k-Circulant Supersaturated Designs

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A class of supersaturated designs called *k-circulant designs* is explored. These designs are constructed from cyclic generators by cycling k elements at a time. The class of designs includes many Es^2-optimal designs, some of which are already known and some of which are more efficient than known designs for model estimation under factor sparsity. Generators for the most efficient designs are listed, and projection properties of some of the designs are explored. We also illustrate that some k-circulant supersaturated designs can be augmented with interaction columns to produce efficient designs for a larger number of factors or for estimating interactions.

KEY WORDS: Cyclic generator; D optimality; Es^2 optimality; Factor sparsity; Interactions; Projection efficiency; Screening.

1. INTRODUCTION

In the early stages of industrial experimentation on a product or process, numerous factors may have been identified as possibly having an influence on the response. It is quite common, however, that only a few of these actually have a *substantial* effect—a situation known as *factor sparsity* (see Box and Meyer 1986). The small number of active, or influential, factors can often be identified through a *screening experiment.* Various design strategies have been proposed for screening under factor sparsity. These strategies include search designs (see Srivastava 1975; Ghosh 1996; Li and Nachtsheim 2000), group screening (e.g., Watson 1961; Kleijnen 1987; Morris 1987; Lewis and Dean 2001), and supersaturated designs (e.g., Booth and Cox 1962; Lin 1993a, 1995; Wu 1993; Nguyen 1996; Liu and Zhang 2000; Butler, Mead, Eskridge, and Gilmour 2001; Eskridge, Gilmour, Mead, Butler, and Travnicek 2001; Bulutoglu and Cheng 2002).

Supersaturated designs are used mainly for the estimation of main effects, although in some designs, it is possible to estimate a small number of interactions also. Supersaturated designs require very few observations in comparison with the number of factors being investigated. The small number of observations results in least squares estimates of some pairs of main effects being correlated, and this can sometimes lead to erroneous conclusions being drawn about which factor effects are nonzero (see Abraham, Chipman, and Vijayan 1999). However, supersaturated designs have been advocated when the proportion of active factors is small, when the active factors have large main effects, and when interactions are negligible (see, e.g., Chen and Lin 1998). Methods of analysis that have been investigated for supersaturated designs include forward selection in regression (Westfall, Young, and Lin 1998), all-subsets selection (Abraham et al., 1999) and various Bayesian methods (e.g., Chipman, Hamada, and Wu 1997; Beattie, Fong, and Lin 2002).

A number of different methods of construction of supersaturated designs have been proposed in the literature. Lin (1993a) constructed supersaturated designs for $m = 2n - 1$ factors in n runs by selecting half the rows of a Hadamard matrix with $2n - 1$ columns and $2n$ rows according to the values in a "branching column." Wu (1993) obtained designs with up to 66 factors in 12 runs and up to 124 factors in 20 runs by appending interaction columns to a Hadamard matrix. The designs of Tang and Wu (1997) are formed by adjoining the columns of distinct Hadamard matrices. Design search by computer algorithm was done by Lin (1995), Nguyen (1996), and Li and Wu (1997). The designs obtained by these authors are optimal or highly efficient under the Es^2 criterion (defined in Sec. 2). A method proposed by Nguyen (1996) of constructing Es^2-optimal supersaturated designs was to adjoin the incidence matrices of pairs of cyclic balanced incomplete block designs (BIBDs). A more thorough investigation of optimal or near-optimal designs obtained by a generalization of this type of construction was given by Liu and Zhang (2000) and Eskridge et al. (2001).

In this article we show that optimal or efficient supersaturated designs with n runs for $m = k(n - 1)$ factors, each having two levels, can be constructed very simply by the cyclic development of a generator. Our method is a generalizaton of that of Plackett and Burman (1946), who introduced the use of cyclic generators for constructing orthogonal saturated main effect plans for $m = n - 1$ factors in n runs, when $n = 0 \bmod 4$. This type of construction was also used by Dean and Draper (1999) and Crosier (2000) for constructing saturated designs when $n = 2 \bmod 4$. The cyclic generation method of obtaining a design matrix is very flexible in that a large number of distinct designs with varied properties can be obtained.

Our class of designs, which we call *k-circulant supersaturated designs*, include the cyclic BIBD-based supersaturated designs of Nguyen (1996) and Liu and Zhang (2000) and the cyclic BIBD or regular graph-based supersaturated designs of Eskridge et al. (2001) as special cases (see Sec. 3). The class also includes alternative designs that are better than previously known designs under at least one of the optimality criteria defined in Section 2.

In Section 3 we present generators for efficient k-circulant designs with factors at two levels, called the "high" and "low" levels. We consider only "balanced" designs in which each factor is observed the same number of times at the high and low levels so that the columns of the design matrix form the main effect contrasts and are independent of the mean. Because of the

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numerous uses of the word "balanced" in the literature, we call such a design *mean-orthogonal.* Our k-circulant designs have an even number, $n = 2t$ runs (observations), and $m = k(2t - 1)$ factors, for positive integer t. We give generators for k-circulant supersaturated designs with parameters in the range $m \leq 55$, $n \leq 22$, and $2 \leq k \leq 6$. For k even, or for k odd and t even, all but one of the designs listed are Es^2 optimal. For k and t both odd, our listed designs are at least 97.8% efficient relative to the lower bound for Es^2 derived by Bulutoglu and Cheng (2002). We discuss projection efficiencies of these designs in Section 4. In Section 5 we investigate the addition of interaction columns to basic k-circulant designs. Such columns can be used either to measure interactions or to extend a design so that the main effects of a larger number of factors can be examined in the same number of runs. It is also possible to obtain efficient supersaturated designs by deleting columns sequentially from a k-circulant design; we will communicate these details in a future article.

2. OPTIMALITY OF SUPERSATURATED DESIGNS

A supersaturated design, d, can be represented by a "design matrix" or "treatment matrix," $\mathbf{T}$, whose rows represent the n treatment combinations to be observed and whose columns designate the m factors to be examined. The (i, f)th element of $\mathbf{T}$ determines the level at which factor f is to be observed in the ith treatment combination. We code the high and low levels of the factors as $+1$ and -1 in $\mathbf{T}$. Before the observations are taken, the rows of $\mathbf{T}$ should be randomly ordered.

The $(m+1) \times n$ "model matrix," $\mathbf{X}$, is formed from the design matrix, $\mathbf{T}$, by appending a column of 1's as the first column. This first column measures the mean and, in a mean-orthogonal design, column $(f + 1)$ is a contrast that measures the main effect of factor f, $(f = 1, \ldots, m)$. The model is

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\Phi} + \boldsymbol{\epsilon},$$

where $\mathbf{Y}$ and $\boldsymbol{\epsilon}$ are the vectors of responses and random error variables, and $\boldsymbol{\Phi}$ is the vector of the mean and the factorial main effect contrasts of the m factors.

If all pairs of columns of the model matrix $\mathbf{X}$ are orthogonal, then the estimators of the main effects of all pairs of factors are uncorrelated, and the design is optimal under all criteria (such as D and A optimality) in which no preference is given to any factor. In a supersaturated design, it is not possible to achieve complete orthogonality, and some of the main-effect estimators are correlated. To minimize errors in identifying the active factors, the number of orthogonal pairs of columns should be maximized or the maximum correlation and the average correlation between nonorthogonal pairs of columns should be minimized (see, e.g., Booth and Cox 1962; Lin 1993a). If interactions are also to be measured, then interaction columns, whose elements are products of the elements in the corresponding main-effects columns, would be added to the model matrix.

The foregoing considerations lead to the following optimality criteria, which are generally used to compare supersaturated designs for factors at two levels. The most commonly used criterion is the minimum Es^2 [or Ave(s^2)] criterion, which was first suggested by Booth and Cox (1962) and is defined as follows. Let s_{ij} be the element in the ith row and jth column of $\mathbf{T}'\mathbf{T}$ $(i, j = 1, \ldots, m)$. Then, $s_{ij} = \sum_{w=1}^{n} t_{wi} \times t_{wj}$, where t_{wj} is the element in row w and column j of $\mathbf{T}$. A supersaturated design is Es^2 optimal if it has the minimum value of $Es^2 = \sum_{i<j} s_{ij}^2 / \binom{m}{2}$ among all competing designs of the same size. In a mean-orthogonal design, the mean and standard deviation of the elements in each column are 0 and 1.0, so Es^2 is n^2 times the average squared correlation between pairs of columns. Although minimizing Es^2 (which depends on the entries of $\mathbf{T}'\mathbf{T}$) is not identical to the criterion of minimizing the average correlation between the contrast estimators (which depends on the entries of $[\mathbf{X}'\mathbf{X}]^{-1}$), it does lead to highly efficient designs under this latter criterion, provided that the designs are mean-orthogonal. Note that when a design is not mean-orthogonal or when the factors have more than two levels, the contrast estimators are not identical to the columns of the design matrix $\mathbf{T}$ (see, e.g., Dean and Draper 1999; Wu and Hamada 2000, sec. 5.6), and the calculations in this article would need to be modified accordingly.

For a mean-orthogonal design with m factors at two levels each and n runs, Nguyen (1996) and Tang and Wu (1997) independently derived a lower bound,

$$n^2(m - n + 1)/(n - 1)(m - 1) \tag{1}$$

for Es^2. Necessary (but not sufficient) conditions for this bound to be attained are that $m = k(n-1)$, n is even, and nk is divisible by 4. Nguyen (1996) further showed that the Es^2 lower bound is attained if and only if $\mathbf{T}\mathbf{T}'$ is of the form $(m - x)\mathbf{I}_n + x\mathbf{J}_n$, where $x = -m/(n - 1)$, $\mathbf{I}_n$ is the identity matrix and $\mathbf{J}_n$ is a matrix of 1's, both of order n. Equivalently, Es^2 can attain its minimum if and only if a BIBD with $n - 1$ treatments and $k(n - 1)$ blocks of size $(n/2) - 1$ exists (see Cheng 1997).

Liu and Zhang (2000, thm. 2.1) have given lower bounds for Es^2, applicable for $m = q(n - 1) + r$ factors, where q is a positive integer, n is even, nq is a multiple of 4, and $|r| < 2$. Bounds for the case $|r| < n/2$ were derived by Butler et al. (2001, thm. 1). Further bounds for Es^2 have been given by Bulutoglu and Cheng (2002, thm 3.1) with no restriction on n. Where the parameter values match, these coincide with the bounds given by Liu and Zhang (2000) and Butler et al. (2001). Otherwise, their bounds are new. We use special cases of the bound (1) for $n = 0 \bmod 4$ and of the bound of Bulutoglu and Cheng (2002, thm. 3.1.3) for $n = 2 \bmod 4$ to assess our designs in Sections 3–5.

An alternative to Es^2 is the criterion of maximizing the percentage of pairs of main effects that can be estimated independently of each other; that is, the percentage $(\mathbf{f})_0$ of uncorrelated pairs of columns of $\mathbf{T}$. This criterion (cf. Booth and Cox 1962; Wang and Wu 1992) maximizes the percentage of zero off-diagonal elements of $\mathbf{T}'\mathbf{T}$. When the number of runs is not a multiple of 4, no pair of columns can be orthogonal, so $(\mathbf{f})_0 = 0$, and instead we maximize $(\mathbf{f})_2$, the percentage of off-diagonal elements of $\mathbf{T}'\mathbf{T}$ whose absolute values are equal to 2. For a finer comparison of designs, other $(\mathbf{f})_i$ can be compared, where $(\mathbf{f})_i$ is the percentage of off-diagonal elements of $\mathbf{T}'\mathbf{T}$ with absolute value i.

Because it is desirable to control the maximum correlation between any pair of main-effect estimators, a third criterion is the minimization of $r_{\max} = S_{\max}/n$, where $S_{\max} = \max |s_{ij}|$. Rarely will a single supersaturated design be optimal under all of these criteria, and it may be possible to compromise by minimizing Es^2 or maximizing $(\mathbf{f})_0$ or $(\mathbf{f})_2$ under the restriction that $S_{\max} \le c$ or $r_{\max} \le c/n$, for some specified constant c.

The foregoing criteria not only aim to reduce the number of errors made in the identification of correct active factors, but also aim to increase the *projection efficiency*; that is, they aim to increase the efficiency of a design for estimating any model containing a small number of potentially active factors. Projection efficiencies (as defined in Lin 1993b) are examined in Section 4.

In general, if there is no information concerning which factors are more likely to be important than others, then it is advisible to select an Es^2-optimal design with low $r_{\max}$. However, if it is thought that certain factors are more likely than others to be active, then it may be preferable to select a design with large $(\mathbf{f})_0$ or $(\mathbf{f})_2$ and assign the potentially more important factors in such a way that the projection designs involving these factors are the more efficient.

3. CONSTRUCTION AND EFFICIENCY OF *k*-CIRCULANT DESIGNS

Plackett and Burman (1946) introduced the use of cyclic generators for constructing orthogonal saturated main-effects plans for $m = n - 1$ two-level factors in n runs, when $n = 0 \bmod 4$. Dean and Draper (1999) and Crosier (2000) showed that efficient (nonorthogonal) saturated main-effects plans can also be obtained by this method when $n = 2 \bmod 4$. In these articles, the listed generator for each design provides the first row of the design matrix $\mathbf{T}$. Each remaining row of $\mathbf{T}$ is obtained from the previous row by cycling the elements one column to the right and moving the last element to the first column so that $\mathbf{T}$ is a circulant matrix. A row of $+1$'s is added to the design to give n runs in total.

To accommodate the large numbers of factors of a supersaturated design, we propose cycling the elements of a generator k elements at a time (which is equivalent to selecting rows 1, $k+1, \dots, m-k+1$ from a circulant saturated design matrix). We call such a design a *k-circulant design*. In this notation, the saturated designs of the foregoing three references are all 1-circulant designs. The following example illustrates the construction of a 3-circulant design.

Example 1. An Es^2-optimal 3-circulant design for $m = 21$ factors in $n = 8$ runs can be obtained from the generator

$$(-1\ -1\ -1\ -1\ -1\ -1\ -1\ 1\ 1\ -1\ -1\ -1\ 1\ -1\ 1\ 1\ 1\ -1\ 1\ 1\ 1)$$

by repeatedly cycling elements $k = 3$ positions to the right and moving the last three elements to the first three positions. The generator provides the first row of the design matrix and, after cycling three elements to the right, the second row is

$$(1\ 1\ 1\ -1\ -1\ -1\ -1\ -1\ -1\ -1\ 1\ 1\ -1\ -1\ -1\ 1\ -1\ 1\ 1\ 1\ -1).$$

Five more rows can be obtained in this manner before the first row is reached again. This gives the first seven rows of the design matrix. An eighth row of $+1$'s is then added to give the design as shown in Table 1. Note that each column contains four $+1$'s and four -1's, so that the design is mean-orthogonal.

Among the 210 choices of pairs of columns from Table 1, 126 pairs contain orthogonal columns (e.g., column pairs 1, 6; 2, 7; and 3, 8), so $(\mathbf{f})_0 = 60\%$. The other 84 pairs of columns have $|s_{ij}| = 4$, so $(\mathbf{f})_4 = 40\%$. Also, $S_{\max} = 4$, so that the maximum correlation between two columns is $r_{\max} = 4/8 = .5$. The design achieves the Es^2 lower bound (1) of 6.4.

As mentioned in Section 1, not all of our designs are new. In particular, the supersaturated designs constructed from k cyclic BIBDs or k cyclic regular graph designs discussed by Nguyen (1996), Liu and Zhang (2000), and Eskridge et al. (2001) all belong to the k-circulant class, as does, in fact, *any* design based on a set of k cyclic incomplete block designs. (Note that supersaturated designs based on noncyclically generated incomplete block designs are not part of this class.) The general construction technique used by the aforementioned authors for obtaining a supersaturated design with $m = k(n-1)$ factors and n runs is as follows. A set of $k \ge 2$ initial blocks is selected, each of size $(n-2)/2$, which together form a set of cyclic generating blocks for a BIBD or regular graph design with $n-1$ treatments. If the hth generating block ($1 \le h \le k$) contains treatment i ($1 \le i \le n-1$), then there is a "+1" in position i of column $n(h-1)+1$ of the corresponding supersaturated design. All other positions in this column are filled with -1. Each of the k columns generates a total of $n-1$ columns (including itself) by cycling the elements downward and moving the last element to position 1. This gives $m = k(n-1)$ columns of length $n-1$. It can be verified that each of the sets of $n-1$ columns can also be generated row-wise. Then, if the columns are reordered so that the k generating columns become columns $1, \dots, k$ and the columns are cycled simultaneously, it is straightforward to see that the resulting design is a k-circulant design. Thus, for example, the supersaturated design for $m = 21$ factors and $n = 8$ runs given by Liu and Zhang (2000, table 1) and based on the three cyclic BIBDs with seven treatments and generating blocks (0, 1, 2), (0, 1, 4), and (0, 2, 4) is identical to the 3-circulant supersaturated design of Example 1 after rearranging columns.

Table 1. A 3-Circulant Design With 21 Factors and 8 Runs

	Factors																				
Run	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*	*21*
1	−1	−1	−1	−1	−1	−1	−1	1	1	−1	−1	−1	1	−1	1	1	1	−1	1	1	1
2	1	1	1	−1	−1	−1	−1	−1	−1	−1	1	1	−1	−1	−1	1	−1	1	1	1	−1
3	1	1	−1	1	1	1	−1	−1	−1	−1	−1	−1	−1	1	1	−1	−1	−1	1	−1	1
4	1	−1	1	1	1	−1	1	1	1	−1	−1	−1	−1	−1	−1	−1	1	1	−1	−1	−1
5	−1	−1	−1	1	−1	1	1	1	−1	1	1	1	−1	−1	−1	−1	−1	−1	−1	1	1
6	−1	1	1	−1	−1	−1	1	−1	1	1	1	−1	1	1	1	−1	−1	−1	−1	−1	−1
7	−1	−1	−1	−1	1	1	−1	−1	−1	1	−1	1	1	1	−1	1	1	1	−1	−1	−1
8	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

The following theorem, the proof of which is given in the Appendix, gives necessary and sufficient conditions on n, m, and in order, for a k-circulant supersaturated design to be mean-orthogonal.

Theorem 1. Let d be a k-circulant supersaturated design with n runs and m factors each having two levels, coded $+1$ and -1. Suppose that the rows of design d are obtained from the generator $(g_1, g_2, \dots, g_m)$ by cycling k elements at each step and adding a row of $+1$'s. Necessary and sufficient conditions for d to be mean-orthogonal are

(a) $n = 2t$, $m = (2t-1)k$, for some positive integer t;
(b) the generator contains exactly kt elements equal to -1 and $(kt-k)$ elements equal to $+1$;
(c) $\sum_{u=0}^{2t-2} g_{uk+j} + 1 = 0$ for each $j = 1, 2, \dots, k$.

Using Theorem 1(a), the Es^2 lower bound (1) for mean-orthogonal k-circulant designs with $nk = 0 \bmod 4$ becomes

$$n^2(k-1)/(nk-k-1). \qquad (2)$$

In the case of mean-orthogonal k-circulant designs with $n = 2 \bmod 4$ and k odd, the Es^2 lower bound of Bulutoglu and Cheng (2002, thm. 3.1.3) becomes

$$\max\{4, [k(k-1)(n-1)n^2 + 2n(n-2)]/[k(n-1)(kn-k-1)]\}. \qquad (3)$$

For selecting of good mean-orthogonal designs, we use either the criterion of minimizing Es^2 or that of maximizing $(\mathbf{f})_0$ and $(\mathbf{f})_2$, under the restriction that $r_{\max}$ is reasonably small. The elements s_{ij} of $\mathbf{T}'\mathbf{T}$ are needed for evaluating these criteria, and these can be calculated directly from the design generator without calculating the $\mathbf{T}'\mathbf{T}$ matrix itself, as shown in Theorem 2. Theorem 2 can be easily proved by identifying the elements in the columns of $\mathbf{T}$ exactly as in the proof of Theorem 1(b).

Theorem 2. Suppose that the m elements in the generator of a mean-orthogonal k-circulant supersaturated design are

$$g_1, g_2, \dots, g_k, g_{k+1}, \dots, g_{2k}, \dots, g_{k(2t-2)+1}, \dots, g_{(2t-1)k},$$

which satisfy the requirements of Theorem 1. Then the following apply:

a. The (i,j)th element s_{ij} of the matrix $\mathbf{T}'\mathbf{T}$ can be expressed in terms of the generator elements as

$$s_{ij} = \sum_{q=0}^{2t-2} (g_{qk+i} \times g_{qk+j}) + 1, \qquad i, j = 1, 2, \dots, m.$$

b. $\mathbf{T}'\mathbf{T}$ is circulant, with

$$s_{qk+i,qk+j} = s_{ij}, \qquad i, j = 1, 2, \dots, m, \quad q = 1, 2, \dots, (2t-2),$$

where, if $qk + i > m$, it is replaced by $qk + i - m$.

Tables 2–6 list a selection of generators for k-circulant designs for all values of m factors, n runs, and cycle sizes k over the range $m \le 55$, $n \le 22$, and $2 \le k \le 6$, plus one example with

Table 2. Generators for Es^2-Optimal 2-Circulant Designs With n Runs and m Factors

(n m)	*$(\mathbf{f})_i$*	*r_{max}*	*Generator*
(6 10)*#	$(100.0)_2$	.33	−1 −1 −1 −1 −1 1 1 −1 1 1
(8 14)*#	$(69.2\ 30.8)_0$	.5	−1 −1 −1 −1 −1 −1 1 1 −1 1 1 −1 1 1
(10 18)*#	$(94.1\ 5.9)_2$	.6	−1 −1 −1 −1 −1 −1 −1 1 1 −1 1 1 −1 −1 1 1 1 1
(12 22)	$(71.4\ 23.8\ 4.8)_0$	.67	−1 −1 −1 −1 −1 −1 1 1 −1 1 −1 1 1 −1 −1 1 1 −1 1 −1 1 1
(12 22)*#	$(57.1\ 42.9)_0$	.33	−1 −1 −1 −1 −1 −1 −1 1 1 −1 1 −1 −1 1 1 1 −1 1 1 −1 1 1
(14 26)	$(96.0\ 0\ 4.0)_2$	.71	−1 −1 −1 −1 −1 −1 −1 1 −1 −1 1 −1 1 1 −1 1 1 −1 −1 1 1 −1 1 1 1 1
(14 26)*#	$(88.0\ 12.0)_2$	.43	−1 −1 −1 −1 −1 −1 −1 −1 1 1 −1 1 −1 −1 1 1 1 −1 −1 1 1 −1 1 1 1 1
(16 30)	$(55.2\ 41.4\ 3.4)_0$	.5	−1 −1 −1 −1 −1 −1 −1 −1 1 1 −1 −1 −1 1 1 −1 1 1 −1 −1 −1 1 1 1 1 −1 1 1 1 1
(16 30)*#	$(44.8\ 55.2)_0$	.25	−1 −1 −1 −1 −1 −1 −1 1 −1 −1 1 −1 1 1 −1 1 1 1 −1 −1 1 −1 1 1 1 −1 −1 1 1 1
(18 34)	$(87.9\ 9.1\ 3.0)_2$	.56	−1 −1 −1 −1 −1 −1 −1 −1 −1 1 −1 1 1 −1 1 −1 −1 1 1 1 1 −1 −1 1 1 −1 −1 1 1 −1 1 1 1 1
(18 34)*#	$(81.8\ 18.2)_2$	.33	−1 −1 −1 −1 −1 −1 −1 −1 −1 1 −1 1 1 −1 1 −1 1 1 −1 1 1 −1 −1 1 1 −1 1 1 −1 −1 1 1 1 1
(20 38)*	$(56.8\ 35.1\ 8.1)_0$	.4	−1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 −1 −1 −1 1 1 1 1 −1 1 −1 −1 1 −1 1 1 1 −1 −1 1 1 1 −1 1 1 −1 1 1
(20 38)	$(48.6\ 46.0\ 5.4)_0$	.4	−1 −1 −1 −1 −1 −1 −1 −1 −1 1 −1 −1 1 1 1 −1 −1 1 −1 1 1 −1 1 1 1 −1 −1 −1 1 1 −1 1 1 −1 1 1 1 1
(20 38)	$(40.5\ 56.8\ 2.7)_0$	.4	−1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 −1 1 −1 −1 1 1 −1 1 1 1 1 −1 1 −1 −1 1 1 1 −1 −1 −1 1 1 −1 1 1 1
(20 38)&#	$(32.4\ 67.6)_0$	.2	−1 −1 −1 −1 −1 −1 −1 −1 1 1 −1 1 −1 1 1 −1 −1 1 1 1 1 −1 −1 −1 1 1 1 −1 1 −1 −1 −1 1 1 −1 1 1 1
(22 42)	$(87.8\ 9.8\ 0\ 2.4)_2$	.64	−1 −1 −1 −1 −1 −1 −1 −1 1 1 −1 −1 −1 −1 1 1 −1 1 1 −1 1 1 1 1 1 1 −1 −1 1 1 −1 −1 1 −1 −1 −1 −1 1 1 1 1 1
(22 42)	$(85.3\ 9.8\ 4.9)_2$	.46	−1 −1 −1 −1 −1 −1 −1 −1 −1 1 −1 1 1 −1 −1 1 1 1 1 −1 −1 −1 1 −1 −1 1 1 −1 1 1 1 1 −1 1 1 −1 −1 −1 1 1 1 1
(22 42)	$(80.5\ 17.1\ 2.4)_2$	.46	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 −1 −1 1 1 −1 −1 −1 1 1 1 1 −1 1 1 −1 1 −1 −1 1 −1 1 1 1 −1 −1 1 1 1 1 1
(22 42)*#+	$(75.6\ 24.4)_2$	.27	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 −1 1 1 −1 1 1 1 −1 −1 1 1 1 1 −1 1 1 −1 −1 −1 1 −1 1 1 −1 −1 −1 1 1 1 1

NOTE: Designs marked "*" and "#" have the same $(\mathbf{f})_i$ vector as the BIBD-based designs of Nguyen (1996) and Liu and Zhang (2000). The designs marked "&" and "+" are better than those of Nguyen (1996) and Eskridge et al. (2001) under the $r_{\max}$ criterion.

Table 3. Generators for Es^2*-Optimal or Near-Optimal 3-Circulant Designs With n Runs and m Factors*

(n m)	Es^2	$(\mathbf{f})_i$	r_{max}	Generator
(8 21)#	6.40	$(60.0\ 40.0)_0$	.5	−1 −1 −1 −1 −1 −1 −1 1 1 −1 −1 −1 1 −1 1 1 1 −1 1 1 1
(10 27)@	8.10	$(87.2\ 12.8)_2$	.6	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 1 −1 −1 1 1 1 −1 −1 1 1 1 1 1 1
(12 33)#	9.00	$(56.2\ 39.6\ 4.2)_0$	.67	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 1 −1 −1 −1 1 1 1 1 −1 1 −1 1 −1 −1 1 1 1 1 1 1
(12 33)	9.00	$(50.0\ 47.9\ 2.1)_0$	.67	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 1 −1 −1 1 −1 1 −1 1 −1 1 1 1 −1 −1 1 1 1 1 1 1
(12 33)+	9.00	$(43.8\ 56.2)_0$	.33	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 1 −1 −1 1 1 1 −1 1 −1 1 −1 1 1 −1 1 −1 1 1 1 1
(14 39)@	10.74	$(86.0\ 10.5\ 3.5)_2$	.71	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 1 −1 1 −1 1 −1 1 −1 −1 −1 1 1 1 1 1 1 −1 −1 1 1 1 1 1 1
(14 39)@	10.74	$(82.5\ 15.8\ 1.7)_2$	.71	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 1 −1 −1 −1 1 1 1 −1 1 1 1 −1 1 1 −1 −1 −1 1 1 1 1 1 1 1
(14 39)@	10.74	$(79.0\ 21.0)_2$	.43	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 1 −1 −1 1 −1 1 −1 1 −1 1 1 1 −1 −1 1 1 1 −1 1 1 1 1 1 1
(16 45)	11.64	$(54.5\ 36.4\ 9.1)_0$	.5	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 1 −1 −1 −1 1 1 1 −1 1 1 1 −1 1 −1 1 −1 1 1 −1 1 −1 1 −1 −1 1 1 1 1 1 1
(16 45)	11.64	$(50.0\ 42.4\ 7.6)_0$	.5	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 1 −1 −1 −1 1 1 1 −1 1 1 1 −1 −1 1 −1 1 1 1 −1 −1 1 1 −1 −1 1 1 1 1 1 1
(16 45)#	11.64	$(45.4\ 48.5\ 6.1)_0$	.5	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 1 −1 −1 1 −1 1 −1 1 −1 1 1 1 1 1 −1 −1 −1 1 −1 1 1 1 −1 −1 1 1 1 1 1 1
(16 45)	11.64	$(40.9\ 54.6\ 4.5)_0$	.5	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 1 −1 −1 1 −1 1 1 1 −1 −1 1 1 −1 1 −1 1 −1 1 1 1 1 −1 −1 −1 1 1 1 1 1 1
(18 51)@	13.39	$(81.4\ 13.3\ 5.3)_2$	.56	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 1 −1 −1 1 1 1 −1 −1 1 1 1 −1 −1 −1 1 1 1 1 1 1 1 −1 −1 −1 −1 1 −1 1 1 1 1 1 1 1
(18 51)@	13.39	$(78.7\ 17.3\ 4.0)_2$	.56	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 1 1 1 1 −1 −1 −1 1 1 1 −1 −1 1 1 1 1 −1 1 −1 1 −1 1 1 −1 −1 −1 1 1 1 1 1 1 1
(18 51)@	13.39	$(76.0\ 21.3\ 2.7)_2$	.56	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 1 −1 −1 −1 1 1 1 −1 1 1 1 −1 −1 1 −1 1 1 1 −1 −1 1 1 −1 −1 1 1 1 −1 1 1 1 1 1 1
(18 51)@	13.39	$(73.4\ 25.3\ 1.3)_2$	.56	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 1 −1 −1 1 1 1 −1 −1 1 −1 1 −1 1 1 −1 −1 −1 1 1 1 1 1 −1 1 −1 1 −1 1 1 1 1 1 1 1
(18 51)@+	13.39	$(70.7\ 29.3)_2$	.33	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 1 −1 −1 1 1 1 −1 −1 1 −1 1 −1 1 −1 1 1 1 1 −1 1 −1 −1 −1 1 1 1 −1 1 1 1 1 1 1 1

NOTE: Designs marked "#" have the same $(\mathbf{f})_i$ vector as Liu and Zhang's (2000) BIBD-based designs. The near-optimal designs marked "@" have the same Es^2 values as the cyclic supersaturated designs described by Eskridge et al. (2001) and efficiency at least 0.978 relative to the bound (3). The designs marked "+" are better than the cyclic supersaturated designs of Eskridge et al. (2001) and Liu and Zhang (2000) under the r_{max} criterion.

a larger number of factors ($m = 66$ and $n = 12$). The designs are grouped by the value of $k = m/(n-1)$. The listed generators have been selected via a complete search of all possible cyclic generators and, with the possible exception of the designs in Table 6, the listed designs are the best available within the k-circulant class.

For larger m, a complete search is not currently feasible because of time constraints, but a partial search can yield a wide variety of efficient designs, as illustrated for $m = 66$ factors in $n = 12$ runs in Table 6. When m is not a multiple of $(n-1)$, efficient designs often can be found by appending interaction columns (see Sec. 5) to a basic k-circulant design, or by deleting columns from the design. We note the result of Cheng (1997) that, when $m = k(n-1) \pm 1$, any mean-orthogonal column can be added to, or deleted from, an Es^2-optimal design, and the optimality is preserved. Cheng (1997) also proved that when $m = k(n-1) \pm 2$ and $n = 0$ mod 4, the optimality is preserved when any pair of mutually orthogonal mean-orthogonal columns are added to, or deleted from, an Es^2 optimal design. When $m = k(n-1) \pm 2$ and $n = 2$ mod 4, the pair of columns added or deleted must contain $(1, 1)$ and $(-1, -1)$ a total of $(n+2)/4$ times and $(1, -1)$ and $(-1, 1)$ a total of $(n-2)/4$ times to preserve Es^2 optimality. In all of these cases, added columns must be distinct from those already in the design.

For the k-circulant designs in Tables 2–6, the maximum correlation between pairs of columns is listed under $r_{\max}$. When $n = 0$ mod 4, the possible values of the correlation are $0, 4/n, 8/n, \ldots, 1.0$ and when $n = 2$ mod 4, the possible values are $2/n, 6/n, 10/n, \ldots, 1.0$. The percentages of pairs of columns with correlations $i/n, (i+4)/n, (i+8)/n, \ldots$ are listed in the column labeled $(\mathbf{f})_i$, where i is 0 or 2. For example, the entry $(71.4, 23.8, 4.8)_0$ in the fourth row of Table 2 denotes that the corresponding design has 71.4% of pairs of columns with $s_{ij} = 0$, 23.8% of pairs of columns with $s_{ij} = 4$, and 4.8% of pairs of columns with $s_{ij} = 8$. Thus $r_{\max} = 8/n = 8/12$. Similarly, the design in the third row of Table 2 has 94.1% of pairs of columns with $s_{ij} = 2$ and 5.9% of pairs of columns with $s_{ij} = 6$, yielding $r_{\max} = 6/10$. In a supersaturated design, it is not possible to have $r_{\max} = 0$, so those designs that have $r_{\max} = 4/n$ and $n = 0$ mod 4 are $r_{\max}$ optimal in the entire class of supersaturated designs.

Most of the listed designs are Es^2 optimal in the entire class of supersaturated designs; the few nonoptimal designs

Table 4. Generators for Es^2-Optimal or Near-Optimal 4-Circulant Designs With *n* Runs and *m* Factors

$(n\ \ m)$	Es^2	$(\mathbf{f})_i$	r_{max}	Generator
$(8\ \ 28)^{\#}$	8.57	$(55.6\ \ 44.4)_0$	.5	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 −1 1 −1 −1 1 −1 −1 1 1 1 1 −1 1 1 1 1
$(10\ \ 36)^{\#}$	7.11	$(85.7\ \ 14.3)_2$	.6	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 1 1 −1 −1 −1 1 −1 1 1 −1 1 −1 1 1 1 1 −1 1 1 1 1
$(12\ \ 44)$	10.05	$(54.7\ \ 39.5\ \ 5.8)_0$	.67	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 −1 1 −1 −1 1 1 1 1 1 −1 1 1 −1 1 −1 −1 1 −1 −1 1 1 1 1 −1 1 1 1 1
$(12\ \ 44)$	10.05	$(51.2\ \ 44.2\ \ 4.6)_0$	.67	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 −1 1 −1 −1 1 1 1 1 1 −1 −1 1 1 1 1 −1 −1 −1 1 −1 1 1 −1 1 1 1 1 1
$(12\ \ 44)$	10.05	$(47.7\ \ 48.8\ \ 3.5)_0$	.67	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 −1 1 −1 −1 1 1 1 1 1 −1 −1 1 −1 1 1 −1 1 −1 1 −1 1 1 −1 1 1 1 1 1
$(12\ \ 44)^{\#}$	10.05	$(44.2\ \ 53.5\ \ 2.3)_0$	.67	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 −1 1 −1 1 1 −1 1 −1 1 1 −1 1 −1 1 1 −1 1 −1 −1 −1 1 1 1 1 1 1 1 1
$(12\ \ 44)^{@}$	10.42	$(34.9\ \ 65.1)_0$	.33	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 1 1 −1 1 −1 1 −1 −1 1 1 1 −1 −1 1 1 1 1 −1 1 −1 1 −1 −1 1 1 1 1 1
$(14\ \ 52)$	11.53	$(84.3\ \ 11.8\ \ 3.9)_2$	.71	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 1 1 −1 −1 1 1 1 1 −1 −1 1 1 1 1 −1 1 1 1 −1 −1 −1 −1 1 1 −1 1 1 −1 1 −1 1 1 1 1
$(14\ \ 52)$	11.53	$(82.4\ \ 14.7\ \ 2.9)_2$	.71	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 −1 1 1 −1 1 −1 1 −1 1 −1 −1 1 −1 1 1 1 1 1 −1 −1 −1 −1 1 1 1 1 −1 −1 1 1 1 1 1 1 1 1
$(14\ \ 52)$	11.53	$(80.4\ \ 17.6\ \ 2.0)_2$	.71	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 1 1 −1 −1 −1 1 −1 1 1 −1 1 −1 1 −1 1 1 −1 1 −1 −1 1 1 1 1 −1 −1 1 1 1 1 1 1 1 1
$(14\ \ 52)^{\#}$	11.53	$(78.4\ \ 20.6\ \ 1.0)_2$	.71	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 1 1 −1 −1 −1 1 1 1 1 −1 −1 1 1 −1 1 −1 1 1 −1 1 −1 1 1 1 −1 −1 1 −1 1 1 1 1 1 1
$(14\ \ 52)^{+}$	11.53	$(76.5\ \ 23.5)_2$	.43	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 1 1 −1 −1 1 1 1 1 −1 −1 −1 1 −1 1 1 −1 1 1 −1 1 1 −1 1 1 1 −1 1 −1 −1 1 1 1 1 1

NOTE: Designs marked "#" have the same $(\mathbf{f})_i$ vector as Liu and Zhang's (2000) BIBD-based designs. The Es^2-optimal design marked "+" is better than the cyclic supersaturated design of Eskridge et al. (2001) and Liu and Zhang (2000) under the r_{max} criterion. The design marked "@" has smaller r_{max} than the Es^2-optimal designs and has efficiency .964 under the Es^2 criterion relative to the bound (2).

are marked "@" and have efficiency of at least .978 relative to the bound (3). For example, generators are listed in Table 4 for five different 4-circulant designs with $m = 44$ factors in $n = 12$ runs. The first four of these designs are Es^2 optimal with $Es^2 = 10.047$ and attain bound (2). The first of the four designs has the largest value of $(\mathbf{f})_0$, with 54.7% of its 946 pairs of columns orthogonal. The fifth design (marked with "@") is r_{max} optimal and nearly Es^2 optimal. The cost of reducing r_{max} from $8/12$ to $4/12$ is a slight increase in Es^2, from 10.047 to 10.419, and a large decrease in $(\mathbf{f})_0$ from 54.7% to 34.9%. Designs 2, 3 and 4 are less good than design 5 with respect to r_{max} but better with respect to $(\mathbf{f})_0$ and $(\mathbf{f})_4$. These designs are

Table 5. Generators for Es^2-Optimal or Near-Optimal 5-Circulant Designs With *n* Runs and *m* Factors

$(n\ \ m)$	Es^2	$(\mathbf{f})_i$	r_{max}	Generator
$(8\ \ 35)^{\#}$	7.53	$(52.9\ \ 47.1)_0$	.5	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 −1 1 1 −1 −1 1 −1 1 −1 1 1 1 −1 1 −1 1 1 1 1 1
$(10\ \ 45)^{@}$	9.24	$(83.6\ \ 16.4)_2$	.6	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 1 1 1 −1 −1 −1 −1 1 −1 1 1 1 −1 1 −1 1 1 1 1 1 −1 1 1 1 1 1
$(12\ \ 55)$	10.67	$(53.3\ \ 40.0\ \ 6.7)_0$	.67	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 −1 1 1 −1 −1 1 −1 1 1 1 1 1 −1 −1 1 1 1 1 1 −1 −1 −1 −1 −1 −1 1 1 1 1 1 1 1 1 1 1
$(12\ \ 55)$	10.67	$(51.1\ \ 43.0\ \ 5.9)_0$	.67	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 −1 1 1 −1 −1 1 −1 1 1 1 1 1 −1 −1 1 −1 1 1 1 −1 1 −1 −1 −1 −1 1 1 1 1 1 1 1 1 1 1
$(12\ \ 55)$	10.67	$(48.9\ \ 45.9\ \ 5.2)_0$	.67	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 −1 1 1 −1 −1 1 1 1 1 1 1 −1 −1 −1 1 −1 1 1 1 −1 1 −1 1 −1 −1 1 1 −1 1 1 1 1 1 1 1
$(12\ \ 55)^{\#}$	10.67	$(46.7\ \ 48.9\ \ 4.4)_0$	.67	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 −1 1 1 −1 1 1 −1 1 1 −1 1 1 −1 −1 −1 −1 1 1 1 1 1 −1 −1 −1 1 1 1 1 1 −1 1 1 1 1 1
$(12\ \ 55)^{+}$	10.67	$(33.3\ \ 66.7)_0$	.33	−1 −1 −1 1 −1 1 1 −1 1 −1 1 −1 1 −1 −1 −1 1 −1 −1 1 −1 1 1 −1 1 −1 −1 1 −1 −1 1 −1 1 1 1 −1 1 1 1 −1 −1 −1 −1 1 1 1 1 −1 −1 −1 1 −1 −1 −1 1

NOTE: Designs marked "#" have the same $(\mathbf{f})_i$ vector as Liu and Zhang's (2000) BIBD-based designs. The Es^2-optimal design marked "+" is better than the cyclic supersaturated design of Eskridge et al. (2001) and Liu and Zhang (2000) under the r_{max} criterion. The design marked "@" has the same Es^2 value as the cyclic supersaturated design described by Eskridge et al. (2001) and efficiency .993 relative to the bound (3).

Table 6. Generators for Es^2-Optimal 6-Circulant Designs With n Runs and m Factors

$(n\ m)$	Es^2	$(\mathbf{f})_i$	r_{max}	Generator
$(10\ 54)^{\#}$	9.43	$(83.0\ 17.0)_2$	.6	−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 −1 −1 1 1 1 −1
				1 1 −1 −1 −1 1 −1 1 −1 1 1 −1 1 −1 1 −1 1 −1 1 1 1 1 −1 1
				1 1 1 1 1 1
$(12\ 66)$	11.08	$(53.8\ 38.5\ 7.7)_0$	.67	−1 −1 −1 −1 1 −1 1 1 1 −1 1 −1 1 1 −1 1 −1 −1 1 −1 1 −1 −1 1
				−1 −1 1 1 −1 1 1 −1 −1 1 −1 −1 −1 1 1 1 −1 1 −1 1 −1 −1 1 −1
				1 −1 1 −1 1 1 −1 −1 −1 1 −1 1 −1 1 −1 −1 1 −1
$(12\ 66)$	11.08	$(49.2\ 44.6\ 6.2)_0$	.67	−1 −1 −1 −1 1 −1 1 1 1 −1 1 −1 1 1 −1 1 −1 −1 1 −1 1 −1 −1 1
				−1 −1 1 1 −1 1 1 −1 −1 1 −1 −1 −1 1 −1 1 1 1 −1 1 −1 −1 −1 1
				−1 −1 −1 −1 1 −1 −1 1 1 −1 −1 −1 1 −1 1 1 1 1
$(12\ 66)^{\#}$	11.08	$(43.1\ 52.8\ 4.1)_0$	.67	−1 −1 −1 −1 1 −1 1 1 1 −1 1 −1 1 1 −1 1 −1 −1 1 −1 1 −1 −1 1
				−1 −1 1 1 −1 1 1 −1 −1 1 −1 −1 −1 1 −1 1 1 1 −1 1 1 1 −1 −1
				1 −1 1 −1 1 1 −1 −1 −1 −1 1 −1 −1 1 −1 −1 −1 1
$(12\ 66)$	11.08	$(36.9\ 61.0\ 2.1)_0$	.67	−1 −1 −1 −1 1 −1 1 1 1 −1 1 −1 1 1 −1 1 −1 −1 1 −1 1 −1 −1 1
				−1 −1 1 1 −1 1 1 −1 −1 1 −1 −1 −1 1 −1 1 1 1 −1 −1 1 1 1 1
				1 1 −1 −1 1 −1 −1 −1 1 −1 −1 −1 −1 1 −1 −1 −1 1
$(12\ 66)^{+}$	11.08	$(30.8\ 69.2)_0$	.33	−1 −1 −1 −1 1 −1 1 1 1 −1 1 −1 1 1 −1 1 −1 −1 1 −1 1 −1 −1 1
				−1 −1 1 1 −1 1 1 −1 −1 1 −1 −1 −1 1 −1 1 1 1 −1 −1 1 1 1 −1
				1 −1 −1 −1 1 1 −1 1 1 −1 −1 −1 −1 1 −1 −1 −1 1

NOTE: Designs marked "#" have the same $(\mathbf{f})_i$ vector as Liu and Zhang's (2000) BIBD-based designs. The Es^2-optimal design marked "+" is better than the cyclic supersaturated design of Liu and Zhang (2000) under the r_{max} criterion. There are many different 6-circulant Es^2-optimal designs with $m = 66$; the designs given in this table are only a sample.

less good than design 1 with respect to $(\mathbf{f})_0$ and $(\mathbf{f})_4$, but they have fewer pairs of columns with maximum correlation .67. If the factors most likely to be important can be identified in advance, then design 1 may be preferred, because the orthogonal pairs of columns can be allocated to the more important pairs of factors. Otherwise, one might wish to minimize the maximum correlation between pairs of columns, and thus design 5, which has $(\mathbf{f})_8 = 0\%$, might be a better choice. On the other hand, design 3 has many more orthogonal pairs of columns than design 5 and fewer pairs of columns with $s_{ij} = 8$ (correlation .667) than design 1, and so may provide an attractive compromise.

The Es^2-optimal 2-circulant designs in Table 2 marked with "*" have the same $(\mathbf{f})_i$ values as the designs of Nguyen (1996), and the designs in Tables 3–6 marked with "#" have the same $(\mathbf{f})_i$ values as the designs of Liu and Zhang (2000). The design marked "&" in Table 2 is better than Nguyen's design under the r_{max} criterion. The designs marked "+" in Tables 2–6 have smaller r_{max} than the designs of Liu and Zhang (2000) and Eskridge et al. (2001).

As an alternative to combinatorial construction, a search algorithm, such as that of Li and Wu (1997), may be used. Search algorithms have the advantage of flexibility of choice of parameter values and so they can handle, for example, $m \neq k(n-1)$, but they may be susceptible to local maxima. We have compared the performance of our designs with designs obtained from runs of the algorithms of Li and Wu (1997) for the following parameter values: $n = 8$, $m = 18, 25, 34$; $n = 12$, $m = 33, 44, 55, 66$; $n = 20$, $m = 38$; and $n = 22$, $m = 42$. One of their algorithms selects the designs with the best value of (Es^2, r_{max}), whereas the second algorithm selects designs with the best values of (Es^2, $(\mathbf{f})_0$). We have not done an extensive evaluation of the Li and Wu algorithms, but we selected the best designs that we found from a few runs of both algorithms with random starting designs. For $n = 8$, the algorithm gave designs with identical values of Es^2, $(\mathbf{f})_0$, and r_{max}. But for $n = 12, 20$ and 22, the resulting Es^2 values were a little higher than ours, and the r_{max} value tended to be considerably higher, in some cases reaching $r_{max} = 1$. We anticipate that the algorithm might perform better by combining all three criteria, Es^2, r_{max}, and $(\mathbf{f})_0$, as suggested by Li and Wu (1997) and by running it with a larger number of starting designs. The algorithm would be useful for parameter values not considered in this article.

4. PROJECTION EFFICIENCIES OF 2-CIRCULANT DESIGNS

The purpose of a supersaturated design is to screen out inactive factors from consideration in further experimentation. A good design not only should be able to do this with few errors, but also should be able to estimate with accuracy the main effects of the p factors that turn out to be active. Because neither this set of p factors, nor the value of p, is known in advance, the design should be efficient when projected onto every set of p factors for the most likely values of p. Projection properties of Plackett and Burman designs have been discussed by a number of different authors, including Lin and Draper (1992), Wang and Wu (1995), Box and Tyssedal (1996) and Cheng (1998). Efficiencies of nonorthogonal saturated main-effects designs projected into p factors were discussed by Lin (1993b), Dean and Draper (1999), and Crosier (2000). In similar vein, we now examine and compare the projection D efficiencies of k-circulant designs.

For a main-effects model and any given projection of design d into $p < n - 1$ of its factors, we calculate the D efficiency of the projected design as

$$E_{D,d,p} = n^{-1}[\det(\mathbf{X}_p'\mathbf{X}_p)]^{\frac{1}{p+1}}.$$

Here $\mathbf{X}_p$ is the model matrix containing columns for the mean and the main-effects contrasts of the p selected factors. We note in passing that $E_{D,d,p} \leq 1.0$, because the maximum value of $\det(\mathbf{X}_p'\mathbf{X}_p)$ is n^{p+1}, and this maximum is achieved by a projection design with orthogonal columns.

In an Es^2-optimal design, the average "nonorthogonality" among the columns of the design is minimized. However, Es^2 optimal designs of the same size are not necessarily equivalent in terms of the D efficiencies of their projected designs. Therefore, the distribution of values of $E_{D,d,p}$ for each design d

needs to be examined in order to compare the performances of different designs in terms of their ability to estimate models containing any subset of p factors. We define $E^q_{D,d,p}$ to be the $100q$th percentile of the distribution of $E_{D,d,p}$ for projections of design d into p factors. When $q = 0$, we define $E^q_{D,d,p}$ to be the minimum value of $E_{D,d,p}$. Ideally, $E_{D,d,p}$ should be as close to 1.0 as possible for all projected designs. Thus design d_1 is better than design d_2 for detecting all sets of p active factors if $E^q_{D,d_1,p} \geq E^q_{D,d_2,p}$ for all q, and with strict inequality for at least one value of q.

In general, there are $m!/p!(m-p)!$ possible projections into p dimensions for a supersaturated design with m factors. However, as we show later, the number of possible projection designs that need to be examined for a k-circulant design can be reduced dramatically because of the cyclic nature of the design. First, label the columns of the design matrix of design d as $\mathbf{T} = \{c_1, c_2, \ldots, c_m\}$. Suppose that d^u_p is a particular projection of d into the p factors $u, i_2, \ldots, i_p$ ($u < i_2 < \cdots < i_p$), so that the corresponding design matrix of d^u_p is $\mathbf{T}_p = \{c_u, c_{i_2}, \ldots, c_{i_p}\}$, and where $1 \leq u \leq k$, $ak < i_p \leq (a+1)k$, for some $0 \leq a \leq 2t-2$. Thus, the first column of $\mathbf{T}_p$ is one of the first k columns of $\mathbf{T}$ and the last column of $\mathbf{T}_p$ lies in the previously specified range of columns of $\mathbf{T}$. Designs obtained by cycling the column numbers by a multiple of k all have the same projection D efficiency; that is, projected designs d^{u+rk}_p ($r = 0, 1, \ldots, 2t-a-2$) with columns $\{c_{u+rk}, c_{i_2+rk}, \ldots, c_{i_p+rk}\}$ all have the same value of $E_{D,d,p}$. Consequently, we need only calculate $E_{D,d,p}$ for the designs d^u_p, $u = 1, \ldots, k$ and $i_p = ak, \ldots, (a+1)k$ for each value of $a = 0, \ldots, 2t-2$, and each of these designs represents a total of $(2t-a-1)$ projection designs. For each value of a and u, the number of such designs is equal to the number of ways of selecting $i_2, \ldots i_p$ from column numbers $u+1, \ldots, (a+1)k$, and this number is

$$\binom{(a+1)k-u}{p-1} - \binom{ak-u}{p-1}. \qquad (4)$$

Thus the total number of projection designs that need to be evaluated is

$$\sum_{u=1}^{k}\sum_{a=0}^{2t-2}\binom{(a+1)k-u}{p-1} - \binom{ak-u}{p-1} = \sum_{u=1}^{k}\binom{m-u}{p-1}. \qquad (5)$$

For example, instead of examining the D-efficiency values for all 861 two-dimensional projections of a 2-circulant design d with $m = 42$ factors and $n = 22$ runs, we need only examine $41 + 40 = 81$ projection designs, d^u_p, where the first column of each is either column $u = 1$ or column $u = 2$ of d and each represents $(21 - a)$ projection designs, where a is determined by the position of the last column of d^u_p as described earlier.

Table 7 listes the minimum and maximum values of $E_{D,d,p}$ as well as the 25th, 50th, and 75th percentiles of the $E_{D,d,p}$ distribution for the Es^2-optimal 2-circulant designs listed in Table 2 with $m = 30, 34, 38$, and 42. As an example, consider the two 2-circulant designs, d_1 and d_2, listed for $m = 30$ factors, both of which are Es^2 optimal. Design d_2 has the same frequency vector $(f)_i$ as the BIBD-based design of Nguyen (1996) and is better than d_1 under the $r_{\max}$ criterion. However, design d_1 has $(f)_0 = 55.2\%$ of the 435 pairs of contrast columns orthogonal, whereas d_2 has only 44.8% of these. If two of the 30 factors turn out to be active, then the efficiency of the corresponding $p = 2$-dimensional projection design will need to be high. The minimum projection efficiency $E^0_{D,d,2}$ is higher for d_2 than for d_1 (.979 compared with .909), reflecting the lower maximum correlation, $r_{\max}$. However, more than 50% of projections of d_1 into two factors are fully efficient, whereas the best 50% of projections of d_2 have efficiencies in the range .979–1.00, reflecting the smaller $(f)_0$.

As a second example, four generators are listed for $m = 38$ in Table 2, resulting in designs d_5–d_8 in Table 7 that have slightly different properties. Design d_5 has the same frequency vector $(f)_i$ as the BIBD-based design of Nguyen (1996). If three factors turn out to be active, then one of the projections into $p = 3$ dimensions will need to be efficient. Design d_8 has the highest minimum efficiency ($E^0_{D,d,3} = .964$), reflecting the lowest $r_{\max}$. The other three designs all have $r_{\max} = .4$ but different minimum $E^0_{D,d,3}$ values (i.e., .871 for d_5, .908 for d_6, and .924 for d_7). Design d_5 has the largest percentage of orthogonal columns [$(\mathbf{f})_0 = 56.8\%$], and more than 50% of its projections into $p = 3$ factors have efficiency .990–1.00, as compared with .979–1.00 for the other three designs.

To save space, we have not listed the projection properties for all our designs, but in general the foregoing patterns can be expected. Designs with the lowest $r_{\max}$ will yield the highest minimum projection efficiency $E^0_{D,d,p}$. Designs with larger $(\mathbf{f})_0$ will have a larger proportion of projection designs with efficiency 1.0. For a large number of factors, there is an extremely small possibility that a projection design is disconnected. For example, the four 5-circulant designs with $n = 12$, $m = 55$ and $r_{\max} = .67$ in Table 5 have a small probability ($< .0015$) of yielding a disconnected projection design in four or five dimensions.

5. INTERACTION COLUMNS

Most methods described in the literature for constructing efficient supersaturated designs are restricted to limited sets of values of m and n. Therefore few authors have considered adding extra columns to, or deleting columns from, a basic supersaturated design. For example, Cheng (1997) obtained Es^2-optimal designs for $n = 8$ runs and $m = 8, \ldots, 35$ factors by adding or deleting columns from k cyclic BIBD-based supersaturated designs ($k = 1, \ldots, 5$). Wu (1993) suggested an alternative approach of appending interaction columns to saturated designs obtained from Hadamard matrices.

In general, an interaction column can be used either to represent an additional factor or to measure the interaction between two factors. In the latter setting, the definition of projection efficiency would need to be modified so that each projection design contains only interaction columns that represent the interaction between those factors present in the projection. Nevertheless, Es^2 can still be used to give a measure of orthogonality between the columns of the entire design. In this section, we investigate the addition of interaction columns to basic k-circulant designs. We approach this from two perspectives: first, from the standpoint of measuring interactions between the original $m = k(n-1)$ factors, and second, from the standpoint of enlarging the range of possible factors so that $m = k(n-1) + r$ factors can be accommodated in n runs for some $r = 1, 2, \ldots, n-2$. In either case, the first task is to ascertain which interaction columns can be added to the design.

Table 7. Comparison of Projection Properties of Es^2-Optimal 2-Circulant Designs of Table 2

Design	n	m	$(\mathbf{f})_i$	r_{max}	p	$E_D{}^0$	$E_D{}^{1/4}$	$E_D{}^{1/2}$	$E_D{}^{3/4}$	$E_D{}^1$
d_1 New	16	30	$(55.2\ 41.4\ 3.4)_0$	.5	2	.909	.979	1.000	1.000	1.000
					3	.866	.967	.984	.984	1.000
					4	.822	.937	.959	.974	1.000
					5	.772	.913	.942	.956	1.000
d_2 Nguyen	16	30	$(44.8\ 55.2)_0$	.25	2	.979	.979	.979	1.000	1.000
					3	.940	.967	.967	.984	1.000
					4	.866	.944	.960	.960	1.000
					5	.836	.926	.935	.949	1.000
d_3 New	18	34	$(87.9\ 9.1\ 3.0)_2$	.56	2	.884	.996	.996	.996	.996
					3	.852	.966	.990	.991	.991
					4	.795	.949	.966	.985	.987
					5	.755	.919	.956	.965	.983
d_4 Nguyen	18	34	$(81.8\ 18.2)_2$	.33	2	.961	.996	.996	.996	.996
					3	.928	.962	.990	.991	.991
					4	.883	.946	.963	.982	.987
					5	.837	.929	.945	.960	.983
d_5 Nguyen	20	38	$(56.8\ 35.1\ 8.1)_0$	.4	2	.944	.986	1.000	1.000	1.000
					3	.871	.973	.990	.990	1.000
					4	.842	.948	.971	.983	1.000
					5	.805	.934	.950	.971	1.000
d_6 New	20	38	$(48.6\ 46.0\ 5.4)_0$	.4	2	.944	.986	.986	1.000	1.000
					3	.908	.979	.979	.990	1.000
					4	.829	.954	.970	.983	1.000
					5	.807	.936	.955	.968	.993
d_7 New	20	38	$(4.5\ 56.8\ 2.7)_0$	.4	2	.944	.986	.986	1.000	1.000
					3	.924	.973	.979	.990	1.000
					4	.883	.958	.970	.975	1.000
					5	.855	.938	.953	.963	.993
d_8 New	20	38	$(32.4\ 67.6)_0$	.2	2	.986	.986	.986	1.000	1.000
					3	.964	.973	.979	.979	1.000
					4	.929	.958	.966	.975	1.000
					5	.864	.944	.950	.958	.993
d_9 New	22	42	$(87.8\ 9.8\ 0\ 2.4)_2$	.64	2	.841	.997	.997	.997	.997
					3	.833	.978	.994	.994	.994
					4	.764	.973	.977	.990	.991
					5	.752	.948	.972	.978	.988
d_{10} New	22	42	$(85.3\ 9.8\ 4.9)_2$	.46	2	.926	.997	.997	.997	.997
					3	.858	.975	.994	.994	.994
					4	.780	.949	.976	.990	.991
					5	.732	.938	.961	.975	.988
d_{11} New	22	42	$(8.5\ 17.1\ 2.4)_2$	.46	2	.926	.997	.997	.997	.997
					3	.911	.975	.993	.993	.993
					4	.864	.958	.976	.989	.991
					5	.828	.941	.960	.974	.988
d_{12} Nguyen	22	42	$(75.6\ 24.4)_2$	.27	2	.975	.997	.997	.997	.997
					3	.926	.975	.978	.994	.994
					4	.872	.959	.975	.978	.991
					5	.833	.947	.957	.966	.988

We denote the ith column of the design matrix $\mathbf{T}$ by $\mathbf{c}_i = (t_{1i}, t_{2i}, \ldots, t_{ni})'$, and the interaction column is formed from the elementwise product of columns $\mathbf{c}_i$ and $\mathbf{c}_j$ as $\mathbf{c}_{i\cdot j} = (t_{1i} \times t_{1j}, t_{2i} \times t_{2j}, \ldots, t_{ni} \times t_{nj})'$. We use the term *augmented k-circulant design* to denote a basic design with one or more interaction columns appended. To ensure that the augmented design is mean-orthogonal, we require that $\sum_{w=1}^{n} t_{wi} \times t_{wj} = 0$. This means that we may append only interaction columns formed from products of orthogonal (uncorrelated) main-effects columns. For a k-circulant design with $m = (2t-1)k$ factors, no orthogonal columns exist when t is odd. Consequently, mean-orthogonal interaction columns can be appended only to k-circulant designs that have run sizes $n = 0$ mod 4. A second consideration is that interaction column $\mathbf{c}_{i\cdot j}$ $(1 \le i < j \le m)$ can be appended only to a basic or augmented k-circulant design if it is distinct from the columns already present. So we must have $\mathbf{c}_{i\cdot j} \ne \mathbf{c}_h$ for columns $\mathbf{c}_h$ already in the design, or, equivalently, the elementwise product of columns $\mathbf{c}_{i\cdot j}$ and $\mathbf{c}_h$ should not be a column of 1's. For ascertaining which interaction columns satisfy both conditions, we need only investigate $\sum_{i=1}^{k}(m-i) = k(2m-k-1)/2$ interactions among the $m(m-1)/2$ possibilities. This is a consequence of the cyclic structure of the design matrix, which is summarized in Theorem 3. A third consideration is that the interaction columns to be added must also be distinct from each other.

Theorem 3. If column $\mathbf{c}_{i\cdot j}$ $(i = 1, \ldots, k; j = i, \ldots, m)$ satisfies the conditions for being mean-orthogonal and distinct from the columns in a basic or augmented k-circulant design d, then the columns in the cyclically generated set $\{\mathbf{c}_{(i+zk)\cdot(j+zk)};\ z = 1, \ldots, 2t-2\}$ also satisfy these conditions. If column $\mathbf{c}_{i\cdot j}$ is not mean-orthogonal, then neither are any of the columns in the cyclically generated set. Similarly, if column $\mathbf{c}_{i\cdot j}$ repeats one of the columns in d, then each column in the cyclically generated set repeats a column in d.

5.1 Measuring Interactions

In the case of a saturated design with $n = 0 \bmod 4$ constructed from a Hadamard matrix, all interactions can be measured by appending all interaction columns $\mathbf{c}_{i\cdot j}$ (cf. Wu 1993). However, not all interactions can be measured for the factors in a k-circulant design, as discussed earlier. Because a supersaturated design is used only in the presence of factor sparsity, this is not a problem provided that one can assess the most likely interactions in advance and assign the factor labels to the design accordingly.

Example 2. Suppose that we want to ascertain which interaction columns $\mathbf{c}_{i\cdot j}$ can be appended to the basic 2-circulant design in Table 2 with $m = 14$ factors and $n = 8$ runs. Theorem 3 implies that we need only check the $k(2m - k - 1)/2 = 25$ interactions $\mathbf{c}_{1\cdot 2}$, $\mathbf{c}_{1\cdot j}$, and $\mathbf{c}_{2\cdot j}$, $j = 3, 4, \ldots, 14$, among the 91 possibilities. From these 25, we need to screen out the interaction columns that are not mean-orthognal and those that duplicate columns already present. Column 1 is not orthogonal to columns 2, 4, 6, and 12, and column 2 is not orthogonal to columns 1, 5, 11, and 13. Consequently, the seven columns $\mathbf{c}_{1\cdot 2}$, $\mathbf{c}_{1\cdot 4}$, $\mathbf{c}_{1\cdot 6}$, $\mathbf{c}_{1\cdot 12}$, $\mathbf{c}_{2\cdot 5}$, $\mathbf{c}_{2\cdot 11}$, and $\mathbf{c}_{2\cdot 13}$, and their cyclically generated sets of columns, would not be orthogonal to the mean and cannot be added to the design matrix. Further, each of the columnwise products, $(\mathbf{c}_1 \times \mathbf{c}_3 \times \mathbf{c}_{11})$, $(\mathbf{c}_1 \times \mathbf{c}_5 \times \mathbf{c}_7)$, $(\mathbf{c}_1 \times \mathbf{c}_9 \times \mathbf{c}_{13})$, $(\mathbf{c}_2 \times \mathbf{c}_4 \times \mathbf{c}_8)$, $(\mathbf{c}_2 \times \mathbf{c}_6 \times \mathbf{c}_{14})$, and $(\mathbf{c}_2 \times \mathbf{c}_{10} \times \mathbf{c}_{12})$, produces a column of all 1's. Consequently, interaction columns $\mathbf{c}_{1\cdot 3}$, $\mathbf{c}_{1\cdot 11}$, $\mathbf{c}_{1\cdot 5}$, $\mathbf{c}_{1\cdot 7}$, $\mathbf{c}_{1\cdot 9}$, $\mathbf{c}_{1\cdot 13}$, $\mathbf{c}_{2\cdot 4}$, $\mathbf{c}_{2\cdot 8}$, $\mathbf{c}_{2\cdot 6}$, $\mathbf{c}_{2\cdot 14}$, $\mathbf{c}_{2\cdot 10}$, and $\mathbf{c}_{2\cdot 12}$, and their cyclically generated sets of columns, cannot be appended to the 2-circulant design, because they are already present in the basic k-circulant design.

As a result, we conclude that among the 25 interactions $\mathbf{c}_{1\cdot 2}$, $\mathbf{c}_{1\cdot j}$, and $\mathbf{c}_{2\cdot j}$ ($j = 3, 4, \ldots, 14$), only $\mathbf{c}_{1\cdot 8}$, $\mathbf{c}_{1\cdot 10}$, $\mathbf{c}_{1\cdot 14}$, $\mathbf{c}_{2\cdot 3}$, $\mathbf{c}_{2\cdot 7}$, and $\mathbf{c}_{2\cdot 9}$, and their cyclically generated sets of columns, can be added to the basic 2-circulant design, provided that the corresponding cyclically generated sets of columns are distinct. The six cyclic sets are

$\{\mathbf{c}_{1\cdot 8}; \mathbf{c}_{3\cdot 10}; \mathbf{c}_{5\cdot 12}; \mathbf{c}_{7\cdot 14}\}$ $\quad$ $\{\mathbf{c}_{2\cdot 9}; \mathbf{c}_{4\cdot 11}; \mathbf{c}_{6\cdot 13}\}$

$\{\mathbf{c}_{1\cdot 10}; \mathbf{c}_{3\cdot 12}; \mathbf{c}_{5\cdot 14}\}$ $\quad$ $\{\mathbf{c}_{2\cdot 7}; \mathbf{c}_{4\cdot 9}; \mathbf{c}_{6\cdot 11}; \mathbf{c}_{8\cdot 13}\}$

$\{\mathbf{c}_{1\cdot 14}\}$ $\quad$ $\{\mathbf{c}_{2\cdot 3}; \mathbf{c}_{4\cdot 5}; \mathbf{c}_{6\cdot 7}; \mathbf{c}_{8\cdot 9}; \mathbf{c}_{10\cdot 11}; \mathbf{c}_{12\cdot 13}\}$.

All of these columns are distinct, and so any of the corresponding interactions can be measured. Following Cheng (1997), the addition of any one of these interaction columns to the basic Es^2 optimal k-circulant design still results in an Es^2-optimal design, as does the addition of any orthogonal pair, such as $\mathbf{c}_{12\cdot 13}$ and $\mathbf{c}_{1\cdot 11}$. Examples of Es^2-optimal designs for the addition of up to six interaction columns are shown in the top part of Table 8.

The middle part of Table 8 shows Es^2-optimal designs for the addition of up to six interaction columns to the Es^2 optimal 3-circulant design of Table 3 with 21 factors and 8 runs. Similarly, the bottom part of Table 8 shows Es^2-optimal designs for the addition of up to seven interaction columns to the 4-circulant design of Table 4 with 28 factors and 8 runs.

5.2 Enlarging the Number of Factors

Cheng (1997) showed how to obtain Es^2-optimal designs by addition and deletion of one, two, or three columns with particular properties from k cyclic BIBD-based supersaturated designs (which form a subset of the k-circulant class). Cheng gave a complete solution for Es^2-optimal designs for $m = 8, \ldots, 35$ factors in $n = 8$ runs. The designs listed in Table 8 form an al-

Table 8. Es^2-Optimal Designs for $n = 8$ Constructed by Adding Interaction Columns From The k-Circulant Supersaturated Design in Tables 2–4

m	Interaction columns	$(\mathbf{f})_0$	r_{max}	Es^2
14		$(69.2\ \ 3.8)_0$	.5	4.923
15	$\mathbf{c}_{12\cdot 13}$	$(65.7\ \ 34.3)_0$	.5	5.486
16	$\mathbf{c}_{12\cdot 13}, \mathbf{c}_{10\cdot 11}$	$(63.3\ \ 36.7)_0$	.5	5.867
17	$\mathbf{c}_{12\cdot 13}, \mathbf{c}_{10\cdot 11}, \mathbf{c}_{8\cdot 13}$	$(61.8\ \ 38.2)_0$	.5	6.118
18	$\mathbf{c}_{12\cdot 13}, \mathbf{c}_{10\cdot 11}, \mathbf{c}_{8\cdot 13}, \mathbf{c}_{2\cdot 7}$	$(6.8\ \ 39.2)_0$	.5	6.275
19	$\mathbf{c}_{6\cdot 13}, \mathbf{c}_{5\cdot 14}, \mathbf{c}_{8\cdot 13}, \mathbf{c}_{2\cdot 3}, \mathbf{c}_{4\cdot 5}$	$(6.2\ \ 39.8)_0$	.5	6.363
20	$\mathbf{c}_{2\cdot 9}, \mathbf{c}_{6\cdot 13}, \mathbf{c}_{5\cdot 14}, \mathbf{c}_{6\cdot 11}, \mathbf{c}_{1\cdot 14}, \mathbf{c}_{2\cdot 3}$	$(60\ \ 40)_0$	.5	6.400
21		$(60\ \ 40)_0$	.5	6.400
22	$\mathbf{c}_{9\cdot 16}$	$(58.4\ \ 41.6)_0$	.5	6.649
23	$\mathbf{c}_{9\cdot 16}, \mathbf{c}_{3\cdot 16}$	$(57.3\ \ 42.7)_0$	.5	6.830
24	$\mathbf{c}_{9\cdot 16}, \mathbf{c}_{3\cdot 16}, \mathbf{c}_{13\cdot 21}$	$(56.5\ \ 43.5)_0$	.5	6.957
25	$\mathbf{c}_{9\cdot 16}, \mathbf{c}_{3\cdot 16}, \mathbf{c}_{13\cdot 21}, \mathbf{c}_{10\cdot 18}$	$(56\ \ 44)_0$	.5	7.040
26	$\mathbf{c}_{9\cdot 16}, \mathbf{c}_{3\cdot 16}, \mathbf{c}_{13\cdot 21}, \mathbf{c}_{10\cdot 18}, \mathbf{c}_{7\cdot 15}$	$(55.7\ \ 44.3)_0$	.5	7.089
27	$\mathbf{c}_{9\cdot 16}, \mathbf{c}_{3\cdot 16}, \mathbf{c}_{13\cdot 21}, \mathbf{c}_{10\cdot 18}, \mathbf{c}_{7\cdot 15}, \mathbf{c}_{4\cdot 12}$	$(55.6\ \ 44.4)_0$	.5	7.111
28		$(55.6\ \ 44.4)_0$	.5	7.111
29	$\mathbf{c}_{7\cdot 25}$	$(54.7\ \ 45.3)_0$	.5	7.251
30	$\mathbf{c}_{7\cdot 25}, \mathbf{c}_{3\cdot 21}$	$(54\ \ 46)_0$	.5	7.356
31	$\mathbf{c}_{7\cdot 25}, \mathbf{c}_{3\cdot 21}, \mathbf{c}_{17\cdot 27}$	$(53.5\ \ 46.5)_0$	.5	7.432
32	$\mathbf{c}_{7\cdot 25}, \mathbf{c}_{3\cdot 21}, \mathbf{c}_{17\cdot 27}, \mathbf{c}_{13\cdot 23}$	$(53.2\ \ 46.8)_0$	.5	7.484
33	$\mathbf{c}_{7\cdot 25}, \mathbf{c}_{3\cdot 21}, \mathbf{c}_{17\cdot 27}, \mathbf{c}_{13\cdot 23}, \mathbf{c}_{9\cdot 19}$	$(53\ \ 47)_0$	.5	7.515
34	$\mathbf{c}_{7\cdot 25}, \mathbf{c}_{3\cdot 21}, \mathbf{c}_{17\cdot 27}, \mathbf{c}_{13\cdot 23}, \mathbf{c}_{9\cdot 19}, \mathbf{c}_{5\cdot 15}$	$(52.9\ \ 47.1)_0$	.5	7.529
35	$\mathbf{c}_{7\cdot 25}, \mathbf{c}_{3\cdot 21}, \mathbf{c}_{17\cdot 27}, \mathbf{c}_{13\cdot 23}, \mathbf{c}_{9\cdot 19}, \mathbf{c}_{5\cdot 15}, \mathbf{c}_{1\cdot 11}$	$(52.9\ \ 47.1)_0$	.5	7.529

NOTE: The choices of interaction columns are not unique; the interaction columns in the table are only one example. Our Es^2 optimality comes from the comparision with the improved bound formula given by Butler et al. (2001).

ternative solution obtained by addition of columns only.

Designs for $m = k(n-1) + r$ factors ($r = 1, \dots, n-2$) can be constructed in a similar way for other values of n, by appending linearly independent interaction columns to a basic k-circulant design. If the basic k-circulant design is Es^2 optimal and two orthogonal interaction columns are appended, then Cheng's (1997) result guarantees that the design for $m = k(n-1) + 2$ factors is also Es^2 optimal. In future work we will investigate the deletion of columns from k-circulant designs.

6. CONCLUSION

In this article, we have defined a class of k-circulant supersaturated designs that are constructed by cycling a generator k elements at a time. We have presented generators that lead to designs that are either Es^2 optimal and near $r_{\max}$ optimal or $r_{\max}$ optimal and near Es^2 optimal. We have shown that Es^2 optimal 2-circulant designs differ in terms of their projection efficiencies and have listed a selection of designs with different properties. We have also shown that certain interaction columns can be added to k-circulant designs that have orthogonal pairs of columns and we have shown by example that the extended design is efficient under the Es^2 criterion. Further results on interaction columns will be discussed in a future article.

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APPENDIX: PROOF OF THEOREM 1

Necessity

(a) The first row of the design matrix $\mathbf{T}$ of design d contains the same elements as the generator. Each of the succeeding $n-2$ rows of $\mathbf{T}$ is obtained from the previous row by cycling k elements to the right. The nth row consists of $+1$'s. Since the first row of $\mathbf{T}$ is identical to the generator, the first k elements in the generator become elements $k+1, k+2, \dots, 2k$ of row 2, elements $2k+1, 2k+2, \dots, 3k$ of row 3, and so on, and finally become elements $(n-2)k+1, \dots, (n-1)k$ in the $(n-1)$st row. Thus $m = (n-1)k$. The design is mean-orthogonal if and only if each factor is observed the same number t of times at each of its two levels; that is, if and only if $n = 2t$ and $m = (2t-1)k$.

(b) The first column of the design matrix $\mathbf{T}$, which corresponds to the first factor, has last element 1 (due the last row of all 1's). The other $n-1$ elements are elements g_1, $g_{k+1}, \dots, g_{(2t-2)k+1}$ of the generator. For the first column to have an equal number of $+1$ and -1, there must be one more -1 than $+1$ among these elements of the generator. Similarly, the first $(n-1)$ elements of column j ($2 \le j \le k$) of $\mathbf{T}$ consist of elements $g_j, g_{k+j}, \dots, g_{(2t-2)k+j}$ of the generator. Thus these also must include one more -1 than $+1$. Columns $k+j, 2k+j, \dots, (2t-2)k+j$ consist of cyclic permutations of the elements in column j and do not place additional requirements on the generator. Therefore, the design is mean-orthogonal if and only if there are k more elements -1 than $+1$ among the $m = (2t-1)k$ elements in the generator.

(c) In a mean-orthogonal design, the sum of the elements in each column of $\mathbf{T}$ is 0. The result follows from the proof of (b), because the sum of elements in column j ($1 \le j \le k$) is $\sum_{u=0}^{2t-2} g_{uk+j} + 1$ and each remaining column of $\mathbf{T}$ contains a permutation of elements of one of the first k columns.

Sufficiency

From (a), and the construction of a k-circulant design, the jth column of $\mathbf{T}$ must be $(g_j, g_{j+k}, \dots, g_{j+(2t-2)k}, 1)'$ (with any $j + rk > m$ replaced by $j + rk - m$). From (c), $\sum_{u=0}^{2t-2} g_{uk+j} + 1 = 0$, so the jth column contains an equal number of $+1$ and -1, so that the design is mean-orthogonal.

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