Math 522 - Spring 09

Selected solutions.

Please note: When the question "why" appears, that indicates a detail you should think through carefully.

1. p193, #13.

This nice and simple solution was suggested by a student in the class:

We have

$$d_n f(a) = \lim_{t\to 0} \frac{f(a+tn) - f(a)}{t} = \lim_{t\downarrow 0} \frac{f(a+tn) - f(a)}{t}, \tag{1}$$

where the second equality holds since the first limit exists by assumption. Since f has a maximum at a

$$\frac{f(a+tn) - f(a)}{t} \leq 0 \text{ for } t > 0 \text{ small}, \tag{2}$$

so the limit is ≤ 0 (why?).//

2. p. 193, #10.

Note: To be consistent with the definition on p. 189, the problem should have included the assumption that f is continuous at a (yet the problem makes sense without that assumption).

Suppose $f : \mathbb{R}^N \to \mathbb{R}^M$ is tangent at $a \in \mathbb{R}^N$ to a "linear" (really *affine*) function $L : \mathbb{R}^N \to \mathbb{R}^M$ as in #12 on p. 193. We have $L = (L_1, ..., L_M)$ and $f = (f_1, ..., f_M)$, where all components are real valued functions. Then given $\epsilon > 0$ there exists $\delta > 0$ such that

$$d_M(f(x), L(x)) \leq \epsilon d_N(x, a) = \epsilon|x-a| \text{ for } 0 < |x-a| < \delta. \tag{3}$$

This implies that $\lim_{x\to a} f(x) = L(a)$, since L is continuous at a. (If we assume f is continuous at a, this implies $f(a) = L(a)$. Otherwise, redefine f at a to be $f(a) = L(a)$; now f is continuous at a. In either case the limit defining "tangency" makes sense.) Now (3) implies

$$|f_i(x) - L_i(x)| \leq \epsilon d(x, a) \text{ for } 0 < |x-a| < \delta \text{ and for all } i. \text{ (why?)} \tag{4}$$

Thus, L_i is tangent to f_i at a and we can apply Theorem 7.9 to see that the L_i are unique. Hence, L is unique.//

Aside: Quick proof of Theorem 7.9 (the case $M = 1$): Suppose $L_1 = b + c\cdot(x-a)$ and $L_2 = d + e\cdot(x-a)$ are both tangent to f at a (here b and d belong to $\mathbb{R}$, while c and e belong to $\mathbb{R}^N$). We showed above that $L_1(a) = L_2(a) = f(a)$, so $b = d$. L_1 and L_2 are tangent to each other at a (why?), so given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|L_1(x) - L_2(x)| = |(c-e)\cdot(x-a)| \leq \epsilon|x-a| \text{ when } |x-a| \leq \delta. \tag{5}$$

Choose $x - a = \delta e_k$, where $e_k = (0, ...0, 1, 0, ...0)$ (with 1 in the kth slot). Then (5) gives

$$|(c_k - e_k)\delta| \leq \epsilon\delta. \tag{6}$$

Since ϵ is arbitrary, $c_k = e_k$. Since k is arbitrary, $c = e$ and hence $L_1 = L_2$.//

3. Page 188, #10, proof of Thm. 7.8 (c). (Part (c) is slightly misstated. The assumption should have been that $Q(h)$ has a positive max and a negative min on S^{N-1}. Equivalently, Q changes sign on $\mathbb{R}^N$.)

There exist vectors b_1 and b_2 such that $Q(b_1) > 0$ and $Q(b_2) < 0$. We have for $t \in \mathbb{R}$

$$\begin{aligned} f(a+tb_1) - f(a) &= \nabla f(a) \cdot tb_1 + Q(tb_1) + \sum_{|\alpha|=2} \frac{1}{\alpha!} \left(D^\alpha f(\xi) - D^\alpha f(a)\right) (tb_1)^\alpha \\ &= t^2 \left(Q(b_1) + \sum_{|\alpha|=2} \frac{1}{\alpha!} (D^\alpha f(\xi) - D^\alpha f(a))(b_1)^\alpha \right), \end{aligned} \tag{7}$$

where ξ lies on the segment $(a, a+tb_1)$. Since f is C^2 the sum inside parentheses in the second line approaches 0 as $t \to 0$, so for $|t| \neq 0$ and small enough

$$f(a+tb_1) - f(a) \geq t^2 Q(b_1)/2 > 0. \tag{8}$$

Similarly, for $|t| \neq 0$ and small enough

$$f(a+tb_2) - f(a) \leq t^2 Q(b_2)/2 < 0. \tag{9}$$

4. p. 178, #2. H is a "double composition" so we must apply the chain rule twice. Let f have argument $y = (y_1, \ldots, y_N)$ and let $g = (g^1, \ldots, g^N)$ have argument $z = (z_1, \ldots, z_M)$. Set $h = (h^1, \ldots, h^M)$. Then $H = f \circ (g \circ h)$. We have

$$\begin{aligned} &\partial_{x_i} H(x) = \sum_{k=1}^N \partial_{y_k} f(g \circ h(x)) \partial_{x_i} (g^k \circ h)(x) = \\ &\sum_{k=1}^N \partial_{y_k} f(g \circ h(x)) \left(\sum_{j=1}^M \partial_{z_j} g^k(h(x)) \partial_{x_i} h^j(x) \right). \end{aligned} \tag{10}$$

5. p. 188 #3. We have

$$f_1(0, x_2) = \lim_{h \to 0} \frac{f(h, x_2) - f(0, x_2)}{h} = -x_2 \ \text{(check)}. \tag{11}$$

So $f_{1,2}(0,0) = -1$. Similarly, $f_2(x_1, 0) = x_1$, so $f_{2,1}(0,0) = 1$. Note that the product and quotient rules should not be used in this problem.

6. HW4 - #1 S is linear so it suffices to prove continuity at 0 (why?). We will show: given $\epsilon > 0$ there exists $\delta > 0$ such that if $|a, b, c, d| < \delta$ then $\|A\| < \epsilon$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We have for $|x| \leq 1$,

(12)

$$|Ax| = |(ax_1 + bx_2, cx_1 + dx_2)| \leq |(|a| + |b|, |c| + |d|)| \leq \sqrt{|a|^2 + |b|^2 + |c|^2 + |d|^2 + 2|a||b| + 2|c||d|}$$
$$\leq \sqrt{2} |a, b, c, d|.$$

Thus, given ϵ, choose $\delta < \epsilon/\sqrt{2}$.

S^{-1} is linear so it suffices to prove continuity at 0. We will show: given $\epsilon > 0$ there exists $\delta > 0$ such that if $\|A\| < \delta$ then $|a, b, c, d| < \epsilon$, for A as above. $\|A\| < \delta$ implies

$$
\begin{aligned}
|A\begin{pmatrix}1\\0\end{pmatrix}| &= |a, c| < \delta \\
|A\begin{pmatrix}0\\1\end{pmatrix}| &= |b, d| < \delta.
\end{aligned}
\tag{13}
$$

So for a given $\epsilon > 0$ take $\delta < \epsilon/\sqrt{2}$.

7. HW4 - #2. To show continuity at x we must show: given $\epsilon > 0$ there exists $\delta > 0$ such that $|x - y| < \delta$ implies $\|df(x) - df(y)\| < \epsilon$. Since $f = (f^1, f^2)$ is C^1, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|\nabla f^1(x) - \nabla f^1(y), \nabla f^2(x) - \nabla f^2(y)| < \epsilon/\sqrt{2}$. By the previous problem we conclude $|x - y| < \delta$ implies $\|df(x) - df(y)\| < \epsilon$ (why?).

8. HW5- p335 #7. First we must check that $g(x) = f(x) + x = \frac{1}{4} + \frac{3}{4}x - \frac{1}{40}x^5$ maps I to I. Note that $g(0) = \frac{1}{4}$, $g(1) = \frac{39}{40}$, and $g'(x) > 0$ on I, so $g : I \to I$. We have

$$
\begin{aligned}
|g(x) - g(y)| &= |\frac{3}{4}(x-y) - \frac{1}{40}(x^5 - y^5)| = |\frac{3}{4}(x-y) - \frac{1}{40}[(x-y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)]| \\
&= |x-y||\frac{3}{4} - \frac{1}{40}(x^4 + x^3y + x^2y^2 + xy^3 + y^4)| < \frac{3}{4}|x-y|
\end{aligned}
\tag{14}
$$

since $\frac{1}{8} \geq \frac{1}{40}(x^4 + x^3y + x^2y^2 + xy^3 + y^4) \geq 0$ for $x \in I$, $y \in I$. By the CMT $g(x)$ has a fixed point in I, so $f(x)$ has a zero in I. The fixed point must lie in $[\frac{1}{4}, \frac{39}{40}]$ (why?). So compute, for example, $x_0 = \frac{1}{2}$, $x_1 = g(x_0)$, $x_2 = g(x_1)$.

9. HW p. 443, #12 Let A, B be points in D and suppose $f_0(t)$, $f_1(t)$, $t \in [a, b]$ are paths in D joining A to B starting at A and ending at B.

(a) A homotopy (as in the definition on p. 440) is given by $f(t, s) = (1 - s)f_0(t) + sf_1(t)$, $s \in [0, 1]$, $t \in [a, b]$. The convexity of D implies $f(t, s) \in D$ for all (t, s).

(b) A theorem of Brouwer on "invariance of domain" says that a 1-1, continuous map defined on *open* subset of $\mathbb{R}^n$ is actually a homeomorphism onto its image; so h is actually a homeomorphism of D onto D_1 (openness is part of the text's definition of "domain"). The author could not assume that you know Brouwer's theorem, so the problem should have stated that h is a homeomorphism.

Given paths f_0, f_1 as above, but now in D_1, let g_0, g_1 be the paths in D given by $g_i(t) = h^{-1}(f_i(t))$. Let $g(t, s)$ be a homotopy of these paths in D. Then $f(t, s) := h(g(t, s))$ is a homotopy of the paths in D_1.

Thanks to Bevin Maultsby for pointing out Brouwer's theorem.

Important Observation: A continuous 1-1 map need not in general be a homeomorphism onto its range. That is, it need not have a continuous inverse. Consider the example: $f : [0, 2\pi) \to S^1$ given by $f(\theta) = (\cos\theta, \sin\theta)$. Here f is a continuous and 1-1 map onto the circle, but the inverse is not continuous (why?). Thanks to Matt Hernandez for pointing out this example.

10. Curve straightening problem, HW 6. Choose a vector $c \in \mathbb{R}^3$ such that the 3×3 matrix

$$\begin{pmatrix} \nabla\phi_1(a) \\ \nabla\phi_2(a) \\ c \end{pmatrix} \tag{15}$$

has nonzero determinant. This can be done, e.g., by taking c to be the cross product of the first two rows (why?). Define

$$(u, v, w) = \psi(x) := (\phi_1(x), \phi_2(x), c \cdot (x - a)). \tag{16}$$

Then $\psi(a) = 0$ and $d\psi(a)$ is the matrix (15). By the inverse function theorem there exist neighborhoods U containing a and V containing 0 such that $\psi : U \to V$ is $1-1$, onto, and C^1 with C^1 inverse. In addition we have

$$\mathcal{C} \cap U = \{x \in U : u(x) = v(x) = 0\}, \tag{17}$$

so

$$\psi(\mathcal{C} \cap U) = \{(u, v, w) \in V : u = v = 0\}. \tag{18}$$

11. Clarification of proof given in class that: *curl* $v = 0$ **in a simply connected region implies line integrals of** v **are independent of path.**

Suppose $v : \mathcal{O} \to \mathbb{R}^2$ is a C^1 vector field in a connected and simply connected open set $\mathcal{O}$ such that *curl* $v = 0$ in $\mathcal{O}$. Fix any two points A and B in $\mathcal{O}$ and consider two oriented C^1 paths C_1 and C_2 in $\mathcal{O}$ starting at A and ending at B. We want to show

$$\int_{C_1} v \cdot dr = \int_{C_2} v \cdot dr \tag{19}$$

by applying Green's theorem to the closed path $C_1 - C_2$. Let the region in $\mathbb{R}^2$ enclosed by $C_1 - C_2$ be called D. We would like to say that Green's Theorem implies

$$\int_{C_1 - C_2} v \cdot dr = \int_D curl\ v\ dA = 0. \tag{20}$$

To say this we need to know that *curl* $v = 0$ in D. This is true because $\mathcal{O}$ is simply connected and therefore $D \subset \mathcal{O}$. Since *curl* $v = 0$ in $\mathcal{O}$, we have *curl* $v = 0$ in D. If $\mathcal{O}$ were not simply connected, some points in D (which was defined as the subset of $\mathbb{R}^2$ enclosed by the curves, *not* the subset of $\mathcal{O}$ enclosed by the curves!) might not lie in $\mathcal{O}$, and for such points we could not assert that *curl* $v = 0$.

The following example shows what can happen if $\mathcal{O}$ is not simply connected. Consider

$$v = \left(\frac{-x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2} \right) \in C^1(\mathcal{O}) \tag{21}$$

where $\mathcal{O} = \{x \in \mathbb{R}^2 : x \neq 0\}$. Note *curl* $v = 0$ in $\mathcal{O}$ but line integrals are not independent of path, since if C is a circle of radius one centered at $(0, 0)$ oriented counterclockwise,

$$\int_C v \cdot dr = 2\pi. \tag{22}$$

Note that if C_1 is the constant path at $(1,0)$ and if $C_2 = C$ (starting and ending at $(1,0)$), the region D in this case is $\{x : |x| < 1\}$ and D is *not* a subset of $\mathcal{O}$. (Other choices of C_1 and C_2 lying on C can be made here; for example, both can start at $(0,-1)$ and end at $(0,1)$).

12. HW9, #3. Fix $\epsilon > 0$. It is enough to show that there exists a subdivision Δ of the cube C such that

$$S_+(f,\Delta) - S_-(f,\Delta) < \epsilon. \tag{23}$$

There exists $M > 0$ such that $|f(x)| \le M$ on C. Choose a finite number of disjoint open cubes $\{C_1, C_2, \ldots, C_K\}$ contained in the interior of C, whose union contains all the discontinuities of f, and such that the volume of the union is $< \frac{\epsilon}{4M}$.

The function f is uniformly continuous on the set $D := C \setminus \cup_{j=1}^K C_j$ (why?). Thus, there exists $\delta > 0$ such that for $x, y \in D$, $|x-y| \le \delta \Rightarrow |f(x) - f(y)| < \frac{\epsilon}{2V(C)}$. Let $\{C_{K+1}, \ldots, C_M\}$ be a subdivision of D of mesh $< \delta$ and set

$$\Delta = \{C_1, \ldots, C_K, C_{K+1}, \ldots, C_M\}. \tag{24}$$

We have

$$\begin{aligned}
S_+(f,\Delta) - S_-(f,\Delta) &= \sum_{j=1}^K (M_j - m_j)V(C_j) + \sum_{j=K+1}^M (M_j - m_j)V(C_j) \\
&\le 2M \sum_{j=1}^K V(C_j) + \sum_{j=K+1}^M \frac{\epsilon}{2V(C)} V(C_j) \ \ \text{(why?)} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \text{ (why?)}.
\end{aligned} \tag{25}$$

13. **HW10, #9b**. Define $f_+(x) = \max(f(x), 0)$ and $f_-(x) = \max(-f(x), 0)$. Note that

$$|f| = f_+ + f_-, \ ; f = f_+ - f_-. \tag{26}$$

It suffices to show that $f_\pm$ are both integrable (why?).

Fix $\epsilon > 0$. To show f_+ is integrable we show that there exists a subdivision Δ' of F such that

$$S_+(f_+, \Delta') - S_-(f_+, \Delta') < \epsilon. \tag{27}$$

Since f is integrable there exists a subdivision $\Delta = \{F_1, \ldots, F_K\}$ such that $S_+(f,\Delta) - S_-(f,\Delta) < \epsilon$. We claim that we can take $\Delta' = \Delta$ in (27). Let

$$\begin{aligned}
M_i &= \sup_{F_i} f, \ m_i = \inf_{F_i} f \\
M_{+i} &= \sup_{F_i} f_+, \ m_{+i} = \inf_{F_i} f_+
\end{aligned} \tag{28}$$

and note that for all i, $M_{+i} - m_{+i} \le M_i - m_i$ (why? : consider the three cases $f \ge 0$ on F_i, $f \le 0$ on F_i, f takes both positive and negative values on F_i). This gives (27) with $\Delta' = \Delta$. The argument for f_- is similar. (The decompositions in (26) are often used for other purposes as well.)