=0}^u dF(u-t) dM(t) & (15)\\
&= \int_{u=0}^W e^{-\alpha u} [dM(u) - dF(u)] & (16)\\
&= \int_{u=0}^W e^{-\alpha u} dM(u) - \int_{u=0}^W e^{-\alpha u} dF(u) & (17)
\end{aligned}$$

where the first equality comes from letting $u = t + x$ and the second one is gotten by using

$$dM(u) = dF(u) + \int_0^u dM(u-t) dF(t). \tag{18}$$

Therefore, we get equation (13). ■

Lemma 2 *:*

$$\begin{aligned}
&\int_{x=0}^W \int_{t=0}^{W-x} e^{-2\alpha x} e^{-\alpha t} dM(t) dF(x) + \int_{u=0}^W \int_{x=0}^{W-u} \int_{t=0}^{W-u-x} e^{-2\alpha(u+x)} e^{-\alpha t} dM(t) dF(x) dM(u)\\
&= \int_{v=0}^W \int_{u=v}^W e^{-\alpha(u+v)} dM(u-v) dM(v). \qquad (19)
\end{aligned}$$

Proof. :

$$\begin{aligned}
&\int_{u=0}^W \int_{x=0}^{W-u} \int_{t=0}^{W-u-x} e^{-2\alpha(u+x)} e^{-\alpha t} dM(t) dF(x) dM(u)\\
&= \int_{u=0}^W \int_{v=u}^W \int_{t=0}^{W-v} e^{-2\alpha v} e^{-\alpha t} dM(t) dF(v-u) dM(u) \qquad (20)
\end{aligned}$$

$$\begin{aligned}
&= \int_{v=0}^{W}\int_{u=0}^{v}\int_{t=0}^{W-v} e^{-2\alpha v}e^{-\alpha t}dM(t)dF(v-u)dM(u) && (21)\\
&= \int_{v=0}^{W}\int_{t=0}^{W-v}\int_{u=0}^{v} e^{-2\alpha v}e^{-\alpha t}dM(t)dF(v-u)dM(u) && (22)\\
&= \int_{v=0}^{W}\int_{t=0}^{W-v} e^{-2\alpha v}e^{-\alpha t}dM(t)[\int_{u=0}^{v} dF(v-u)dM(u)] && (23)\\
&= \int_{v=0}^{W}\int_{t=0}^{W-v} e^{-2\alpha v}e^{-\alpha t}dM(t)dM(v) - \int_{v=0}^{W}\int_{t=0}^{W-v} e^{-2\alpha v}e^{-\alpha t}dM(t)dF(v) && (24)\\
&= \int_{v=0}^{W}\int_{u=v}^{W} e^{-\alpha(u+v)}dM(u-v)dM(v) - \int_{v=0}^{W}\int_{t=0}^{W-v} e^{-2\alpha v}e^{-\alpha t}dM(t)dF(v) && (25)\\
& && (26)
\end{aligned}$$

where the first equality comes from letting $v = x+u$, the fifth one is from equation (18), and the last one is gotten by letting $t = u - v$. Therefore, we get equation (13). ■

Theorem 2 : *Under non-renewable free-replacement warranty policy,*

$$E[Z(W)] = E[D]\int_0^W e^{-\alpha t}dM(t), \tag{27}$$

and

$$\begin{aligned}
E[Z^2(W)] &= 2E^2[D]\int_{v=0}^{W}\int_{u=v}^{W} e^{-\alpha(u+v)}dM(u-v)dM(v)\\
&\quad +E[D^2]\int_0^W e^{-2\alpha u}dM(u).
\end{aligned} \tag{28}$$

Proof. : By conditioning on the time of the first failure, X_1, and first replacement cost, D_1, we get

$$E[Z(W) \mid X_1 = x, D_1 = d] = \begin{cases} de^{-\alpha x} + e^{-\alpha x}E[Z(W-x)] & \text{if } 0 < x \le W \\ 0 & \text{if } x > W \end{cases} \tag{29}$$

which yields

$$E[Z(W)] = E[D]\int_0^W e^{-\alpha x}dF(x) + \int_0^W e^{-\alpha x}E[Z(W-x)]dF(x). \tag{30}$$

By multiplying $e^{\alpha W}$ to both sides of the above equation, we have

$$e^{\alpha W}E[Z(W)] = E[D]\int_0^W e^{\alpha(W-x)}dF(x) + \int_0^W e^{\alpha(W-x)}E[Z(W-x)]dF(x). \tag{31}$$

This is a standard renewal equation for $e^{\alpha W}E[Z(W)]$. Hence from Theorem 8.10 in Kulkarni[6], we get the following solution

$$e^{\alpha W}E[Z(W)] = E[D]\int_0^W e^{\alpha(W-x)}dF(x) + \int_0^W E[D]\int_0^{W-t} e^{\alpha(W-t-x)}dF(x)dM(t). \tag{32}$$

Therefore,

$$E[Z(W)] = E[D][\int_0^W e^{-\alpha x} dF(x) + \int_0^W \int_0^{W-t} e^{-\alpha(t+x)} dF(x)dM(t)]. \quad (33)$$

By using Lemma 1, we get equation (27). The second moment can be also derived by the renewal argument as the above. We have

$$E[Z^2(W) \mid X_1 = x, D_1 = d] = \begin{cases} E[(de^{-\alpha x} + e^{-\alpha x} Z(W-x))^2] & \text{if } 0 < x \le W \\ 0 & \text{if } x > W \end{cases} \quad (34)$$

$$= \begin{cases} e^{-2\alpha x} E[d^2 + 2dZ(W-x) + Z^2(W-x)] & \text{if } 0 < x \le W \\ 0 & \text{if } x > W \end{cases} \quad (35)$$

which yields

$$\begin{aligned} E[Z^2(W)] &= E[D^2] \int_0^W e^{-2\alpha x} dF(x) + 2E[D] \int_0^W e^{-2\alpha x} E[Z(W-x)] dF(x) \\ &+ \int_0^W e^{-2\alpha x} E[Z^2(W-x)] dF(x). \end{aligned} \quad (36)$$

By getting $E[Z(W-x)]$ from equation (27) and substituting it in equation (36), we get

$$\begin{aligned} E[Z^2(W)] &= E[D^2] \int_0^W e^{-2\alpha x} dF(x) + 2E^2[D] \int_0^W e^{-2\alpha x} \int_0^{W-x} e^{-\alpha t} dM(t) dF(x) \\ &+ \int_0^W e^{-2\alpha x} E[Z^2(W-x)] dF(x). \end{aligned} \quad (37)$$

By multiplying $e^{2\alpha W}$ to both sides of the above equation, we have

$$\begin{aligned} e^{2\alpha W} E[Z^2(W)] &= E[D^2] \int_0^W e^{2\alpha(W-x)} dF(x) + 2E^2[D] \int_0^W \int_0^{W-x} e^{2\alpha(W-x)} e^{-\alpha t} dM(t) dF(x) \\ &+ \int_0^W e^{2\alpha(W-x)} E[Z^2(W-x)] dF(x). \end{aligned} \quad (38)$$

This is a standard renewal equation for $e^{\alpha W} E[Z^2(W)]$. Hence from Theorem 8.10 in Kulkarni[6], we get the following solution

$$\begin{aligned} e^{2\alpha W} E[Z^2(W)] &= E[D^2] \int_0^W e^{2\alpha(W-x)} dF(x) + 2E^2[D] \int_{x=0}^W \int_{t=0}^{W-x} e^{2\alpha(W-x)} e^{-\alpha t} dM(t) dF(x) \\ &+ \int_0^W E[D^2] \int_0^{W-u} e^{2\alpha(W-u-x)} dF(x) dM(u) \\ &+ 2E^2[D] \int_{u=0}^W \int_{x=0}^{W-u} \int_{t=0}^{W-u-x} e^{2\alpha(W-u-x)} e^{-\alpha t} dM(t) dF(x) dM(u). \end{aligned} \quad (39)$$

Therefore,

$$\begin{aligned} E[Z^2(W)] &= E[D^2][\int_0^W e^{-2\alpha x} dF(x) + \int_0^W \int_0^{W-u} e^{-2\alpha(u+x)} dF(x) dM(u)] \\ &+ 2E^2[D][\int_{x=0}^W \int_{t=0}^{W-x} e^{-2\alpha x} e^{-\alpha t} dM(t) dF(x) \\ &+ \int_{u=0}^W \int_{x=0}^{W-u} \int_{t=0}^{W-u-x} e^{-2\alpha(u+x)} e^{-\alpha t} dM(t) dF(x) dM(u)]. \end{aligned} \quad (40)$$

By using Lemma 1 and 2, we get equation (28) ■

Remark : Theorem 2 can be also derived by an alternative method. At first we discretize the time horizon $[0, W]$ by N equal small intervals. We assume that N is large enough so that we can ignore the possibility of more than one failure incurring in a given interval of length $\frac{W}{N}$. Let

$$Y(n) = \begin{cases} 1 & \text{if there is a failure during } (\frac{(n-1)W}{N}, \frac{nW}{N}] \\ 0 & \text{otherwise} \end{cases} \tag{41}$$

$$Z(W) = \sum_{n=1}^{N} e^{-\alpha n} Y(n) D_n, \tag{42}$$

$$\begin{aligned} E[Z(W)] &= E[\sum_{1}^{N} e^{-\alpha n} Y(n) D_n] \\ &= E[D] \sum_{1}^{N} e^{-\alpha n} E[Y(n)] \\ &= E[D] \sum_{1}^{N} e^{-\alpha n} [M(\frac{nW}{N}) - M(\frac{(n-1)W}{N})] \end{aligned} \tag{43}$$

Letting $N \to \infty$, the above equation reduces to equation (27). Next we derive the second moment. We have

$$Z^2(W) = \sum_{n=1}^{N} \sum_{i=1}^{N} e^{-\alpha(n+i)} Y(n) Y(i) D_n D_i, \tag{44}$$

$$\begin{aligned} E[Z^2(W)] &= \sum_{n=1}^{N} \sum_{i=1}^{N} e^{-\alpha(n+i)} E[Y(n)Y(i) D_n D_i] \\ &= 2E^2[D] \sum_{n=1}^{N} \sum_{i=n+1}^{N} e^{-\alpha(n+i)} E[Y(n)Y(i)] + E[D^2] \sum_{n=1}^{N} e^{-2\alpha n} E[Y(n)] \\ &= 2E^2[D] \sum_{n=1}^{N} \sum_{i=n+1}^{N} e^{-\alpha(n+i)} [M(\frac{nW}{N}) - M(\frac{(n-1)W}{N})] \\ &\quad \times [M(\frac{(i-n)W}{N}) - M(\frac{(i-n-1)W}{N})] \\ &\quad + E[D^2] \sum_{n=1}^{N} e^{-2\alpha n} [M(\frac{nW}{N}) - M(\frac{(n-1)W}{N})]. \end{aligned} \tag{45}$$

Again, letting $N \to \infty$, the above equation reduces to equation (28). ■

We illustrate the above theorem with a simple example.

Example 2: Consider the lifetime distribution as in Example 1. It is known that

$M(t) = \theta t$. From Theorem 2,

$$\begin{aligned} E[Z(W)] &= E[D] \int_0^W e^{-\alpha t} \theta dt \\ &= \frac{\theta E[D](1 - e^{-\alpha W})}{\alpha}. \end{aligned} \tag{46}$$

$$\begin{aligned} E[Z^2(W)] &= 2E^2[D] \int_0^W \int_u^W e^{-\alpha(u+v)} \theta^2 dv du + E[D^2] \int_0^W e^{-2\alpha u} \theta du \\ &= \frac{E^2[D]\theta^2(1 - e^{-\alpha W})^2}{\alpha^2} + \frac{E[D^2]\theta(1 - e^{-2\alpha W})}{2\alpha}. \end{aligned} \tag{47}$$

■

2.3 Pro Rata Warranty Policy

The pro rata warranty is usually offered for consumer durables, especially for non-repairable items. Let d be the price of the item. Under this kind of warranty the consumer has to pay the full price, d, to get a new one if the item fails after the warranty period, W, or he just has to pay a partial price, prorated to the age of the item when it fails, if the item fails before W. That is, if the item fails at time x and $x < W$, then the consumer buys a new item for price $d\frac{x}{W}$. Thus the producer's warranty cost of this replacement is $d(1 - \frac{x}{W})$. Since the consumer pays for the part he/she used, the warranty period W is renewed after each failure. The next theorem gives the first and second moment of $Z(W)$. We use the notation $\tilde{F}(\alpha, W)$ from equation (1).

Theorem 3 : *Under pro rata warranty policy,*

$$E[Z(W)] = \frac{d \int_0^W (1 - \frac{x}{W}) e^{-\alpha x} dF(x)}{1 - \tilde{F}(\alpha, W)}, \tag{48}$$

and

$$E[Z^2(W)] =$$

$$\frac{d^2(1 - \tilde{F}(\alpha, W)) \int_0^W (1 - \frac{x}{W})^2 e^{-2\alpha x} dF(x) + 2d^2 \int_0^W (1 - \frac{x}{W}) e^{-\alpha x} dF(x) \int_0^W (1 - \frac{x}{W}) e^{-2\alpha x} dF(x)}{(1 - \tilde{F}(\alpha, W))(1 - \tilde{F}(2\alpha, W))}. \tag{49}$$

Proof. : By conditioning on the time of the first failure, X_1, we have

$$E[Z(W) \mid X_1 = x] = \begin{cases} d(1 - \frac{x}{W}) e^{-\alpha x} + e^{-\alpha x} E[Z(W)] & \text{if } 0 < x \le W \\ 0 & \text{if } x > W \end{cases} \tag{50}$$

which yields

$$E[Z(W)] = d\int_0^W (1-\frac{x}{W})e^{-\alpha x}dF(x) + \int_0^W e^{-\alpha x}E[Z(W)]dF(x). \quad (51)$$

Rearranging, we get equation (48). To derive the second moment, we use a similar renewal argument. We have

$$E[Z^2(W) \mid X_1 = x] = \begin{cases} E[((1-\frac{x}{W})de^{-\alpha x} + e^{-\alpha x}Z(W))^2] & \text{if } 0 < x \le W \\ 0 & \text{if } x > W \end{cases} \quad (52)$$

$$= \begin{cases} e^{-2\alpha x}E[((1-\frac{x}{W})d)^2 + 2(1-\frac{x}{W})dZ(W) + Z^2(W)] & \text{if } 0 < x \le W \\ 0 & \text{if } x > W \end{cases} \quad (53)$$

which yields

$$E[Z^2(W)] = d^2\int_0^W (1-\frac{x}{W})^2 e^{-2\alpha x}dF(x) + 2dE[Z(W)]\int_0^W (1-\frac{x}{W})e^{-2\alpha x}dF(x) + \int_0^W e^{-2\alpha x}E[Z^2(W)]dF(x). \quad (54)$$

Therefore,

$$E[Z^2(W)] = \frac{d^2\int_0^W (1-\frac{x}{W})^2 e^{-2\alpha x}dF(x) + 2dE[Z(W)]\int_0^W (1-\frac{x}{W})e^{-2\alpha x}dF(x)}{1-\tilde{F}(2\alpha, W)}. \quad (55)$$

We can substitute $E[Z(W)]$ into the above equation and get equation (49). ■

We illustrate the above theorem with a simple example.

Example 3: Consider the lifetime distribution as in Example 1. It is known that

$$\tilde{F}(\alpha, W) = \frac{\theta}{\alpha+\theta}(1-e^{-(\alpha+\theta)W}),$$

$$\int_0^W (1-\frac{x}{W})e^{-\alpha x}dF(x) = \tilde{F}(\alpha, W) + \frac{1}{W}\frac{d}{d\alpha}\tilde{F}(\alpha, W),$$

and

$$\int_0^W (1-\frac{x}{W})^2 e^{-2\alpha x}dF(x) = \tilde{F}(2\alpha, W) + \frac{1}{W}\frac{d}{d\alpha}\tilde{F}(2\alpha, W) + \frac{1}{4W^2}\frac{d^2}{d^2\alpha}\tilde{F}(2\alpha, W).$$

By substituting in equation (48) and (49), we can get the results of this example. ■

2.4 Non-Renewable Minimal-Repair Warranty Policy

The minimal-repair warranty policy is usually offered for complex and expensive products since the repair usually involves only a small part of the product. Under this policy, the producer has to repair the failure happening during the warranty period $(0, W]$ and pay for the repair cost. However, the failure rate of the failed item remains unchanged after each repair. The warranty period W is not renewed after each repair. Let $r(t)$ be the failure rate.

Theorem 4 *: Under minimal-repair warranty policy with non-renewed W,*

$$E[Z(W)] = E[D]\int_0^W r(t)e^{-\alpha t}dt, \tag{56}$$

and

$$E[Z^2(W)] = E[D^2]\int_0^W r(t)e^{-2\alpha t}dt + E^2[D](\int_0^W r(t)e^{-\alpha t}dt)^2. \tag{57}$$

Proof. : Let $N(t)$ be the number of failures in $[0,t]$. It is known that $\{N(t), t \geq 0\}$ is a nonhomogeneous Poisson process with rate function $r(t)$ under this policy. Let

$$R(t) = \int_0^t r(s)ds. \tag{58}$$

Since $\{N(t), t \geq 0\}$ is a nonhomogeneous Poisson process, by Theorem 5.12 in Kulkarni[6], we know the distributions of the failure times given the number of failures. Therefore,

$$E[Z(W) \mid N(W) = 1, D_1 = d_1] = \int_0^W d_1 \frac{r(t)}{R(W)} e^{-\alpha t}dt \tag{59}$$

and

$$E[Z(W) \mid N(W) = n, D_1 = d_1, \cdots, D_n = d_n] = \sum_{i=1}^n \int_0^W d_i \frac{r(t)}{R(W)} e^{-\alpha t}dt. \tag{60}$$

Since $\{D_i, i = 1, \cdots\}$ are i.i.d., we have

$$E[Z(W) \mid N(W) = n] = nE[D]\int_0^W \frac{r(t)}{R(W)} e^{-\alpha t}dt. \tag{61}$$

We also know

$$Pr\{N(W) = n\} = e^{-R(W)}\frac{(R(W))^n}{n!}. \tag{62}$$

Hence, using equation (61) and (62), we get

$$E[Z(W)] = E[D]\sum_{n=0}^{\infty} e^{-R(W)}\frac{(R(W))^(n-1)}{(n-1)!}\int_0^W r(t)e^{-\alpha t}dt \tag{63}$$

which reduces to equation (56). The second moment can be derived similarly. We have

$$E[Z^2(W) \mid N(W) = 1, D_1 = d_1] = \int_0^W d_1^2 \frac{r(t)}{R(W)} e^{-2\alpha t}dt \tag{64}$$

and

$$E[Z^2(W) \mid N(W) = n, D_1 = d_1, \cdots, D_n = d_n]$$
$$= \sum_{i=1}^n \int_0^W d_i^2 \frac{r(t)}{\mathcal{R}(W)} e^{-2\alpha t}dt + 2\sum_{i=1}^n \sum_{j=i+1}^n (\int_0^W d_i \frac{r(t)}{\mathcal{R}(W)} e^{-\alpha t}dt)(\int_0^W d_j \frac{r(t)}{\mathcal{R}(W)} e^{-\alpha t}dt) \tag{65}$$

Therefore,

$$E[Z^2(W) \mid N(W) = n] = nE[D^2] \int_0^W \frac{r(t)}{R(W)} e^{-2\alpha t} dt + n(n-1)E^2[D] (\int_0^W \frac{r(t)}{R(W)} e^{-\alpha t} dt)^2. \tag{66}$$

By using equation (62) and unconditioning on $N(W)$, we get equation (57). ■

We illustrate the above theorem with a simple example.

Example 4: As in example 1, suppose the lifetime of the product is an exponential random variable with mean $\frac{1}{\theta}$. Then the failure rate is a constant θ. Substituting in equation (56) and (57), we get the results that match those the example 2. This is expected since minimal repair in an exponential case is same as replacement. ■

3 Analysis: Aggregate Sales

In this section, we use the single sale analysis in the previous section to build a model of multiple sales over a given time period. Let $\lambda(t)$ be the instantaneous selling rate at time t. Let $S(t)$ be the number of items sold during $[0, t]$. We assume that $\{S(t), t \geq 0\}$ is a nonhomogeneous Poisson process with rate function $\lambda(t)$. Let L be the life cycle of this item, i.e. the manufacture and sales of the item are discontinued after time L. However, the warranty liabilities of all items sold upto L are honored. Thus the manufacturer is free of warranty liabilities at time $L+W$ at the latest. Let $Z(W)$ be the total discounted warranty cost of single sale computed by using appropriate results from section 2 and $T(L)$ be the total discounted warranty cost of all the sales during a life cycle L. Here we compute the mean and variance of $T(L)$. The results are given in terms of $E[Z(W)]$ and $E[Z^2(W)]$.

Theorem 5 *:*

$$E[T(L)] = E[Z(W)] \int_0^L \lambda(t) e^{-\alpha t} dt, \tag{67}$$

and

$$Var(T(L)) = E[Z^2(W)] \int_0^L \lambda(t) e^{-2\alpha t} dt \tag{68}$$

Proof. : Since $\{S(t), t \geq 0\}$ is a nonhomogeneous Poisson process, we can derive the mean by the same method as in section 2.4. We condition on the number of item sold in $[0, L]$ and get

$$E[T(L) \mid S(L) = n] = n \frac{E[Z(W)]}{\Lambda(L)} \int_0^L \lambda(t) e^{-\alpha t} dt, n \geq 0, \tag{69}$$

where $\Lambda(L) = \int_0^L \lambda(s)ds$. We also know

$$Pr\{S(L) = n\} = e^{-\Lambda(L)} \frac{(\Lambda(L))^n}{n!}. \tag{70}$$

Unconditioning on $S(L)$, we get equation (67). Since we need the second moment of $T(L)$ to compute the variance, we derive $E[T^2(L)]$ first. We have

$$\begin{aligned} E[T^2(L) \mid S(L) = n] &= nE[Z^2(W)] \int_0^W \frac{\lambda(t)}{\Lambda(L)} e^{-2\alpha t} dt \\ &\quad + n(n-1) \frac{(E[Z(W)] \int_0^L \lambda(t) e^{-\alpha t} dt)^2}{\Lambda(L)}, n \geq 0. \end{aligned} \tag{71}$$

Therefore,

$$E[T^2(L)] = E[Z^2(W)] \int_0^L \lambda(t) e^{-2\alpha t} dt + E^2[Z(W)] (\int_0^L \lambda(t) e^{-\alpha t} dt)^2. \tag{72}$$

Substituting in $Var(T(L)) = E[T^2(L)] - E^2[T(L)]$, we get equation (68). ■

Corollary 1 *: For infinite horizon, i.e. $L \to \infty$, we have*

$$E[T(\infty)] = E[Z(W)] \lambda^*(\alpha) \tag{73}$$

and

$$Var(T(\infty)) = E[Z^2(W)] \lambda^*(2\alpha) \tag{74}$$

where $\lambda^*(\alpha)$ is the Laplace Transform of $\lambda(\cdot)$. ■

We illustrate the above theorem with a simple example.

Example 5: Suppose the selling rate is a constant λ. Each item sold carries a non-renewable minimal-repair warranty for a period of length W. From Theorem 5, we have

$$\begin{aligned} E[T(L)] &= E[Z(W)] \int_0^L \lambda e^{-\alpha t} dt \\ &= E[Z(W)] \frac{\lambda(1 - e^{-\alpha L})}{\alpha} \end{aligned} \tag{75}$$

and

$$\begin{aligned} Var(T(L)) &= E[Z^2(W)] \int_0^L \lambda e^{-2\alpha t} dt \\ &= E[Z^2(W)] \frac{\lambda(1 - e^{-2\alpha L})}{2\alpha} \end{aligned} \tag{76}$$

Furthermore, suppose the lifetime of the items are i.i.d. exponential random variables with mean $\frac{1}{\theta}$. Thus we use the results in Example 2 for $E[Z(W)]$ and $E[Z^2(W)]$. Substituting in equation (75) and (76), we get

$$E[T(L)] = \frac{\theta\lambda E[D](1-e^{-\alpha W})(1-e^{-\alpha L})}{\alpha^2} \tag{77}$$

and

$$Var(T(L)) = [\frac{E^2[D]\theta^2(1-e^{-\alpha W})^2}{\alpha^2} + \frac{E[D^2]\theta(1-e^{-2\alpha W})}{2\alpha}]\frac{\lambda(1-e^{-2\alpha L})}{2\alpha}. \tag{78}$$

■

4 Warranty Reserve

It is a common practice in industry to set aside a fixed sum of money, called warranty reserve, to cover the future warranty liabilities. Suppose we start with a warranty reserve of size x. Suppose the warranty reserve accrues interest continuously so that one dollar in the reserve at time 0 grows to $e^{\alpha t}$ dollars at time t, in the absence of any withdrawals. Let $X(t)$ be the amount of money in the reserve at time t, after accounting for all the warranty costs incurred upto time t and all the interest accrued upto time t. The next theorem relates $X(t)$ to $U(t)$, the discounted warranty cost incurred upto time t.

Theorem 6 *: Let x be the initial warranty reserve. Then*

$$X(t) = e^{\alpha t}[x - U(t)] \tag{79}$$

Proof. : Let X_i be the time between the $(i-1)$st and ith claim, S_n be the time at which the nth claim happens $(= \sum_{i=1}^{n} X_i)$, D_i be the cost of the ith claim, and $N(t)$ be the number of claims incurring in $[0, t]$. Now, if $N(t) = 1$, we have

$$\begin{aligned} X(t) &= (xe^{\alpha X_1} - D_1)e^{\alpha(t-S_1)} \\ &= xe^{\alpha t} - D_1 e^{\alpha(t-S_1)} \\ &= e^{\alpha t}[x - D_1 e^{-\alpha S_1}]. \end{aligned}$$

If $N(t) = 2$, we have

$$\begin{aligned} X(t) &= [(xe^{\alpha X_1} - D_1)e^{\alpha X_2} - D_2]e^{\alpha(t-S_2)} \\ &= xe^{\alpha t} - D_1 e^{\alpha(t-S_1)} - D_2 e^{\alpha(t-S_2)} \\ &= e^{\alpha t}[x - D_1 e^{-\alpha S_1} - D_2 e^{-\alpha S_2}]. \end{aligned}$$

In general,

$$X(t) = e^{\alpha t}[x - D_1 e^{-\alpha S_1} - D_2 e^{-\alpha S_2} - \cdots - D_{N(t)} e^{-\alpha S_{N(t)}}]. \tag{80}$$

However, the discounted warranty cost incurred upto time t is given by

$$U(t) = D_1 e^{-\alpha S_1} + D_2 e^{-\alpha S_2} + \cdots + D_{N(t)} e^{-\alpha S_{N(t)}}. \tag{81}$$

Substituting in equation (80), we get equation (79). ■

Note that $U(t)$ is a non-decreasing function of t. Furthermore, under the non-renewable free-replacement and minimal-repair warranty policies, no warranty cost is incurred after time $L + W$ and hence $U(t) = U(L + W)$ for all $t \geq L + W$. For the renewable free-replacement and pro rata warranty policies, warranty costs may be incurred over an unlimited time period. Hence $U(t) \uparrow U(\infty)$ in this case. In both cases, $U(\infty)$ equals $T(L)$, the total expected discounted warranty cost of all items sold over $[0, L]$. We next compute the probability, denoted by $\psi(x)$, that the warranty reserve is never depleted, i.e. $X(t) \geq 0$ for all $t \geq 0$, given that $X(0) = x$.

Theorem 7 *:*

$$\psi(x) = Pr\{T(L) \leq x\} \tag{82}$$

Proof. : We have

$$\begin{aligned}
\psi(x) &= Pr\{X(t) \geq 0, \text{ for all } t \geq 0 \mid X(0) = x\} \\
&= Pr\{e^{\alpha(t)}[x - U(t)] \geq 0, \text{ for all } t \geq 0\} \\
&= Pr\{U(t) \leq x, \text{ for all } t \geq 0\} \\
&= Pr\{U(\infty) \leq x\} \\
&= Pr\{T(L) \leq x\}.
\end{aligned}$$

■

In practice, we are interested in setting x such that $\psi(x) \geq 1 - \beta$ for some small β (say 0.05). To compute such a x requires the knowledge of the distribution of $T(L)$, which is too complicated. To circumvent this difficulty, we approximate the distribution of $T(L)$ by a Normal distribution with mean $E[T(L)]$ and variance $Var(T(L))$ (from equation (67) and (68)). This approximation is expected to work well as long as the expected number of sales upto L, given by $\Lambda(L)$, is large enough. We assume this to be the case. Now let Z_β be the number such that

$$1 - \Phi(Z_\beta) = \beta$$

where $\Phi(\cdot)$ is the standard Normal cdf (zero mean and variance equal to one).

Theorem 8 *: If $x \geq E[T(L)] + Z_\beta\sqrt{Var(T(L))}$, then*

$$\psi(x) \geq 1 - \beta. \tag{83}$$

Proof. : ¿From the previous theorem, we know

$$\psi(x) = Pr\{T(L) \leq x\}.$$

By using Normal approximation, we get inequality (83). ∎

In the other words, if the producer puts $x = E[T(L)] + Z_\beta\sqrt{Var(T(L))}$ amount in the bank at time 0, then the probability that the reserve never drops below zero is $1 - \beta$. Therefore, once the producer decides on the risk β, that he/she is willing to assume, the warranty reserve is given by

$$WR(L) = E[T(L)] + Z_\beta\sqrt{Var(T(L))}.$$

We illustrate with an example.

Example 6:

Suppose a company produces microwave ovens whose lifetimes are i.i.d. exponential random variables with mean one year (θ=1). The company provides a warranty policy on the ovens with one year warranty period (W=1). We consider three warranty policies: (1) non-renewable free-replacement, (2) renewable free-replacement, and (3) pro rata. For policies 1 and 2, we assume that the successive replacement costs are i.i.d. $U[50, 150]$. Then we use $E[D] = 100$ and $E[D^2] = 32500/3$ in equation (2), (3), (27), and (28) to compute $E[Z(W)]$ and $E[Z^2(W)]$ for these two warranty policies. For policy 3, we assume that each oven sells for one hundred and fifty dollars (d=150). The selling rate function is $\lambda(t) = 400e^{-t} + 600$ (ovens/per year), the discount factor is $\alpha = 0.068$,and $\beta = 0.05$. We plot the warranty reserves for these three warranty policies as a function of L in the following plot.

For example, if $L = 5$, then the warranty reserve should be $\$485,613$ for policy 1, $\$293,407$ for policy 2, and $\$428,243$ for policy 3. ■

5 Summary

In this paper, we derived the first and second moments of the producer's total discounted warranty cost of single sale for single-component items under four different kinds of warranty policies: renewable free-replacement, non-renewable free-replacement, pro rata, and non-renewable minimal-repair warranty policy. We also computed the mean and variance of the producer's total discounted warranty cost of the aggregate sales. Finally, we used those quantities to derive the warranty reserve by applying the Normal approximation. Several simple examples are also given in this paper.

References

[1] H. N. Amato and E. E. Anderson. *Determination of warranty reserves: An Extension.* Management Science, 22, No.12, 1391–1394, 1976

[2] W. R. Blischke and D. N. P. Murthy. *Product warranty management - I: a taxonomy for warranty policies .* European Journal of Operations Research, 62, 127–148, 1992.

[3] Wallace R. Blischke and Ernest M. Scheuer. *Calculation of the cost of warranty policies as a function of estimated life distributions.* Naval Research Logistic Quarterly, 22, No.4, 681–696, 1975.

[4] S. S. Chukova, B. N. Dimitrov and V. V. Rykov. *Warranty analysis.* Journal of Soviet Mathematics, 67, No.6, 3486–3508, 1993.

[5] Jehoshua Eliashberg, Nozer D. Singpurwalla, and Simon P. Wilson. *Calculating the warranty reserve for time and usage indexed warranty.* Management Science, 43, No.7, 966–975, 1997

[6] V. G. Kulkarni. *Modeling and analysis of stochastic systems.* Chapman and Hall, 1995.

[7] John W. Mamer. *Cost analysis of pro rata and free-replacement warranties.* Naval Research Logistic Quarterly, 29, No.2, 345–356, 1982.

[8] John W. Mamer. *Discounted and per unit costs of product warranty.* Management Science, 33, No.7, 916–930, 1987.

[9] W. W. Menke. *Determination of warranty reserves.* Management Science, 15, No.10, 542–549, 1969.

[10] D. N. P. Murthy and W. R. Blischke. *Product warranty management - II: an integrated framework for study.* European Journal of Operations Research, 62, 261–281, 1992.

[11] D. N. P. Murthy and W. R. Blischke. *Product warranty management - III: a review of mathematical models .* European Journal of Operations Research, 62, 1–34, 1992.

[12] D. G. Nguyen and D. N. P. Murthy. *Cost analysis of warranty policies.* Naval Research Logistic Quarterly, 31, 525–541, 1984.

[13] D. G. Nguyen and D. N. P. Murthy. *A general model for estimating warranty costs for repairable products.* IIE Transactions, 16, No.4, 379–386, 1984.

[14] J. G. Patankar and G. H. Worms. *Prediction intervals for warranty reserves and cash flow.* Management Science, 27, No.2, 237–241, 1981.

[15] C. S. Tapiero and M. J. Posner. *Warranty reserving.* Naval Research Logistics, 35, 473–479, 1988.

[16] M. U. Thomas. *A predict model for manufacturer warranty reserves.* Management Science, 35, 1515–1519, 1989.