Here is a correction to the proof given in class on Tues. (Feb. 24) that if $S$ is a figure, $A(\partial S) = 0$. My proof did involve some illegal shifting of grid squares as Dan Wielunski suspected. Thanks to Dan (and also Matt Hernandez) for pointing out the error.

Proposition:

If $S \subset \mathbb{R}^2$ is a figure, then $A(\partial S) = 0$.

*Proof.* The Proposition follows from the (not immediately obvious) claim: For any $n$, $\partial S$ is contained in the union of the boundary squares of $S$ for the $n$th grid. Denote that union by $B_n S$. We then have

$$A_n^+(\partial S) \leq A(B_n S) = A_n^+(S) - A_n^-(S) \tag{1}$$

and the right side goes to 0 as $n \to \infty$ since $S$ is a figure.

It remains to prove the claim that $\partial S \subset B_n S$. Let $P \in \partial S$. Then $P$ is in some covering square$_n$ of $\partial S$, call it $C_n$. If $P \in C_n^o$ (interior of $C_n$), then $C_n$ is a boundary square of $S$ (why?), so $P \in B_n S$. So if $C_n$ is not a boundary square of $S$, then $P$ must lie on the boundary of $C_n$ *and* either $C_n \subset S$ or $C_n \subset S^c$. Consider the square or squares of the $n$th grid, other than $C_n$, that also contain $P$ (we are talking about either 1 or 3 squares here; e.g., 3 if $P$ is a corner point of $C_n$). Since $P \in \partial S$ at least one of those squares must be a boundary square of $S$ (why? Use the fact that the union of these squares with $C_n$ has $P$ in its interior.). Thus, again $P \in B_n S$.

□

Of course, the analogue holds for volumes in $\mathbb{R}^n$ by the same kind of proof.

Also: Please note that if $S = \{(x, y) \in [0, 1] \times [0, 1] : x, y \text{ rational}\}$, then $S \subset \partial S$, but it is not true that $S = \partial S$, as stated in class.