Pricing and Operational Recourse in Co-production Systems

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Abstract

Co-production systems, in which multiple products are produced simultaneously in a single production run, are prevalent in many industries. Such systems typically produce a random quantity of vertically differentiated products. This product hierarchy enables the firm to fill demand for a lower-quality product by converting a higher-quality product. In addition to the challenges presented by random yields and multiple products, co-production systems often serve multiple customer classes that differ in their product valuations. Furthermore, the sizes of these classes are uncertain. Employing a utility-maximizing customer model, we investigate the production, pricing, downconversion and allocation decisions in a two-class, stochastic-demand, stochastic-yield co-production system. For the single-class case, we establish that downconversion will not occur if prices are set optimally. In contrast, we show that downconversion can be optimal in the two-class case even if prices are set optimally. We consider the benefit of postponing certain operational decisions, e.g., the pricing or allocation-rule decisions, until after uncertainties are resolved. We use the term recourse to denote actions taken after uncertainties have been resolved. We find that recourse pricing benefits the firm much more than either downconversion or recourse allocation do, implying that recourse demand-management is more valuable than recourse supply-management. Special cases of our model include the single-class and two-class random-yield newsvendor models.

Keywords: Flexibility, random yield, utility-maximizing customers.

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1 Introduction

A key feature of many biochemical, chemical and material-processing operations is that multiple products are produced simultaneously in a single production run. Such systems are sometimes referred to as co-production systems (Bitran and Leong (1992) and Bitran and Gilbert (1994)). In the energy industry the term co-production is sometimes used to refer to the practice in which consumers of electricity also produce energy that is routed to the grid. That usage differs from the intended meaning of co-production in this paper, that being the simultaneous production of multiple products.

Semiconductor manufacturing is one example of a co-production system. The fabrication process produces random quantities of devices of varying speeds. (A device’s speed is the result of random events that occur during fabrication.) Devices are then tested and binned according to their processor speed, with higher bins containing higher-speed devices. This results in a nested product structure that gives the firm some flexibility in meeting demand: “depending on the demand in the marketplace, fast devices can be configured to run more slowly, but slow devices can not be enticed to run faster,” Kempf (2004). Filling demand for a low-speed device with a reconfigured high-speed device is sometimes referred to as downbinning. We note that downbinning “a product whereby its speed is lowered by blowing fuses in the chip” (Johnson (2005)) is a fast operation, on the order of days, whereas fabrication can take up to three months (Johnson, 2005; Wu et al., 2006). As such, the fabrication quantity decision is made before yield and demand uncertainties are realized, but the downbinning decision occurs after uncertainties are realized (Johnson, 2005).

Downbinning is valuable to the firm when it has an excess of high-speed devices but a shortage of low-speed ones. However, why would a firm incur the downbinning (reconfiguration) cost? It could, instead, provide the low-speed customer a high-speed device at the low-speed price (and the customer would willingly accept this direct substitution). Such direct substitution, however, cannibalizes the high-speed demand as some customers may be willing to purchase the high-speed device at the high-speed price if the low-speed device is sold out. There are also strategic reasons for firms to reconfigure high-speed devices before using them to fill low-speed demands. For example, reconfiguration prevents opportunistic reselling of high-speed devices that were obtained at low-speed prices. Reconfiguration is a common practice in the semiconductor industry. See Kempf
Semiconductor devices vary greatly in their life cycles depending on their end use. “The life cycle for custom ICs on cellular phones is quite different from that for the motor controller ICs for hard drives” (Wu et al., 2006), with the life cycle of devices used in fashion-products, e.g., phones, being very short - on the order of months. For such devices, the life cycle can be short enough relative to the production lead time that firms cannot replenish their inventory during the life cycle. For other devices, the life cycle is long enough to allow periodic replenishment.

A similar story plays out in many other co-production systems. A single production run results in random quantities of vertically differentiated products. By vertical differentiation, we mean that the products differ along a key performance dimension, e.g., speed, for which all customers agree that more is better (at the same price.) The nested product structure allows the firm to fill demand for a lower-quality product by either converting a high-quality product or by directly substituting the higher-quality product. We use the term downconversion to refer to the practice of converting a higher-quality product to a lower-quality one, and we use the term downgrading to refer to the practice of direct substitution.

The management of co-production systems is complicated by the fact that firms typically sell to heterogenous customers, that is, customers differ in their valuation of quality. Customers with the same valuation can be thought of as being members of a particular customer class. The number of customers in each class may be uncertain. Customers not receiving their preferred product may choose to purchase some other product that is available.

Existing literature on co-production systems has concerned itself with the production-quantity and substitution decisions under the following assumptions; prices are exogenous, the preferred product of each customer class is exogenous, each customer class has a different preferred product, unfilled demand does not spill over to other products, and there is no cost to downconversion. We note that zero-cost downconversion and downgrading are equivalent in these models as unfilled demand does not spill over. Bitran and Dasu (1992), Bitran and Leong (1992), and Bitran and Gilbert (1994) all propose various heuristic solutions to the multiple-period, deterministic-demand, N-product problem. Gerchak et al. (1996) investigate the structural properties of the optimal solution to the single-period, deterministic-demand, two-product problem. Hsu and Bassok (1999) and Rao et al. (2004) develop algorithms and heuristics for optimizing the single-period, stochastic-
In this paper, we consider the pricing decision in co-production systems and allow for spill-over of unmet demand. Furthermore, we allow for costly downconversion. We note that when prices are endogenous and/or when customer spill-over is considered, multiple customer classes might wish to purchase the same grade of product, a situation that cannot occur in the papers cited above. This highlights a further complication in managing co-production systems; firms must decide how to allocate their inventories (after downconversion) among competing customer classes. Thus, there are four key decisions in managing a co-production system; product pricing, production quantity, downconversion quantities, and the allocation rule.

With both supply and demand uncertainties present in co-production systems, mismatches in realized supply and demand are to be expected. After uncertainties have been resolved, the firm might be able to take some action to better match realized supply and demand. We use the term operational recourse to describe any action taken after uncertainty is resolved. Downconversion is one such operational recourse; it allows the firm to adjust the proportion of the various products after uncertainties are resolved. The firm may have other operational recourses available to it. Rather than pricing in advance of uncertainty resolution, the firm may choose to postpone pricing until after uncertainties are resolved. Likewise, it may choose to postpone its decision regarding the allocation rule. Postponement of decisions until uncertainties are resolved are sometimes classified as operational hedges (Van Mieghem, 2003; Ding et al., 2004). As ours is a risk-neutral setting, we prefer to use the term operational recourse as it more precisely reflects the salient feature of these uncertainty-management approaches. The implementation of recourse actions is not the only option for improving the performance of co-production systems. A firm might instead (or in addition) consider operational-improvement actions such as yield-uncertainty reduction, product-quality improvement and production-cost reduction. Therefore, in measuring the value of any given recourse action (for example moving from advance pricing to recourse pricing), we compare the value obtained to that obtained by operational-improvement actions.

In this paper, we embed a demand model of utility-maximizing customers in a single-period, two-product, two customer class, stochastic-demand co-production system. Our utility-maximizing demand model ensures that a customer’s purchasing behavior is completely consistent with her valuation of quality and the product prices. We establish that downconversion is not optimal in
the single-class case if prices are set optimally (but downconversion can be optimal if prices are not set optimally.) In contrast, downconversion can be optimal in the two-class case even if prices are set optimally. We show that a priority-based allocation policy is optimal for recourse allocation regardless of the timing of the pricing decision. Of the three potential recourse actions, we find that recourse-pricing delivers significantly more benefit than downconversion or recourse-allocation. In addition, the value of either downconversion or recourse allocation is significantly reduced if recourse pricing is implemented. Relative to operational improvement actions, numeric results show that implementing recourse pricing delivers approximately the same benefit as completely eliminating the yield uncertainty, and a benefit equivalent to a 9% reduction in production cost, a 7% increase in expected yield of the high-quality product, or a 38% increase in the perceived quality of the low-grade product.

The rest of the paper is organized as follows. Section 2 surveys the existing related literature. Section 3 introduces the model. We consider a single customer-class version of the model in §4, and then the two customer-class version in §5. In §6 we consider some alternative models. For example, we consider downgrading and investigate whether it can be preferred to downconversion. Conclusions and opportunities for future research are discussed in §7. Proofs of all theorems can be found in Appendix A1. An unabridged version of this paper containing some additional results is available upon request.

2 Literature

As the existing co-production literature has been discussed in the introduction, we now focus on three other streams of literature related to this paper; the joint quantity-and-price setting problem, investment in flexible resources, and utility-maximizing demand models in operations.

While the joint quantity-and-price setting problem has been studied quite extensively in the literature, most of the papers assume perfectly-reliable supply and a single product. Petruzzi and Dada (1999), Van Mieghem and Dada (1999) and Dana and Petruzzi (2001) all consider the quantity-and-price setting problem in a single-product, perfect-supply, newsvendor setting. Of these papers, only Van Mieghem and Dada (1999) consider operational recourse actions. Van Mieghem and Dada (1999) show that production postponement is of little value if the firm also engages in

The joint quantity-and-price setting problem with multiple products has received much less attention. Bish and Wang (2004) and Chod and Rudi (2005) consider the quantity-and-price setting problem in a perfect-supply, two-product, newsvendor model, where the quantity is set before demand uncertainty is resolved but price is set after the demand uncertainty is resolved. Advance-pricing is not considered. Bish and Wang (2004) consider dedicated and flexible resources, and they show that it can be optimal for the firm to invest in flexible resource even with perfectly correlated demands. Chod and Rudi (2005) focus on a single flexible resource, and they prove that optimal resource level is increasing in both demand variability and correlation. Both of these papers use price-dependent aggregate demand models, and implicitly assume horizontal rather than vertical product differentiation. In Bish and Wang (2004), demand for one product is not influenced by the price of the other product. In essence, they implicitly assume that there are two customer classes, and each class is willing to buy one product and one product only. Chod and Rudi (2005) do allow for cross-price demand dependencies. Aggregate demand models, as used in these two papers, would not be reasonable in the case of vertical-differentiation, as such models would not a-priori rule out unreasonable situations such as there being demand for a lower-quality product even if its price is higher than that of a higher-quality product. Our utility-maximizing demand model rules out any such inconsistencies.

To the best of our knowledge, Li and Zheng (2006) is the only paper to consider the joint quantity-and-price setting problem in the presence of supply-uncertainty. They investigate a single product, periodic-review model, where inventory replenishment and pricing decisions are set at the beginning of each period. They do not consider multiple products nor operational recourses. We note that Kazaz (2004) considers a single-product production-planning problem under yield uncertainty, with emergency procurement as a recourse action, but price is an exogenous function of realized yield.

Mix flexibility, whereby a resource (inventory or capacity) has the ability to produce multiple products, has been the subject of much attention in the operations literature. A resource may be
totally flexible (i.e., it can fill demand for any product), partially flexible (i.e., it can fill demand for a subset of products), or dedicated (i.e., it can only fill demand for a particular product). Many papers (Fine and Freund, 1990; Gupta et al., 1992; Li and Tirupati, 1994, 1995, 1997; Van Mieghem, 1998, 2004; Tomlin and Wang, 2005) focus on investments in dedicated and totally flexible resources. Jordan and Graves (1995) and Graves and Tomlin (2003) investigate how the structure of partial-flexibility in supply chains influences performance. These papers all assume exogenous prices. Bish and Wang (2004) and Chod and Rudi (2005) consider the product-pricing decision in the context of totally flexible resources.

Co-production systems with vertical differentiation result in a special case of partial flexibility, whereby products are downwardly flexible. Netessine et al. (2002) investigate the resource-investment problem in a downwardly flexible system in the context of services such as rental cars. Unlike in co-production systems, resource investments are not subject to yield uncertainty, and each resource requires a separate investment. Downgrading rather than downconversion is assumed, although these specific terms are not used. Prices are exogenous in Netessine et al. (2002). We note that all of the flexibility papers cited are single-period models.

The operations management literature has typically assumed that demand is independent of a firm’s operating decisions. Linking demand to operating decisions can be achieved with aggregate demand functions, as done by Bish and Wang (2004) and Chod and Rudi (2005), or by explicitly modeling individual customers as utility-maximizing entities. This latter approach has been used by van Ryzin and Mahajan (1999) and Mahajan and van Ryzin (2001b) to determine the optimal assortment and stocking levels of products in a single-period problem with exogenous prices. The first paper assumes customers not receiving their first-choice products are lost. The second paper allows consumer to spill over to other products. Mahajan and van Ryzin (2001a) adopt a consumer choice model to study inventory competition in a multiple-firm, single-period problem with exogenous prices. Talluri and van Ryzin (2004) use a consumer-choice model in an airline revenue-management setting to study the assortment (fares to make available) problem in a finite-horizon, fixed-capacity problem. Dana and Petruzzi (2001) is the only paper of which we are aware that uses a utility-maximizing customer model to investigate a joint quantity-and-pricing problem. They study a single-product, perfect-supply newsvendor problem, in which customers decide between purchasing a firm’s product (at price $p$) or some outside option. Ex ante, all customers are
identical, that is, they have the same deterministic valuation $V$ for the firm’s product and the same distribution $G(u)$ [density $g(u)$] for the utility of their outside option. As such there is a single class of customers. $G(u)$ is assumed to satisfy the condition $R'(\cdot) < R(\cdot)^2$, where $R(\cdot)$ is the reverse hazard rate function, i.e., $R(\cdot) = g(\cdot)/G(\cdot)$. Ex post, customers differ depending on the realized utility of their outside option. A consumer will purchase the product (if available) if $V - p \geq u$, but will walk away otherwise. Our paper builds upon this utility-maximizing model to allow for two products and two customer classes.

We now summarize the contribution of this paper. In addition to extending the co-production literature by considering pricing, downconversion and allocation, our paper extends the other three streams of literature as follows. To the best of our knowledge, it is the first paper to consider the joint quantity-and-price setting problem for multiple products in the presence of supply-uncertainty, the first to consider the joint production-and-price setting problem in the class of downwardly flexible systems, and the first to use a utility-maximizing customer model for the joint quantity-and-price setting problem in either uncertain-supply or multi-product settings.

3 The Model

We consider a monopolist newsvendor that sells two products (H and L) in a market with two classes of utility-maximizing customers. The two products differ in their quality level, with H being a higher quality than L. We use the term quality in the broad sense of any performance dimension for which all customers agree that more is better, i.e. we assume a vertical differentiation between the products. The firm sells product H at a price of $p_H$ and product L at a price of $p_L$. We assume that the firm cannot price discriminate between customer classes, that is, it has to offer the same price to both classes. We will discuss the timing of the firm’s pricing decision after we have first described the firm’s production system and the utility-maximizing model that specifies customer-purchasing behaviors.

The firm operates a co-production system in which the two products are produced simultaneously in a single production run. A production run of size $Q$ produces $q_H = yQ$ units of product H and $q_L = (1 - y)Q$ units of product L, where $y$ is the realization of a yield random variable $\tilde{Y}$. The yield random variable has a distribution function $F_Y(\cdot)$ with support between 0 and 1. Production
thus results in a random split between the two products. Our yield random variable is sometimes referred to as the split factor in the semiconductor industry, e.g., Wang et al. (2004). The total production cost is linear in the quantity launched, with the marginal production cost given by $c_P$. We note that all results carry through if the production cost is convex in $Q$. After production, the firm has the opportunity to downconvert product H to product L at a marginal cost of $c_D$ per unit converted. One can think of product H as being a flexible or a customizable product. After downconversion, the firm then has an inventory of $q_H - q_D$ units of product H and $q_L + q_D$ units of product L, where $q_D \leq q_H$ is the quantity of H converted to L.

The market consists of two customer classes (1 and 2). The market potential (or size) of each class is uncertain, with $F_X(\cdot)$ denoting the joint distribution function for the market potentials. Customers within a class are infinitesimal and are homogeneous with respect to their valuations of the two products. Customer class $i = 1, 2$ has a valuation of $a_{iH}$ for product H and $a_{iL}$ for product L. We assume that $a_{iH} \geq a_{iL}$ for $i = 1, 2$; reflecting the fact that product H is of higher quality. Without loss of generality we index the classes such that $a_{1H} \geq a_{2H}$. The utility that a class $i$ customer derives from purchasing product H (or L) is then $a_{iH} - p_H$ (or $a_{iL} - p_L$). Each customer also has an outside option that provides her with a utility of $u \in \mathbb{R}^+$. Ex ante, all customers within a class are identical but ex post they differ in the utility of their outside option, and hence differ in their willingness to pay. Let the distribution of outside-option utilities for class $i$ customers be $G_i(\cdot)$. We assume that $G_1(\cdot)$ and $G_2(\cdot)$ have independent, continuous distributions but allow them to be non-identically distributed. In addition, we assume that $R_i'(\cdot) < R_i(\cdot)^2$ for $i = 1, 2$, where $R_i(\cdot) = g_i(\cdot)/G_i(\cdot)$. We note that $R_i'(\cdot) < 0$ is equivalent to $G_i(\cdot)$ being log-concave, and so all log-concave distribution functions satisfy $R_i'(\cdot) < R_i(\cdot)^2$. Log concave distribution functions include the Uniform, Normal, Weibull, Gamma and many other common distributions. Truncations of log-concave distributions are also log-concave (Bagnoli and Bergstrom, 2005).

Customers observe their outside utility before making their purchasing decision. In an infinite-supply environment, a customer would choose the option (purchase H, purchase L, or take the outside option) that maximizes her utility. In a limited-supply environment, a customer may not get her first-choice option due to inventory limitations. We assume the outside option is always available, and therefore supply limitations are driven by the final quantities of H and L that result from the firm’s production realization and downconversion decision. If a customer’s first-choice is
unavailable, she then chooses among her remaining two options to again maximize her utility. If
her second-choice is unavailable, the customer is then lost to the firm.

The firm has four decisions to make in managing this system; the production quantity $Q$ to
launch, the post-production downconversion quantity $q_D$, the pricing vector $p = (p_H, p_L)$ to charge,
and the allocation or rationing rule to use if both customer classes demand the same product. We
discuss each decision in turn.

- **Production quantity, $Q$:** This decision must be made before market uncertainty, i.e., the size
  of each customer-class, is resolved.

- **Downconversion quantity, $q_D$:** This decision is made after both production-yield and market-
  size uncertainties have been resolved, and can be used to better balance the supply and
demand for the two products. In Section §6.2, we consider a model in which there is some
residual market uncertainty when downconversion occurs. We note that the firm cannot sim-
ply downgrade product H, that is, sell H to a customer at the price of product L; it must first
convert H to L. Downgrading is considered in §6.1. We assume that all customer purchasing
decisions are made instantaneously, and so the firm must complete any downconversion before
customer purchasing begins. (An alternative model that relaxes this assumption is considered
in the unabridged version of the paper.)

- **Price vector $p = (p_H, p_L)$:** We consider two alternatives for the timing of the pricing deci-
  sion. In the advance-pricing case, prices are set before either yield uncertainty or market-size
  uncertainty has been resolved. In the recourse-pricing case, prices are set after both yield and
  market-size uncertainties have been resolved. This provides the firm with another lever to bal-
  ance supply and demand. We note that the optimal prices must satisfy $p_L^* < \max\{a_{1L}, a_{2L}\}$
  and $p_H^* < a_{1H}$ as otherwise no customer will purchase L or H. We, therefore, restrict at-
ten tion to $p_L < \max\{a_{1L}, a_{2L}\}$ and $p_H < a_{1H}$ in all that follows. For a given price vector
  $p = (p_H, p_L)$, customer class $i = 1, 2$ prefers H to L if $a_{iH} - p_H \geq a_{iL} - p_L$, but prefers
  L to H if $a_{iH} - p_H < a_{iL} - p_L$. We use the convention that a customer who is indifferent
  between H and L will first try to purchase H. Therefore, we say that class $i$ prefers H to L
  iff $a_{iH} - p_H \geq a_{iL} - p_L$. The distribution for the utility of class $i$ customers’ outside option
  is represented by $G_i(\cdot)$, and so for a given price vector $p = (p_H, p_L)$, the fraction of cus-
tomers who prefer H to their outside option is \( G_i(a_{iH} - p_H) \). For product L the fraction is \( G_i(a_{iL} - p_L) \). If \( a_{iH} - p_H \geq a_{iL} - p_L \), then \( \frac{G(a_{iL} - p_L)}{G(a_{iH} - p_H)} \) is the fraction of those class \( i \) customers whose first-choice is H who also prefer L to their outside option, i.e., it is the fraction of class \( i \) customers who spill over from H to L. If \( a_{iL} - p_L > a_{iH} - p_H \), then the fraction of class \( i \) customers who spill over from L to H is \( \frac{G(a_{iH} - p_H)}{G(a_{iL} - p_L)} \).

- **Allocation:** In the event of the firm having insufficient inventory of a particular product to meet demand of both classes, the firm must allocate or ration its inventory. As with pricing, we consider advance allocation and recourse allocation. Under advance allocation, the firm determines its allocation policy before uncertainties are resolved. Under recourse allocation, the firm determines its allocation policy after both production-yield and market-size uncertainties have been resolved.

In closing, we note that this model generalizes a number of newsvendor models that, to the best of our knowledge, have not been studied previously. Please see Figure 1. The lower right quadrant (CPP2) is the general model while special cases are contained in the other quadrants. We analyze CPP1 in §4 and CPP2 in §5. We analyze the random-yield special cases, RYP1 and RYP2, in Appendices A2 and A3 respectively. Deterministic-supply versions of all these cases are obtained by setting the yield random variable \( y \) equal to some \( 0 \leq \overline{y} \leq 1 \) with probability 1.

*** Insert Figure 1 here ***

## 4 A Single-Class, Co-Production Model with Pricing

In this section, we consider the single-class co-production model (labeled CPP1), and therefore remove the class subscript \( i = 1, 2 \) from all parameters. As there is only one class, no allocation decision is needed. The relevant decisions facing the firm are the price vector \( p = (p_H, p_L) \), the production quantity \( Q \) and the downconversion quantity \( q_D \). We first characterize the optimal downconversion quantity for a given price vector \( p \), realized yield \( y \) and realized market potential \( x \). We then proceed to consider the quantity-and-price setting problem. This section concludes with a numerical investigation of the impact of recourse pricing and operational improvements.
4.1 The Optimal Downconversion Quantity

The firm chooses the downconversion quantity $q_D$ after yield and market uncertainties have been resolved. The realized quantity of product H is $q_H = yQ$, the realized quantity of product L is $q_L = (1 - y)Q$, and the realized market potential is $x$. The firm chooses the downconversion quantity $0 \leq q_D \leq q_H$ to maximize $r(q_D) - c_D q_D$, where the revenue function

$$r(q_D) = p_H \min \{xG(a_H - p_H), q_H - q_D\} + p_L \min \left\{ [xG(a_H - p_H) - (q_H - q_D)]^+, \frac{G(a_L - p_L)}{G(a_H - p_H)}, q_L + q_D \right\}$$

(1)

if $a_H - p_H \geq a_L - p_L$ (i.e. customers prefer H to L), and

$$r(q_D) = p_L \min \{xG(a_L - p_L), q_L + q_D\} + p_H \min \left\{ [xG(a_L - p_L) - (q_L + q_D)]^+, \frac{G(a_H - p_H)}{G(a_L - p_L)}, q_H - q_D \right\}$$

(2)

otherwise. The first term in each case is the revenue from sales of the first-choice product and the second term is the revenue from sales of the second-choice product. Define $\alpha = \frac{G(a_H - p_H)}{G(a_L - p_L)}$.

**Theorem 1.** The optimum downconversion quantity $q_D^*$ is specified by one of the following three cases: (i) $a_H - p_H \geq a_L - p_L \Rightarrow q_D^* = 0$, (ii) $a_H - p_H < a_L - p_L$ and $c_D \geq p_L - \alpha p_H \Rightarrow q_D^* = 0$, (iii) $a_H - p_H < a_L - p_L$ and $c_D < p_L - \alpha p_H \Rightarrow q_D^* = \min \{z, [xG(a_L - p_L) - q_L]^+] / (1 - \alpha)\}$, where $z = [xG(a_L - p_L) - q_L]^+$ is the amount of customers whose first choice, L in this case, is not satisfied.

We note that downconversion never occurs if customers prefer H to L, because either all demand is satisfied from the inventory of H, in which case there is no reason to convert, or else all of product H inventory is sold, in which case there is nothing left to convert. Even if customers prefer L to H, then downconversion can occur only in the case where demand for L, $xG(a_L - p_L)$, exceeds $q_L$ and the downconversion cost is sufficiently low.

Figure 2 illustrates the firm’s revenue as a function of the downconversion quantity $q_D$ for this case. As the downconversion quantity increases, the quantity of L increases, and so there is less unsatisfied first-choice demand. At the same time, the quantity of H decreases, and so there is less product available to meet spill-over demand from unsatisfied customers. The revenue from sales
of \( L \) is non-decreasing in \( q_D \), while the revenue from sales of \( H \) is non-increasing. In region 1, i.e.,
\[ q_H - q_D > (z - q_D) \frac{a_H - p_H}{a_L - p_L}, \]
the downconversion quantity is low enough such that all spill-over demand is met and there is left-over inventory of \( H \). The marginal increase in revenue from sales of \( L \) more than offsets the marginal loss in revenue from sales of \( H \), and so revenue is increasing in \( q_D \). In region 2, the downconversion quantity is high enough such that all remaining \( H \) is used and there is unsatisfied spill-over demand. The marginal increase in revenue from sales of \( L \) is less than the marginal loss in revenue from sales of \( H \) because the firm could have sold the converted unit at the higher price of \( p_H \) rather than selling it at \( p_L \). The firm’s revenue is maximized at the boundary of regions 1 and 2, which occurs at \( q_D = [q_H - z\alpha]^+/\alpha \). We note region 1 does not exist if \( q_H - z\alpha \leq 0 \). In this case there is insufficient \( H \) to meet spill-over demand even before downconversion occurs, and so \( q_D^* = 0 \).

*** Insert Figure 2 here ***

4.2 The Quantity-and-Price Problem

In this section we investigate the firm’s quantity-and-price problem. In the recourse-pricing case, the firm sets prices after yield and market uncertainties are resolved. (In Section §6.2, we consider a model in which there is some residual market uncertainty in the recourse-pricing case.) In the advanced-pricing case, the firm sets prices before any uncertainties are resolved. We refer the reader to §A4 for a treatment of advanced pricing, and focus attention here on the recourse-pricing case.

After yield and market uncertainties are resolved, the firm chooses its prices to maximize its revenue less downconversion costs, where the revenue function is given by equations (1) and (2) above, but substituting for the optimum downconversion quantity using Theorem 1.

**Theorem 2.** (a) For any post-downconversion inventory vector \((q_H - q_D, q_L + q_D)\) and any realization of market potential \( x \), the optimal price vector satisfies \( a_H - p_H^* \geq a_L - p_L^* \), (b) For any realization of product quantities \((q_H, q_L)\) and market potential \( x \), (i) No downconversion will occur,
(ii) The optimal recourse price vector uniquely satisfies

\[
p^*_H = \frac{G(a_H - p^*_H)}{g(a_H - p^*_H)}, \quad \quad \quad p^*_L = \frac{G(a_L - p^*_L)}{g(a_L - p^*_L)}, \quad x \in \Omega_0,
\]
\[
p^*_H = a_H - G^{-1}\left(\frac{q_H}{x}\right), \quad \quad \quad p^*_L = \frac{G(a_L - p^*_L)}{g(a_L - p^*_L)}, \quad x \in \Omega_1,
\]
\[
p^*_H = a_H - G^{-1}\left(G(a_L - p^*_L)\sqrt{\frac{g(a_H - p^*_H)}{g(a_L - p^*_L)}}\right), \quad p^*_L = \frac{G(a_L - p^*_L)}{g(a_L - p^*_L)}, \quad x \in \Omega_2,
\]
\[
p^*_H = a_H - G^{-1}\left(\frac{q_H + q_L}{x}\right), \quad \quad \quad p^*_L = a_L - G^{-1}\left(\frac{q_H + q_L}{x}\right), \quad x \in \Omega_3.
\]

where \(\Omega_0, \Omega_1, \Omega_2,\) and \(\Omega_3\) partition the market space, and are given by

\[
\Omega_0 : \quad x \leq \frac{q_H}{\left(a_H - G^{-1}\left(\frac{q_H}{x}\right)\right) g\left(G^{-1}\left(\frac{q_H}{x}\right)\right)},
\]
\[
\Omega_1 : \quad \frac{q_H}{\left(a_H - G^{-1}\left(\frac{q_H}{x}\right)\right) g\left(G^{-1}\left(\frac{q_H}{x}\right)\right)} < x \leq \frac{q_H}{\sqrt{\left(G^{-1}\left(\frac{q_H}{x}\right)\right) \nu_L G(a_L - \nu_L)}},
\]
\[
\Omega_2 : \quad \frac{q_H}{\sqrt{g\left(G^{-1}\left(\frac{q_H}{x}\right)\right) \nu_L G(a_L - \nu_L)}} < x \leq \frac{q_L + q_H / \sqrt{\frac{g(a_H - p_H)}{g(a_L - \nu_L)}}}{G(a_L - \nu_L)},
\]
\[
\Omega_3 : \quad x > \frac{q_L + q_H / \sqrt{\frac{g(a_H - p_H)}{g(a_L - \nu_L)}}}{G(a_L - \nu_L)},
\]

where \(p_H = \max\left\{0, a_H - G^{-1}\left(\frac{q_H}{x} - \frac{q_L}{G(a_L - \nu_L)}\right)\right\}\) and \(\nu_L\) is the unique solution to \(\nu = G(a_L - \nu) / g(a_L - \nu)\).

We note that the optimal prices are increasing in the market potential, reflecting the fact that the firm can charge higher prices when demand is high relative to supply.

**Corollary 1.** Under recourse pricing, if the realized market potential \(x \in \Omega_0 \cup \Omega_1\), then it is optimal for the firm to price the two products such that customers strictly prefer \(H\) to \(L\). Otherwise, when the market potential \(x \in \Omega_2 \cup \Omega_3\), the firm prices the products such that the customers are indifferent.

The above corollary tells us that, regardless of the product-quantity or market-potential realizations, it is always optimal to price the products to induce a preference for \(H\) over \(L\). As a consequence, downconversion will never occur under recourse pricing if prices are set optimally.

For the rest of this subsection, we turn our attention to the special case of the outside utility
\(G(\cdot)\) being uniformly distributed. Without loss of generality, we assume \(G(\cdot) \sim U(0, 1)\) and \(0 \leq a_L \leq a_H \leq 1\).

**Corollary 2.** If \(G(\cdot) \sim U(0, 1)\), the optimal recourse price vector \(p^*(q_H, q_L, x)\) is given by

\[
\begin{align*}
p^*(q_H, q_L, x) &= \left(\frac{a_H}{2}, \frac{a_L}{2}\right), \quad \Omega_0 : \quad x \leq \frac{2q_H}{a_H}, \\
p^*(q_H, q_L, x) &= \left(a_H - \frac{q_H}{x}, \frac{a_L}{2}\right), \quad \Omega_1 : \quad \frac{2q_H}{a_H} < x \leq \frac{2q_H}{a_L}, \\
p^*(q_H, q_L, x) &= \left(a_H - \frac{a_L}{2}, \frac{a_L}{2}\right), \quad \Omega_2 : \quad \frac{2q_H}{a_L} < x \leq \frac{2(q_H + q_L)}{a_L}, \\
p^*(q_H, q_L, x) &= \left(a_H - \frac{q_H + q_L}{x}, a_L - \frac{q_H + q_L}{x}\right), \quad \Omega_3 : \quad x > \frac{2(q_H + q_L)}{a_L}.
\end{align*}
\]

The above corollary demonstrates that the optimal prices are non-decreasing in the realized market potential and that the optimal prices always induce the customer preference for product H over L. We can use the optimal prices from Corollary 2 to develop an expression for the optimal (recourse) revenue \(r^*_r(q_H, q_L, x)\) as a function of the realized quantities and market potential,

\[
\begin{align*}
r^*_r(q_H, q_L, x) &= \left(\frac{a_H^2}{4}\right) x, \quad x \in \Omega_0, \\
r^*_r(q_H, q_L, x) &= \left(a_H - \frac{q_H}{x}\right) q_H, \quad x \in \Omega_1, \\
r^*_r(q_H, q_L, x) &= \left(a_H - a_L\right) q_H + \left(\frac{a_L^2}{4}\right) x, \quad x \in \Omega_2, \\
r^*_r(q_H, q_L, x) &= \left(a_H - \frac{q_H + q_L}{x}\right) q_H + \left(a_L - \frac{q_H + q_L}{x}\right) q_L, \quad x \in \Omega_3.
\end{align*}
\]

As one would expect, the optimal revenue is non-decreasing in the market potential \(x\), the realized quantity \(q_H\), and the customer valuations \(a_H\) and \(a_L\). In addition, the optimal revenue is non-increasing in the realized quantity \(q_L\) since an increase in \(q_L\) necessitates a decrease in \(q_H\). Recalling that \(q_H = yQ\) and \(q_L = (1 - y)Q\), we can express the optimal revenue as a function of the
production quantity $Q$, realized yield $y$ and market uncertainty $x$ as

$$r^*_r(Q, y, x) = \left( \frac{a_H^2}{4} \right) x, \quad 0 \leq x \leq \frac{2yQ}{a_H}$$

$$r^*_r(Q, y, x) = \left( a_H - \frac{yQ}{x} \right) yQ, \quad \frac{2yQ}{a_H} < x \leq \frac{2yQ}{a_L}$$

$$r^*_r(Q, y, x) = \left( a_H - a_L \right) yQ + \left( \frac{a_L^2}{4} \right) x, \quad \frac{2yQ}{a_H} < x \leq \frac{2Q}{a_L}$$

$$r^*_r(Q, y, x) = \left( a_H - \frac{Q}{x} \right) yQ + \left( a_L - \frac{Q}{x} \right) (1 - y) Q, \quad x > \frac{2Q}{a_L}.$$

We are now in a position to characterize the firm’s optimal production-quantity $Q^*$. The firm chooses $Q$ to maximize its expected profit,

$$\Pi_r(Q) = -c_P Q + E_{\bar{Y}, \bar{X}} [r^*_r(Q, y, x)].$$

The first term is the production cost, and the second term is the expected revenue, where the expectation is taken over the yield and market potential random variables. We note that there is no cost incurred after uncertainties are resolved, as we proved above that downconversion will not occur if prices are set optimally. Using the above expressions for $r^*_r(Q, y, x)$, we can write the expected profit function as

$$\Pi_r(Q) = -c_P Q + \int_0^{2yQ/a_H} \int_0^{2yQ/a_H} \left( \frac{a_H^2}{4} \right) x dF_X(x) dF_Y(y)$$

$$+ \int_0^{2yQ/a_H} \int_{2yQ/a_H}^{2yQ/a_L} \left( a_H - \frac{yQ}{x} \right) yQ dF_X(x) dF_Y(y)$$

$$+ \int_0^{2yQ/a_L} \int_0^{2yQ/a_H} \left( a_H - a_L \right) yQ + \left( \frac{a_L^2}{4} \right) x dF_X(x) dF_Y(y)$$

$$+ \int_0^{2yQ/a_L} \int_{2yQ/a_L}^{2Q/a_L} \left( a_H - \frac{Q}{x} \right) yQ + \left( a_L - \frac{Q}{x} \right) (1 - y) Q dF_X(x) dF_Y(y).$$

It is relatively straightforward to show that $\Pi_r(Q)$ is concave in $Q$, and so the optimal produc-
tion quantity $Q^*$ is given by the first-order condition. Closed form solution to $Q^*$ will not, however, typically exist.

4.3 The value of recourse pricing and operational improvements

We now investigate the impact of recourse pricing on the firm’s expected profit. (We note that the unabridged version of the paper also investigates the effect of recourse pricing on the optimal production quantity and finds that the effect is quite complex.) Clearly, the optimal profit under recourse pricing is always at least as large as the optimal profit under advance pricing, because recourse pricing can always obtain the same profit as advance pricing by simply setting the recourse price vector equal to the advance price vector for all yield and market realizations.

It is of interest to understand the relative value (i.e., the relative increase in expected profit) that can be gained by implementing recourse pricing, and how firm and market characteristics influence the relative value. To address such questions we designed a comprehensive numeric study to investigate the influence of market uncertainty, expected yield, yield uncertainty, production cost, and customer valuation of the low-quality product.

As is common in the marketing and operations literatures, we scaled the valuations and utilities to lie between 0 and 1 in our numeric study. Therefore, costs and resulting prices are relative to these scaled utilities. We used a doubly truncated (at 0 and 1) normal distribution $DTN(\mu = 0.5, \sigma = 1)$ for the outside-option utility, a discretized normal distribution for the market potential, and a discretized beta distribution for the yield\(^1\). We conducted a full factorial study using the parameter values illustrated in Table 1. The expected yield and yield uncertainty values are representative of the semiconductor industry (see Wang et al. (2004)). We chose the remaining parameters so that the weighted average profit margin at the mid-point of the parameter space is approximately 100%, a gross margin found in the semiconductor industry (see, for example, the consolidated statements

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\(^1\)Bitran and Leong (1992) and Bitran and Gilbert (1994) also used beta distributions for the yield. Other authors, e.g., Wang et al. (2004), have used a uniform distribution to represent the yield. We note that a Uniform (0,1) distribution is often assumed for consumer utilities in the marketing and operations literatures. We conducted a second study in which we used uniform distributions for the utility, market potential and yield. The magnitude and directional results of this study were very similar, with the exception of the directional effect of the expected yield, discussed later. As such, we do not report the numbers from that second study but they are available upon request.
The average percentage increase in expected profit gained by moving from advance to recourse pricing was 7.98% over all the problem instances, indicating that substantial benefit can be gained by recourse pricing. We now present our findings on the influence of market and firm characteristics on the relative value of recourse pricing over advance pricing. When reporting the relative increase in expected profit for a particular parameter value (or combination of parameter values) in the tables that follow, we present the average percentage increase across all instances that have that particular parameter value (or combination of parameter values).

We found that the value of recourse pricing was typically (but not always) increasing in market uncertainty and in yield uncertainty. The reason for this is that recourse pricing enables the firm to better manage uncertainty, and so recourse pricing is more advantageous if uncertainty increases. The value of recourse pricing was typically increasing in the production cost but we have not been able to identify an intuitive explanation for this effect. While the value of recourse pricing was increasing in the expected yield for a uniformly-distributed yield\(^2\), there was no clear directional effect for the case of a beta-distributed yield. We conjecture that the lack of a clear directional effect is due to the fact that the shape of the beta distribution changes as the mean changes. Tables reporting the relative change in expected profit (due to recourse pricing) as a function of market uncertainty, yield uncertainty, expected yield and production cost can be found in the unabridged version of the paper.

We now consider the influence that the customer valuation of product L, \(a_L\), has on the value of recourse pricing. We first note that an increase in \(a_L\) mitigates the impact of yield uncertainty as a high realization of the “by-product” L is less of an issue if that by-product is highly valued. In fact, in the extreme case where \(a_H = a_L\), then yield uncertainty is irrelevant because production results in a random split of equally valued products; in essence the system becomes a perfect-supply,\(^2\)

\(^2\)The reason that recourse pricing becomes more advantageous as expected yield increases (in the uniform distribution case) is linked to the fact that there is relatively more inventory of H than L as expected yield increases. This gives the firm more latitude in pricing product H but less in pricing product L. The increased latitude in pricing of H appears to be especially beneficial for recourse pricing. Note that the expected profits under both recourse and advance pricing increase in the expected yield and so does the magnitude of their difference. In addition, the relative value of recourse pricing also increases in the expected yield.
single-product system. If this mitigation effect was the only effect of increasing \( a_L \), then the value of recourse-pricing would be decreasing in \( a_L \). There is, however, another effect of increasing \( a_L \); the firm’s inventory is more valuable. The firm will always charge a price for product L that is lower than \( a_L \). Increasing \( a_L \) increases the feasible price region, and this increase enables recourse pricing to be more effective and thus more valuable. We then have two counterbalancing effects; the uncertainty mitigation effect and the pricing-latitude effect. Table 2 presents the relative value of recourse pricing as a function of \( a_L \). We see that for low \( a_L \), the mitigation effect dominates and the value of recourse pricing is decreasing in \( a_L \). As \( a_L \) increases further, uncertainty becomes less significant due to mitigation, and so the pricing-latitude effect starts to dominate with the result that value of recourse pricing starts to increase in \( a_L \).

*** Insert Table 2 here ***

Implementing recourse pricing is not the only option for improving the expected profit of a co-production system. The firm might instead prefer to invest in some operational improvement effort. Operational improvements include reducing the production cost, increasing the expected yield of the high-quality product H, reducing yield uncertainty, or increasing the valuation of the low-quality product L. This last option might reflect an actual increase in the quality of L, or alternatively a marketing effort to increase the customer’s valuation of the original quality of L. In order to compare the value of recourse pricing to operational improvement, we calculated, for every problem instance, the necessary improvement in a given dimension (e.g. production-cost reduction) to deliver an equivalent benefit to recourse pricing. On average, a 8.7% reduction in production cost, or a 6.9% increases in expected yield, or a 37.9% increase in low-product valuation was needed to deliver equivalent benefit to recourse pricing. We note that in 21.02% of the cases, the maximum possible increase in expected yield did not deliver as large a benefit as did recourse pricing. Completely eliminating yield uncertainty did not deliver as large a benefit as did recourse pricing in 97.3% of the instances. In these cases, the benefit of completely eliminating yield uncertainty delivered an average of 89% of the value of recourse pricing. Such numbers indicate that implementing recourse pricing delivers approximately the same benefit as eliminating yield uncertainty, suggesting that

\(^3\)The equivalent number is 0.01% for the study with a uniform yield distribution.
recourse pricing is very effective at managing yield uncertainty. We note that as market uncertainty increases, the operational improvements needed to match recourse pricing also increased. This is because recourse pricing is particularly advantageous when market uncertainty is high. The numbers presented here relate only to the value of operational improvements and recourse pricing, and do not account for the relative difficulty or cost of achieving such improvements or implementing recourse pricing. The preference a firm has for a particular operational improvement effort or for recourse pricing will therefore depend on the various implementation costs.

5 A Two-Class, Co-Production Model with Pricing

In this section, we consider the two-class co-production model (labeled CPP2). Recall that without loss of generality we index the classes \( i = 1, 2 \) such that \( a_{1H} \geq a_{2H} \).

5.1 Allocation Policy

Unlike in the single-class case, the firm now faces an allocation decision, that is, how to ration its inventory among the two classes in the event of a shortage. A natural allocation policy to consider is a priority-based allocation policy, whereby the firm selects one class as being the priority class, and members of that class make their purchasing decisions before members of the non-priority class. In what follows, let \( K \in \{H, L\} \) and let \( \overline{K} \) denote the complement. Recall that \( G_i(a_iK - p_K), i = 1, 2; K \in \{H, L\} \), is the fraction of class \( i \) customers willing to buy product \( K \). For \( a_iK - p_K \geq a_{i\overline{K}} - p_{\overline{K}} \) (i.e., class \( i \) prefers \( K \) to \( \overline{K} \)), denote the spill-over ratio \( \frac{G_i(a_iK - p_K)}{G_i(a_{i\overline{K}} - p_{\overline{K}})} \) as \( s_iK \). In other words, \( s_iK \) is the fraction of class \( i \) customers willing to spill over to product \( \overline{K} \) if their first choice \( K \) runs out.

**Theorem 3.** (a) For a given price vector \( p \), (i) the firm is indifferent between all allocation policies if class 1 and 2 have different product preferences. (ii) If the firm uses a prioritization policy and if both classes prefer \( K \) to \( \overline{K} \), then priority should be given to class 1 if \( s_1K < s_{2\overline{K}} \) and class 2 if \( s_1K > s_{2\overline{K}} \). The firm is indifferent if \( s_1K = s_{2\overline{K}} \). (b) (i) A priority-based allocation rule is optimal (among all allocation policies) for recourse allocation regardless of the timing, advance or recourse, of the pricing decision. (ii) A priority-based allocation rule is optimal (among all allocation policies) for advance allocation under advance pricing.
From the above theorem, we see that the firm is indifferent between allocation policies if the two classes prefer different products. As mentioned in the introduction, the existing co-production literature not only assumed exogenous prices but implicitly assumed that different customer classes preferred different products. This is one of the reasons that the allocation question did not arise in such papers, that, and the fact that spill overs were not allowed.

The theorem also proves that if both classes prefer the same product, say $K \in \{H, L\}$, then the firm should prioritize the class with the lower spill-over ratio. Why is this? If there is insufficient inventory of $K$, and so the firm has to ration $K$, the firm’s revenue is increasing in the number of unsatisfied customers preferring to purchase the other product rather than taking their outside option. Thus, priority is based on the spill-over ratios.

As mentioned in the introduction, we consider both advance allocation in which the firm chooses its allocation policy before yield and market uncertainties are resolved and recourse allocation in which the firm postpones its choice of allocation policy until after uncertainties are resolved. Clearly recourse allocation can be no worse than advance allocation, as the firm could simply choose the same allocation policy as under advance allocation for all realizations of yield and market potentials. Theorem 3 established that a priority policy is optimal under recourse pricing and under advanced pricing if advanced allocation is done. We note that while we have not proven that a priority allocation policy is optimal for advance allocation under recourse pricing, we restrict attention to a prioritization policy (unless otherwise stated), and therefore use the term prioritization rather than allocation in what follows.

Theorem 3 established that the optimal priority depends only on the spill-over ratios. The spill-over ratios depend only on the price vector $p$ and valuations $a_iK$, $i = 1, 2; K \in \{H, L\}$. The valuations are not uncertain. However, in recourse pricing, the optimal $p$ depends on market and yield realizations, and therefore the optimal priority class cannot be determined a-priori. Therefore, recourse prioritization is preferred to advanced prioritization under recourse pricing. In advance pricing, there is no uncertainty about the price vector $p$ and therefore the optimal priority class can be determined a-priori. The following result then follows immediately.

Remark: Recourse prioritization offers no value if the firm makes its pricing decision in advance.

We note that implementation of a prioritization policy assumes that the firm knows a customer’s class. This is a reasonable assumption in many co-production systems because the firm will often
know how a customer uses its product and will therefore be able to infer the customer’s valuation type. However, if the firm is not able to distinguish between customer classes, then randomized allocation is the most likely policy. We refer the reader to Appendix A5 for an analytical and numeric treatment of randomized allocation. Our study found that the value of knowing a customer type, i.e., the ability to implement prioritization, can sometimes be significant. (See Appendix A5.)

We now proceed to develop the firm’s revenue expression given it uses a prioritization policy. We use \( j \) to denote the priority class and \( \overline{j} \) to denote the non-priority class. For notational clarity in what follows, define \( d_{iK} = x_i G_i(a_{iK} - p_K), i = 1, 2; K \in \{H, L\}, \) that is, \( d_{iK} \) is the quantity of class \( i \) customers who prefer \( K \), at price \( p_K \), to their outside option. For a price vector \( \mathbf{p} = (p_H, p_L) \), realized quantities \( \mathbf{q} = (q_H, q_L) \) and market-potential realizations \( \mathbf{x} = (x_1, x_2) \), the firm’s revenue as a function of the downconversion quantity \( q_D \) is

\[
 r(q_D) = p_H \min \left\{ \sum_{i=1}^{2} d_{iH}, (q_H - q_D) \right\} \\
 + p_L \min \left\{ (d_{jH} - (q_H - q_D))^+ s_{jL} + (d_{jH} - ((q_H - q_D) - d_{jH})^+ s_{jL}, (q_L + q_D)) \right\}, \quad \mathbf{p} \in \Gamma_1, \\
 r(q_D) = p_H \min \left\{ d_{jH} + (d_{jH} - (q_H - q_D))^+ s_{jL}, (q_H - q_D) \right\} \\
 + p_L \min \left\{ (d_{jL} - (q_L + q_D))^+ s_{jL}, (q_L + q_D) \right\}, \quad \mathbf{p} \in \Gamma_2, \\
 r(q_D) = p_H \min \left\{ d_{jH} - (q_H - q_D))^+ s_{jH}, (q_H - q_D) \right\} \\
 + p_L \min \left\{ (d_{jL} - (q_L + q_D))^+ s_{jL}, (q_L + q_D) \right\}, \quad \mathbf{p} \in \Gamma_3, \\
 r(q_D) = p_H \min \left\{ (d_{jL} - (q_L + q_D))^+ s_{jH} + (d_{jL} - ((q_L + q_D) - d_{jL})^+ s_{jH}, (q_H - q_D)) \right\} \\
 + p_L \min \left\{ \sum_{i=1}^{2} d_{iL}, (q_L + q_D) \right\}, \quad \mathbf{p} \in \Gamma_4,
\]

where \( \Gamma_1, \Gamma_2, \Gamma_3, \) and \( \Gamma_4 \) partition the pricing space and are given by

\[
 \Gamma_1 : \quad p_H - p_L \leq \min_i \{a_{iH} - a_{iL}\}, \\
 \Gamma_2 : \quad a_{jH} - a_{jL} < p_H - p_L \leq a_{jH} - a_{jL}, \\
 \Gamma_3 : \quad a_{jH} - a_{jL} < p_H - p_L \leq a_{jH} - a_{jL}, \\
 \Gamma_4 : \quad p_H - p_L > \max_i \{a_{iH} - a_{iL}\}.
\]
We note that $\Gamma_2$ and $\Gamma_3$ cannot exist simultaneously, that is, depending on the values of $a_{iK}, i = 1, 2; K \in \{H, L\}$, either $\Gamma_2$ or $\Gamma_3$ will exist. We note that the revenue expression under randomized allocation can be found in Appendix A5.

5.2 Downconversion

Using the above revenue expression, we can now derive the firm’s optimal downconversion quantity $q_D$ under a priority allocation policy. Recall that the downconversion cost is $c_D$ and that $j$ denotes the priority class and $\overline{j}$ the non-priority class.

**Theorem 4.** For a price vector $p = (p_H, p_L)$, realized quantities $q = (q_H, q_L)$ and market-potential realizations $x = (x_1, x_2)$, the optimal downconversion quantity $q_D^* = 0$ if (a) both class of customers prefer product H to L; or (b) class $j$ customers prefer product L and $c_D \geq p_L - p_H s_{jH}$; or (c) class $j$ customers prefer product H and class $\overline{j}$ customers prefer product L and $c_D \geq p_L - p_H s_{jH}$. Otherwise, the optimal downconversion quantity is given by $q_D^* = \min\{z, \hat{q}_D\}$, where

$$\hat{q}_D = \frac{(q_H - d_{jH}) - (d_{jL} - q_L) s_{jH}}{1 - s_{jH}}, \quad p \in \Gamma_2, \quad d_{jH} < q_H \cap d_{jL} > q_L \cap (d_{jL} - q_L) s_{jH} < q_H - d_{jH},$$

$$\hat{q}_D = \frac{(q_H - d_{jH}) - (d_{jL} - q_L) s_{jH}}{1 - s_{jH}}, \quad p \in \Gamma_3, \quad d_{jH} < q_H \cap d_{jL} > q_L \cap (d_{jL} - q_L) s_{jH} < q_H - d_{jH},$$

$$\hat{q}_D = \min\{z_1, \hat{q}_{D1}\} + \min\{z_2, \hat{q}_{D2}\}, \quad p \in \Gamma_4, \quad d_{jL} > q_L \cap (d_{jL} - q_L) s_{jH} + d_{jL} s_{jH} < q_H,$$

$$\hat{q}_D = \frac{q_H - \left(\sum_{i=1}^{2} d_{iL} - q_L\right) s_{jH}}{1 - s_{jH}}, \quad p \in \Gamma_4, \quad d_{jL} \leq q_L \cap \sum_{i=1}^{2} d_{iL} > q_L \cap \left(\sum_{i=1}^{2} d_{iL} - q_L\right) s_{jH} < q_H,$$

$$\hat{q}_D = 0, \quad \text{otherwise},$$

where

$$\hat{q}_{D1} = \frac{q_H - (d_{jL} - q_L) s_{jH} - d_{jL} s_{jH}}{1 - s_{jH}},$$

$$\hat{q}_{D2} = \frac{(q_H - (d_{jL} - q_L - \hat{q}_{D1}) s_{jH} - \hat{q}_{D1} - d_{jL} s_{jH})^+}{1 - s_{jH}}.$$
and

\begin{align*}
z &= \max\{0, d_{jL} - q_L\}, & \quad p \in \Gamma_2, \\
z &= \max\{0, d_{jL} - q_L\}, & \quad p \in \Gamma_3, \\
z_1 &= \max\{0, d_{jL} - q_L\}, \quad z_2 = \max\{0, d_{jL}^*\}, & \quad p \in \Gamma_4, \\
z &= \max\left\{0, \sum_{i=1}^{2} d_{iL} - q_L\right\}, & \quad p \in \Gamma_4.
\end{align*}

An implication of this theorem is that the firm will not downconvert unless there is at least one customer class that prefers product \( L \) to product \( H \), and even then will only downconvert if the cost is not too high. An equivalent theorem for randomized allocation can be found in Appendix A5.

### 5.3 Pricing

In §4 we proved that in the single-class case, the firm prices the products such that \( H \) is preferred to \( L \) and that therefore downconversion does not occur. The following theorem establishes that in the two-class case, the firm may price the products such that one class prefers product \( L \) and that downconversion may occur.

**Theorem 5.** The optimal price vector \( \mathbf{p}^* = (p^*_H, p^*_L) \) may induce one class to prefer \( L \) to \( H \), that is \( a_{iL} - p^*_L > a_{iH} - p^*_H \) for some \( i = 1, 2 \). Furthermore, a strictly positive downconversion quantity may be optimal.

A fundamentally different result therefore holds when we move from the single-class case to the two-class case. The firm’s inability to price discriminate lies at the heart of this result. The firm is constrained to offer the same price vector to both classes and this constraint can result in the best single price vector inducing one class to prefer \( L \) to \( H \). In this case, downconversion can be optimal so long as the downconversion cost is not too high.

For the single-class model (CPP1), we were able to obtain implicit solutions for the optimal recourse price vector (and closed form solutions in the case of a uniform utility distribution.) Unfortunately the two-class model does not in general lend itself to solving for the optimal recourse prices. However, in the special case where \( a_{iL} = 0, i = 1, 2 \), we are able to solve for the optimal
price. This special case (RYP2) is a random-yield, single-product model, and a full analysis of RYP2 can be found in §A3. As mentioned, we have not developed implicit expressions for the optimal recourse prices for the general CPP2 model. However, the optimal prices (recourse and advance) can be readily found numerically.

5.4 The Value of Recourse Actions

We carried out an extensive numeric study to investigate the value of recourse pricing, downconversion and recourse prioritization. We assumed a doubly truncated (at 0 and 1) normal $DTN(0.5,1)$ distributions for the utilities of outside options\(^4\). Yield uncertainty was represented by a discretized beta distribution, where the yield standard deviation was varied from 0.04 to 0.20 in increments of 0.04. The expected yield of product H was varied from 0.4 to 0.8 in increments of 0.1. The expected market potential $\mu_{X_1}$ and product valuations $(a_{1H}, a_{1L})$ for class 1 customers were fixed at $\mu_{X_1} = 100$ and $(a_{1H}, a_{1L}) = (0.8, 0.4)$. The expected market potential for class 2 customers $\mu_{X_2}$ was varied from 40 to 160 in increments of 30. Market uncertainty was represented by three scenarios $(\mu_{X_i} - s_{X_i}, \mu_{X_i}, \mu_{X_i} + s_{X_i})$ with associated probabilities $(0.25, 0.5, 0.25)$, where $s_{X_i}$ is defined as the spread of the distribution. The scenarios were set up such that the two market potentials were independent. The market spread for each class was simultaneously varied from 10 to 50\(^5\) in increments of 10, i.e., the markets have equal variance in each instance. Class 2’s valuation of product H was varied from 0.3 to 0.6 in increments of 0.1. Class 2’s valuation of product L was varied from 0.2 to $a_{2L} - 0.1$ in increments of 0.1. The production cost was fixed at 0.20. Downconversion was either impossible (i.e., infinite cost) or possible at a marginal cost of 0.5% of the production cost\(^6\). The total number of problem instances was thus 12,000. We note that the numbers reported below for advanced prioritization (i.e., a fixed priority is chosen before uncertainties are resolved) reflect the optimal advanced priority choice.

In a separate study, we investigated the effect of market correlation by creating instances with correlation coefficients of $-1$, 0, and 1. We fixed the expected yield at 0.6, the yield spread at 0.12

\(^4\)As with the single-class study, we also carried out a separate investigation when utilities, market potentials and the yield all had uniform distributions. Again, the results were very similar, unless otherwise stated in the paper, and are not reported here. They are available upon request.

\(^5\)We note that in the case of $\mu_{X_2} = 40$, the market spread for class 2 was not increased beyond 40 as this is the maximum spread for a distribution centered at 40.

\(^6\)We also investigated the downconversion cost at 5% of the production cost. The value of downconversion (reported later) was slightly lower but the other results were similar to those for the 0.5% case.
and the production cost at 0.2. All other parameters were varied as described above. This study had 1440 instances.

For advanced (recourse) prioritization, the average value of recourse pricing, i.e. the increase in expected profit over advance pricing, was 10.02\% (10.03\%) when downconversion was possible and 11.06\% (11.07\%) when downconversion was not possible. The influence of production and market characteristics on the value of recourse pricing were similar to the single-class case and so we do not discuss them again here.

For advanced prioritization, the average value of downconversion, i.e. the increase in expected profit over no downconversion, was 1.07\% for advanced pricing and 0.12\% for recourse pricing. Such numbers might suggest that downconversion offers little value. The average value, however, obscures the fact that downconversion can be very valuable in certain instances. The maximum value of downconversion was 12.90\% for the advance pricing case\textsuperscript{7}. The value of downconversion is influenced primarily by two factors. One factor is the probability of mismatches in supply and demand for product L; the more likely such mismatches are, the more likely the firm will need to engage in downconversion. The other factor is the constraint on how much the firm can convert, i.e., the less of product H it has, the less it can convert. Tables 3, 4, and 5 present the average and maximum values of downconversion as market uncertainty (as represented by the spread), yield uncertainty and expected yield increase. We see that the value of downconversion is increasing in market uncertainty, yield uncertainty and expected yield. The value is increasing in market uncertainty and yield uncertainty because supply-demand mismatches are more likely as uncertainty grows. The value is increasing in expected yield because more product H is produced as the expected yield increases. Tables 3, 4, and 5 also show that the value of downconversion is significantly decreased if recourse pricing is used. This finding, in conjunction with the fact that recourse pricing provides a higher average value (11.07\%) than does downconversion (1.07\%), suggests recourse demand-management, i.e., pricing, is more valuable than recourse supply-management, i.e., downconversion.

Our investigation of market correlation found the value of downconversion to be decreasing in market correlation. This makes sense as one of the necessary condition for downconversion to occur is that there be an excess of one product and a shortage of the other. The more positively

\textsuperscript{7}In the study with uniformly distributed utilities, potential and yield, the maximum value was 25.32\% but the average values were essentially the same as those reported above, i.e., 1.03\% as compared to 1.07\%, and 0.13\% as compared to 0.12\%
correlated the markets the less likely are such situations.

*** Insert Table 3 here ***

*** Insert Table 4 here ***

*** Insert Table 5 here ***

We now turn our attention to the value of recourse prioritization, i.e. the value of postponing the class prioritization decision until uncertainties are resolved. Recourse prioritization allows the firm to postpone the prioritization decision until after uncertainties are resolved. We already have proven that recourse prioritization offers no value under advance pricing but that it can add value under recourse pricing (because the priority depends on the prices and the valuations, and the optimal prices cannot be known in advance). However, the other determinant, i.e., the product valuations, is not uncertain, and so one might conjecture that recourse prioritization might be of little value. For recourse pricing, the average (max) value of recourse prioritization, i.e., the increase in expected profit over advanced prioritization, was 0.01% (0.41%) in our numeric study. This number suggests that the value of recourse prioritization is negligible on average, even for high market uncertainties. Customer valuations appear to be a more critical driver of the prioritization decision, and the fact that valuations are certain means that recourse prioritization is of little value.

6 Alternative Models

We now consider two modifications to our earlier model. In §6.1, we consider a model with downgrading and investigate the firm’s preference for either downconversion or downgrading. In §6.2, we consider a model in which market uncertainty is not completely resolved when recourse actions are undertaken.

6.1 Downgrading versus Downconversion

As discussed in the introduction, the firm could downgrade rather than downconvert when filling a low-quality demand from its high-quality inventory. However, downgrading has some tactical and strategic disadvantages. Tactically, the practice of downgrading cannibalizes high-quality de-
mand as it does not extract the high-quality price from those customers willing to spill up to the
high-quality product. Strategically, the practice may promote undesirable reselling on the part
of customers who receive a high-quality product for the price of a low-quality one. Downconver-
sion is therefore a common practice in the semiconductor industry. Downgrading, however, has the
advantage of being free whereas downconversion incurs a cost. Therefore, downgrading may, in the-
ory, be preferable to downconversion. In this section, we investigate the tradeoff between the cost
disadvantage of downconversion and the cannibalization disadvantage of downgrading. Strategic
disadvantages of downgrading are not considered but these would only serve to make downgrading
less desirable.

The practice of downgrading is captured by a slight modification to the model presented in §3:
if low-quality demand, i.e., demand for product L, exceeds the low-quality inventory, \( q_L \), then the
firm fulfills as much as it can of this unsatisfied demand (at price \( p_L \)) using any excess inventory
of product H, i.e., any inventory of H left over after satisfying first-choice demand for H.

**Theorem 6.** (a) Downgrading will not occur in the single customer-class case if prices are set
optimally. (b) In the two customer-class case, (i) downconversion (weakly) dominates downgrading
if the downconversion cost, \( c_D \), is zero, (ii) downgrading can be preferred to downconversion if
\( c_D > 0 \).

The single-class result echoes the earlier result for downconversion. The two-class results sug-
gest that, as one might expect, downgrading is more likely to be preferred as the downconversion
cost increases. We investigate this using the same numeric study as described in §5.4 (restricting
attention to advanced pricing and recourse allocation) but using a wider range of relative down-
conversion costs, from 0% of the production cost up to 7.5%. Table 6 presents, as a function of the
downconversion cost and class 2’s valuation of product H, the percentage of cases in which the firm
1) prefers downconversion, 2) prefers downgrading, or 3) is indifferent between the two practices.

*** Insert Table 6 here ***

As can be seen, downconversion becomes less attractive relative to downgrading as the cost of
downconversion increases. While downgrading was quite often preferred to downconversion, it is
important to note that when downgrading was preferred, the average improvement over downcon-
version was 0.09%, whereas, when downconversion was preferred, the average improvement over
downgrading was 0.67%. Furthermore, the maximum improvement of downgrading over downconversion was 1.52% while the maximum improvement of downconversion over downgrading was 6.50%. These results, combined with the strategic disadvantage of downgrading and the fact that downconversion is inexpensive, help explain why downconversion is a common practice in the semiconductor industry.

Finally we note that downconversion is increasingly preferred as \(a_{2H}\), class-2’s valuation of product H, increases. The reason is as follows. In situations in which the optimal prices induce class 2 to prefer product L, an increase in \(a_{2H}\) makes class-2 customers more willing to spill up to product H if there is insufficient inventory of product L. Downgrading does not take advantage of spill up and, therefore, the cannibalization disadvantage of downgrading is more significant as \(a_{2H}\) increases.

### 6.2 Residual Market Uncertainty

To this point, we have assumed that the firm has perfect market information when making the downconversion and (recourse) pricing decisions. That is, market uncertainty is completely resolved by the end of the production lead time. We note that single-period models in which resource allocation and/or pricing decisions can be postponed until after the market size is perfectly observed are common in both the operations and marketing literatures. Ex-post resource allocation is assumed in the flexibility literature (e.g., Fine and Freund (1990); Van Mieghem (1998, 2004)), in the delayed-differentiation literature (e.g., Swaminathan and Tayur (1998); Anand and Girotra (2006)), in the transshipment literature (e.g., Rudi et al. (2001); Dong and Rudi (2004)), and in the subcontracting literature (e.g., Van Mieghem (1999)). As discussed in §2, ex-post pricing is assumed in Bish and Wang (2004) and Chod and Rudi (2005), and is one of the cases considered in Van Mieghem and Dada (1999). It is also assumed in Desai et al. (2007) who note that “given the lead time needed for manufacturing, the production decision may be based on the firm’s expectations about the market demand [and] thus, subsequent marketing decisions, such as price and advertising, are conditional not only on the realized demand, but also on the production decision made earlier, when the firm did not have complete information about market demand.”

Market-size uncertainty may be reduced over the production lead time as the firm obtains new
information from its interactions with potential customers and/or from observations of economic indicators that influence market size. When prices are set in advance, customers might pre order based on advanced showings and prototype demonstrations. If prices are not set in advance then customer will not pre order; they may, however, signal their interest (with the actual purchase decision dependent on the eventual price.) Economic indicators, such as interest rates, macro-level industrial demand, etc., may also contain useful signals of market sizes. While market uncertainty may not be fully resolved based on customer interactions and economic indicators, the common assumption that postponed decisions can be made with perfect market-size information is an acceptable approximation if market uncertainty reduces significantly over the production lead time.

There may, however, be situations in which market uncertainty is not even close to being fully resolved after production. Yield uncertainty is, of course, resolved after production. In our context, downconversion and (recourse) pricing decisions would then be made with perfect yield information but imperfect market information. Would this residual market uncertainty effect our earlier findings, and, if so, how? To address this question, we focus on the single-class case. Let $F_R(\cdot)$ denote the distribution function for the market potential $x$ at the point in time at which the firm chooses the downconversion quantity (and prices in the recourse case.) In other words, $F_R(\cdot)$ represents the firm’s market forecast after production. If $F_R(\cdot)=F_X(\cdot)$, the original forecast, then no uncertainty is resolved over the lead time.

Recall that the realized inventories of product H and L are given by $q_H = yQ$ and $q_H = (1-y)Q$ respectively. For any price vector $(p_H, p_L)$, Theorem 1 specified the optimal downconversion quantity assuming the firm had perfect market information. The following theorem extends that earlier result to the case in which there is residual market uncertainty when downconversion occurs.

**Theorem 7.** For any realized inventory vector $(q_H, q_L)$ (i) if $a_H - p_H \geq a_L - p_L$ then $q_D^* = 0$, (ii) if $a_H - p_H < a_L - p_L$ then (a) the optimal downconversion quantity $q_D^* = 0$ if

$$c_D \geq p_LF_R\left(\frac{q_L}{G(a_L - p_L)}\right) - p_HG(a_H - p_H)F_R\left(\frac{q_L}{G(a_L - p_L)} + \frac{q_H}{G(a_H - p_H)}\right) - F_R\left(\frac{q_L}{G(a_L - p_L)}\right) - p_HF_R\left(\frac{q_L}{G(a_L - p_L)} + \frac{q_H}{G(a_H - p_H)}\right),$$
and, otherwise, $q_D^*$ is the unique solution to

$$c_D = p_L F_R \left( \frac{q_L + q_D}{G(a_L - p_L)} \right) - p_H G(a_H - p_H) \left( F_R \left( \frac{q_L + q_D}{G(a_L - p_L)} \right) + \frac{q_H - q_D}{G(a_H - p_H)} \right) - F_R \left( \frac{q_L + q_D}{G(a_L - p_L)} \right) - p_H F_R \left( \frac{q_L + q_D}{G(a_L - p_L)} + \frac{q_H - q_D}{G(a_H - p_H)} \right).$$

(b) $0 \leq q_D^* < q_H$, i.e., the firm never downconverts all product $H$ to $L$.

For the case of perfect market information (and a single customer class), we established that the firm does not downconvert if prices are set optimally. While we have not been able to analytically establish the same result when there is residual market uncertainty at the time of downconversion and pricing, extensive numeric investigations did not uncover a single instance when downconversion occurred. This suggests that our earlier finding is robust.

Clearly, recourse pricing will be less beneficial if market uncertainty is not fully resolved over the production lead time. To investigate how residual market uncertainty affects the firm’s expected profit, we model market uncertainty as follows. Let $X = X_L + X_R$, where $X_L$ is realized over the lead time and $X_R$ is a zero-mean random variable representing the residual uncertainty. In particular, let $X_L$ and $X_R$ be normally distributed with $X_L \sim N(\mu, \lambda \sigma)$ and $X_R \sim N(0, (1 - \lambda)\sigma)$. Then, $X \sim N(\mu, \sigma)$. The parameter $\lambda$ represents the fraction of market size uncertainty that is resolved over the production lead time. At $\lambda = 0$, no uncertainty is resolved. The $\lambda = 1$ case corresponds to our original model in which all market uncertainty is resolved.

For the base case scenario described in §4.3, Figure 3 plots the increase in expected profit (relative to the advanced pricing case) as a function of $\lambda$. We see from Figure 3 that the relative benefit of recourse pricing is convex increasing in $\lambda$. However, there is substantial benefit even for low values of $\lambda$, suggesting that recourse pricing is of significant value even with imperfect market information. In fact, at $\lambda = 0$, recourse pricing has no additional market information over advanced pricing, yet there is a 1% increase in profit over advanced pricing. This indicates that recourse pricing is quite beneficial even if only the yield has been observed.

*** Insert Figure 3 here ***
7 Conclusions

Co-production systems are prevalent in many industries. Such systems present many challenges; the firm must make pricing, quantity, downconversion and allocation decisions in an environment of potentially high uncertainties in both supply and demand. Previous literature has focused on the quantity and downconversion decision assuming exogenous prices, differing customer-class preferences, and costless downconversion. We jointly consider the pricing, quantity, downconversion and allocation problem for a single-period, two-product problem, using a utility-maximizing demand model to ensure a customer’s behavior is completely consistent with her product valuations and the announced prices. Furthermore we allow for costly downconversion. We consider both advanced and recourse pricing and advanced and recourse allocation.

We establish that downconversion will never occur in the single-customer class case if prices are set optimally. (The unabridged version of the paper considers an alternative model in which customers are willing to wait for downconversion. We prove that downconversion can occur even if prices are set optimally in that case.) We explicitly characterize the optimal recourse prices and show recourse pricing is very beneficial, delivering the same value as a 9% reduction in production cost or approximately the same value as completely eliminating supply uncertainty. With two customer classes, the firm must decide on an allocation policy. We prove that a priority allocation policy is optimal for recourse allocation. In addition, we show that downconversion can be optimal when there are two customer classes. While the average value of downconversion is not very high, it can be very valuable in environments of high demand and supply uncertainty. Recourse prioritization was found to add little value, primarily because the valuation heavily influences the choice of the priority class and valuations are not uncertain. Recourse pricing was found to be very beneficial, suggesting that recourse demand-management is more valuable than recourse supply-management. It may, however, be harder to implement recourse pricing.

As with the existing stochastic-demand co-production literature, we have limited our attention to the single-period problem. An extension to the multi-period setting (even for the exogenous price case) for stochastic-demand problem would be of great interest, but would also be very challenging. Even the deterministic-demand papers resort to heuristics for the multi-period problem. We conclude by noting that co-production systems, while complex from an analytical nature, are
rich in opportunities for research in risk management and product-variety management. The high degree of supply and demand uncertainty make co-production systems attractive candidates for those interested in the financial and operation hedging of risk. All co-production research to date has assumed risk neutrality. The multi-product nature of co-production systems should make them attractive to those interested in product variety or assortment problems. For example, in many co-production systems the output varies continuously over the quality dimension. A firm then has to decide how to segment the output into products; it might offer a high number of very tightly specified products or a small number of more loosely specified products. All research to date has assumed the product offering decision is exogenous. We hope that future research will address these and other questions.

Acknowledgements

The authors are grateful for the invaluable comments and suggestions provided by the referees, the associate editor, and the department editor on earlier versions of this paper.

References


8 Figures and Tables

<table>
<thead>
<tr>
<th>Single product</th>
<th>Two products</th>
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| **RYP1**
(single-class, random yield model with pricing)

\[ a_{1L} = 0 ; a_{1H} > 0 \]
\[ a_{2L} = 0 ; a_{2H} = 0 \]

| **CPP1**
(single-class, co-production model with pricing)

\[ a_{1L} \geq 0 ; a_{1H} > 0 \]
\[ a_{2L} = 0 ; a_{2H} = 0 \]

| **RYP2**
(two-class, random yield model with pricing)

\[ a_{1L} = 0 ; a_{1H} > 0 \]
\[ a_{2L} = 0 ; a_{2H} \geq 0 \]

| **CPP2**
(two-class, co-production model with pricing)

\[ a_{1L} \geq 0 ; a_{1H} > 0 \]
\[ a_{2L} \geq 0 ; a_{2H} \geq 0 \]

Figure 1: Four Models

\[ r(q_D) \]

\[ q_H - q_D = (z - q_D) \alpha \]

Figure 2: Downconversion
Figure 3: Residual Uncertainty in Market Size

Table 1: Numeric study design (bracket indicate number of scenarios)

<table>
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<tr>
<th></th>
<th>$a_H$</th>
<th>$a_L$</th>
<th>$c_P$</th>
<th>$c_D$</th>
<th>mkt mean</th>
<th>mkt spread</th>
<th>yield mean</th>
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<td>.4</td>
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<td>.16 , .24 (5)</td>
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<td>fixed</td>
<td>10,50 (5)</td>
<td>.40 , .80 (5)</td>
<td>.04 , .20 (5)</td>
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Table 2: Relative value of recourse pricing as a function of $a_L$ (in %)

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<th>.4</th>
<th>.5</th>
<th>.6</th>
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<td>30</td>
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<td>6.97</td>
<td>6.60</td>
<td>7.07</td>
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Table 3: Relative value of downconversion as a function of market uncertainty (in %)
Table 4: Relative value of downconversion as a function of yield uncertainty (in %)

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<td>average</td>
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<td>maximum</td>
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Table 5: Relative value of downconversion as a function of expected yield (in %)

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<th>yield spread</th>
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<td></td>
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<td>maximum</td>
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<td>recourse pricing</td>
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<tr>
<td>average</td>
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<tr>
<td>maximum</td>
<td>.30</td>
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Table 6: Percentage (%) of cases where downconversion (DC) or downgrading (DG) is preferred. (ID denotes indifference). *Relative to production cost.

<table>
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<th>downconversion cost*</th>
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<tr>
<td></td>
<td>0%</td>
<td>0.5%</td>
</tr>
<tr>
<td>0.6</td>
<td>DC</td>
<td>43.08</td>
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<td></td>
<td>ID</td>
<td>56.92</td>
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<tr>
<td></td>
<td>DG</td>
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A1 Appendix - Proofs

We first present a useful technical lemma which is used in some of the subsequent proofs.

**Lemma A1.** (a) Define \( h_i(u) = uG_i(a_i - u) \) in the domain \( 0 \leq u \leq a_i \). A sufficient condition for \( h_i(u) \) to be unimodal in \( u \) is

\[
(T1) \quad G_i(a_i - u)g_i'(a_i - u) - 2g_i^2(a_i - u) < 0
\]

where \( R_i = \frac{g_i'(a_i - u)}{G_i(a_i - u)} \). Note that \((T1) \iff R_i^2 > R_i' \). (b) Define \( h(u) \) as a convex combination of \( h_i(u) \), i.e., \( h(u) = \lambda h_1(u) + (1 - \lambda)h_2(u), 0 \leq \lambda \leq 1 \). A sufficient condition for \( h(u) \) to be unimodal in \( u \) over \( 0 \leq u \leq \min \{a_1, a_2\} \) is

\[
(T2) \quad \lambda^2 (G_1(u_1)g_1'(u_1) - 2g_1^2(u_1)) + (1 - \lambda^2) (G_2(u_2)g_2'(u_2) - 2g_2^2(u_2))
\]

\[
+ \lambda(1 - \lambda) (G_1(u_1)g_2'(u_2) + G_2(u_2)g_1'(u_1) - 4g_1(u_1)g_2(u_2)) < 0,
\]

where \( u_i = a_i - u, i = 1, 2 \). (c) If \( \lambda = 0 \) or 1, then \((T1) \) implies \((T2)\).

**Proof of Lemma A1.** (a) Note \( \frac{\partial h_i(u)}{\partial u} = G_i(a_i - u) - ug_i(a_i - u) \) and \( \frac{\partial^2 h_i(u)}{\partial u^2} = -2g_i(a_i - u) - ug_i'(a_i - u) \). Because \( \frac{\partial h_i(u)}{\partial u} \bigg|_{u=0} > 0 \) and \( \frac{\partial h_i(u)}{\partial u} \bigg|_{u=a_i} \leq 0 \), a sufficient condition for \( h_i(u) \) to be unimodal in \( u \) is \( \frac{\partial^2 h_i(u)}{\partial u^2} < 0 \) whenever \( \frac{\partial h_i(u)}{\partial u} = 0 \). Note \( \frac{\partial h_i(u)}{\partial u} = 0 \Rightarrow u^* = \frac{G_i(a_i - u)}{g_i'(a_i - u)} \Rightarrow \frac{\partial^2 h_i(u)}{\partial u^2} \bigg|_{u=u^*} = (-2g_i^2(a_i - u) + G_i(a_i - u)g_i'(a_i - u))/g_i(a_i - u) < 0 \) by \((T1) \). (b) Follows analogously as (a).

**Proof of Theorem 1.** Define \( \pi(q_D) = r(q_D) - cdq_D \). Also define \( \alpha = \frac{G(a_H - p_H)}{G(a_L - p_L)} \).

(i) In this case, \( a_H - p_H \geq a_L - p_L \). We will show that \( \pi(q_D) \) is decreasing in \( q_D \). Recall \( r(q_D) = p_H \min\{xG(a_H - p_H), q_H - q_D\} + p_L \min\{xG(a_H - p_H) - (q_H - q_D)\}/\alpha, q_L + q_D\} \). For \( q_D \leq [q_H - xG(a_H - p_H)]^+/\alpha, q_L + q_D \). For \( q_D > [q_H - xG(a_H - p_H)]^+/\alpha, \pi(q_D) = p_H xG(a_H - p_H) - cdq_D, \) which is decreasing in \( q_D \). For \( q_L + q_D > xG(a_H - p_H) - (q_H - q_D)/\alpha, \) then \( \pi(q_D) = p_H (q_H - q_D) + p_L (xG(a_H - p_H) - (q_H - q_D))/\alpha - cdq_D \). Note \( \pi(q_D) \) is decreasing in \( q_D \) because \( p_H > p_L > p_L/\alpha - cd \). If \( q_L + q_D \leq xG(a_H - p_H) - (q_H - q_D)/\alpha, \) then \( \pi(q_D) = p_H (q_H - q_D) + p_L (q_L + q_D) - cdq_D \), which is again decreasing in \( q_D \). Since \( \pi(q_D) \) is always decreasing in \( q_D \), it then follows that \( q_D^* = 0 \).
(ii) In this case \(a_H - p_H < a_L - p_L\). Therefore \(\pi(q_D) = p_L \min \{ xG(a_L - p_L), q_L + q_D \} + p_H \min \{ [xG(a_L - p_L) - (q_L + q_D)]^+, q_H - q_D \} - c_Dq_D\). Because \(p_L - c_D \leq p_H \alpha\), \(\pi(q_D)\) is decreasing in \(q_D\). Therefore \(q_D^* = 0\).

(iii) Define \(z = [xG(a_L - p_L) - q_L]^+\). If \(q_D > z\), then \(\pi(q_D) = p_L xG(a_L - p_L) - c_Dq_D\), which is decreasing in \(q_D\). If \(q_D \leq z\), then \(\pi(q_D) = p_L(q_L + q_D) + p_H \min \{ (z - q_D)\alpha, q_H - q_D \} - c_Dq_D\). For \(q_D \leq \frac{[q_H - z]^+}{1 - \alpha}\), \(\pi(q_D) = p_L(q_L + q_D) + p_H(q_H - q_D)\), and \(\pi(q_D)\) is increasing in \(q_D\) because \(c_D < p_L - p_H \alpha\). For \(q_D > \frac{[q_H - z]^+}{1 - \alpha}\), \(\pi(q_D) = p_L(q_L + q_D) + p_H(q_H - q_D)\), which is decreasing in \(q_D\) because \(p_H > p_L + (a_H - a_L) > p_L\). Combining these results, we then have \(q_D^* = \min \{ z, \frac{[q_H - z]^+}{1 - \alpha} \}\).

Proof of Theorem 2. We prove (a) by contradiction. Consider any arbitrary price vector \(p_H'\) and \(p_L'\), such that \(a_L - p_L' > a_H - p_H'\). Define \(q_L' = q_H - q_D\) and \(q_L = q_L + q_D\).

If \(xG(a_L - p_L') \leq q_L'\), then \(r(p_H', p_L') = p_L' xG(a_L - p_L')\). Define \(\hat{p}_H = a_H - a_L + p_L'\). Then \(r(\hat{p}_H, p_L') = \hat{p}_H \min \{ xG(\hat{a}_H - \hat{p}_H), q_H' \} + p_L' \min \{ xG(\hat{a}_H - \hat{p}_H) - q_H', q_L' \}\). Note \(r(p_H', p_L') \leq r(\hat{p}_H, p_L')\) for any value of \(q_H'\) because \(\hat{p}_H \geq p_L'\) and \(xG(\hat{a}_H - \hat{p}_H) = xG(a_L - p_L')\). It then follows that \(a_L - p_L' > a_H - p_H'\) cannot be optimal.

If \(xG(a_L - p_L') > q_L'\), define \(\beta' = \frac{xG(a_L - p_L')}{xG(a_L - p_L')}\). Then \(r(p_H', p_L') = p_L' q_L' + p_H' \min \{ (xG(a_L - p_L') - q_L')^{\beta'}, q_H' \}\). For \(\beta' < \frac{xG(a_L - p_L') - q_L'}{xG(a_L - p_L')}\), \(r(p_H', p_L') = p_L' q_L' + p_H' q_H' \Rightarrow r(\hat{p}_H, p_L')\) is increasing in \(p_L'\). For \(\beta' \geq \frac{xG(a_L - p_L') - q_L'}{xG(a_L - p_L')}\), \(r(p_H', p_L') = p_L' q_L' + p_H'(xG(a_L - p_L') - q_L')^{\beta'}\), which is increasing in \(p_L'\) up to \(p_L' = (a_H - a_L)\). We have now proven that \(r(p_H', p_L')\) is increasing in \(p_L'\) for \(p_L' < p_L' - (a_H - a_L)\). Therefore \(a_H - p_H' < a_L - p_L'\) cannot hold.

(b)(i) By (a), for any post downconversion quantity, the optimal price vector satisfies \(a_H - p_H \geq a_L - p_L\). For any given price vector that satisfies \(a_H - p_H \geq a_L - p_L\), the total revenue is non-decreasing in \(q_H\). Therefore, at the optimal price vector, downconversion will not occur, i.e., \(q_D^* = 0\).

(b)(ii). From (a) and (b)(i), \(a_H - p_H \geq a_L - p_L\) and \(q_D^* = 0\). Therefore, the revenue as a function of price is given by \(r(p_H, p_L) = p_H \min \{ xG(a_H - p_H), q_H \} + p_L \min \{ xG(a_H - p_H) - q_H, q_L \}\).

In the following, we prove theorem statements by establishing an upper bound on \(r(p_H, p_L)\) and then considering \(r(p_H, p_L)\) by different regions of \(p_H\).
Let \( \bar{r}(p_H, p_L) = \lim_{q_H \to -\infty, q_L \to -\infty} r(p_H, p_L) = p_H xG(a_H - p_H) \), then \( \bar{r}(p_H^*, p_L^*) = p_H^* xG(a_H - p_H^*) \) is the upper bound on \( r(p_H, p_L) \). By Lemma A1, \( \bar{r}(p_H, p_L) \) is unimodal in \( p_H \) and by first order condition, \( p_H^* = \frac{G(a_H-p_H)}{g(a_H-p_H)}. \)

To fully characterize \( r(p_H, p_L) \), we consider the revenue function by different regions of \( p_H \). Define \( p_H = \max\{0, a_H - G^{-1}\left(\frac{q_H}{x}\right)\} \) and \( p_H = \max\{0, a_H - G^{-1}\left(\frac{q_H G(a_H-p_H)}{x G(a_H-p_H)-q_L}\right)\} \), where \( \nu_L \) is the unique solution to \( \nu = \frac{G(a_L-\nu)}{g(a_L-\nu)} \) (note \( G(a_L-\nu) \) is unimodal in \( \nu \) for \( \nu \in [0, a_L] \)). Note that \( 0 \leq p_H \leq p_H \leq a_H \). Define \( \Gamma_0 : [p_H, a_H], \Gamma_1 : [p_H, p_H] \), and \( \Gamma_2 : [0, p_H] \) to partition \( p_H \) into three regions. We next show that \( r(p_H, p_L) \) is unimodal in \( p_H \) in all three regions.

For \( p_H \in \Gamma_0, r(p_H, p_L) = p_H xG(a_H - p_H) \), which is unimodal in \( p_H \) by above analysis. For \( p_H \in \Gamma_1, r(p_H, p_L) = p_H qH + p_L xG(a_H - p_H) - qH) G(a_H-p_H) = p_H qH + p_L G(a_H - p_H) \left( x - \frac{q_H}{G(a_H-p_H)} \right) \). Note that the optimal \( p_L \) is independent of \( p_H \). Because \( p_L G(a_H - p_H) \) is unimodal in \( p_L \), we have \( p_L^* = \frac{G(a_L-p_H)}{g(a_L-p_H)}. \) Substitute \( p_L^* \) into the revenue function, we have \( r(p_H, p_L^*) = p_H qH + p_L^* G(a_H - p_L^*)(x - \frac{q_H}{G(a_H-p_H)}) \). Note that \( \frac{\partial r(p_H, p_L^*)}{\partial p_H} = qH \left(1 - p_L^* G(a_H - p_L^*) \right) \frac{q_H}{G(a_H-p_H)} \), and \( \frac{\partial^2 r(p_H, p_L^*)}{\partial p_H^2} = -p_L^* G(a_L-p_H) (-g'(a_H-p_H)G(a_H-p_H) + 2g^2(a_H-p_H)) \) < 0 by \( R^2 > R^0 \) assumption. Therefore, \( r(p_H, p_L) \) is concave in \( p_H \) for \( p_H \in \Gamma_1 \).

For \( p_H \in \Gamma_2, r(p_H, p_L) = p_H qH + p_L qL \), where \( p_L \) satisfies the condition \( G(a_H-p_L) = q_L \). Therefore, given \( p_H, p_L(p_H) = a_H - G^{-1}\left(\frac{q_L}{x}\right) \). Substitute \( p_L(p_H) \) into the revenue function, we have \( r(p_H, p_L(p_H)) = p_H qH + \left(a_H - G^{-1}\left(\frac{q_L}{x}\right) \right) qL \). Note that \( \frac{\partial r(p_H, p_L(p_H))}{\partial p_H} = qH - q_L \frac{G^{-1}(\kappa)}{\partial p_H}, \) and \( \frac{\partial^2 r(p_H, p_L(p_H))}{\partial p_H^2} = -q_L \frac{G^{-1}(\kappa)}{\partial p_H^2}, \) where \( \kappa = \frac{q_L}{x G(a_H-p_H)}. \)

To prove the revenue function is unimodal, it is necessary and sufficient to show that \( \frac{\partial^2 G^{-1}(\kappa)}{\partial p_H^2} > 0 \) whenever \( \frac{\partial r(p_H, p_L(p_H))}{\partial p_H} = 0 \). Note that \( \frac{\partial G^{-1}(\kappa)}{\partial p_H} = \frac{1}{g(a_H-p_L)(x-q_H/G(a_H-p_H))^2 G^2(a_H-p_H)/g(a_H-p_H)} \), and \( \frac{\partial^2 G^{-1}(\kappa)}{\partial p_H^2} = \left(- \frac{g'(a_H-p_H)}{g(a_H-p_L)} + \frac{g(a_H-p_H)}{\left|g(a_H-p_L)\right|} g'(a_H-p_H) \right) \frac{\partial G(a_H-p_H)}{\partial p_H} \)

\[
\frac{q_H q_L}{(x G(a_H-p_H) - q_H)^2} + \frac{g(a_H-p_H) q_H q_L}{(x G(a_H-p_H) - q_H)^2} \cdot \frac{2x g(a_H-p_H) q_H q_L}{(x G(a_H-p_H) - q_H)^2} S, \]

where \( S = -g'(a_H-p_H) + \frac{g(a_H-p_H) q_H q_L}{(x G(a_H-p_H) - q_H)^2} \frac{2x g(a_H-p_H) q_H q_L}{(x G(a_H-p_H) - q_H)^2} \frac{G(a_H-p_H)}{G(a_H-p_H) - q_H} \).

By definition \( \frac{\partial p_L}{\partial p_H} = -G^{-1}(\kappa) \), Setting \( \frac{\partial r(p_H, p_L(p_H))}{\partial p_H} = 0, \) we have \( \frac{\partial p_L}{\partial p_H} = -q_H \). Substitute \( \frac{\partial p_L}{\partial p_H} = -q_H \) into expression \( S \) and recognizing that \( x G(a_H-p_L) = q_L + q_H G(a_H-p_H) \), we have \( S = -g'(a_H-p_H) - \frac{g(a_H-p_H) q_H q_L}{G(a_H-p_H)} + \frac{2g^2(a_H-p_H) G(a_H-p_L)}{q_H G(a_H-p_H)} = -g'(a_H-p_H) - \frac{g(a_H-p_H) G(a_H-p_L) + 2g^2(a_H-p_H) + q_H G(a_H-p_L) G(a_H-p_H)}{G(a_H-p_H)} \).

Note that \( -g'(a_H-p_H) G(a_H-p_H) + 2g^2(a_H-p_H) > 0 \) by \( R^2 > R^0 \).
$R'$ assumption. If we assume $g'(\cdot) \leq 0$, then we have $S > 0 \Rightarrow \frac{\partial^2 G^{-1}(\epsilon)}{\partial \epsilon} > 0 \Rightarrow \frac{\partial^2 r(p_H, p_L(p_H))}{\partial p_H} < 0$ whenever $\frac{\partial p_L}{\partial p_H} = 0$. Without the $g'(\cdot) \leq 0$ assumption, the expression $S$ is not necessarily positive. However, we can say more about the property of $S$.

Next we show that if there exists an optimal solution in $\Gamma_2$, then there exists an equivalent optimal solution that satisfies $a_H - p^*_H = a_L - p^*_L$. We prove this by construction.

First note that $r(p_H, p_L(p_H))$ is a continuous function in $p_H$. Therefore the optimal solution of $p^*_H$ and $p^*_L$ must be local stationary points, which is given by $\frac{\partial r(p_H, p_L(p_H))}{\partial p_H} = 0 \Rightarrow \frac{\partial p_L}{\partial p_H} = -\frac{q_H}{q_L}$. Now, suppose there exists an optimal solution pair $p^*_H$ and $p^*_L$ which does not satisfy $a_H - p^*_H = a_L - p^*_L$. Construct a new solution pair where $p'_H = p^*_H + \epsilon$ and $p'_L = p^*_L - \epsilon \frac{q_H}{q_L}$. Because $r(p'_H, p'_L) = r(p^*_H, p^*_L)$ and $p'_H$ and $p'_L$ satisfy local stationary point condition, the solution $(p'_H, p'_L)$ is at least as good as the existing $(p^*_H, p^*_L)$ solution. If $a_H - p'_H = a_L - p'_L$ then we have found an equivalent optimal solution; if not we can continue increase $p'_H$ by $\epsilon$ until either $p'_H \geq p_H$ or $a_H - p'_H = a_L - p'_L$ is satisfied. In the former case, an equivalent optimal solution exists in region $\Gamma_1$. This proves that in region $\Gamma_2$, one only needs to search solution pairs where $a_H - p_H = a_L - p_L$. Substitute this condition into expression $S$, we have $S = \frac{-g'(a_H-\nu_H)G(a_H-\nu_H)+2g^2(a_H-\nu_H)}{G(a_H-\nu_H)} + \frac{q_H}{q_L} \frac{2g^2(a_H-\nu_H)}{G(a_H-\nu_H)} - g'(a_H - \nu_H) = \frac{-g'(a_H-\nu_H)G(a_H-\nu_H)+2g^2(a_H-\nu_H)}{G(a_H-\nu_H)} (1 + \frac{q_H}{q_L}) > 0$. This proves that in region $\Gamma_2$, $\frac{\partial r(p_H, p_L(p_H))}{\partial p_H} < 0$ whenever $\frac{\partial p_L}{\partial p_H} = 0$.

Finally, we prove that $r(p_H, p_L)$ is continuous at $p_H$ but is in general not continuous at $\bar{p}_H$, i.e., the lower and upper limits of $\frac{\partial (p_H, p_L)}{\partial p_H}$ are identical at $p_H$ but not so at $\bar{p}_H$. Note that $\frac{\partial (p_H, p_L)}{\partial p_H} \mid_{p_H} = \frac{q_H}{q_L} - \frac{G(a_H-\nu_H)}{G(a_L-\nu_L)}$. From here we have $\frac{\partial p_L}{\partial p_H} \mid_{p_H} = \frac{q_H}{q_L} - \frac{G(a_H-\nu_H)}{G(a_L-\nu_L)} \bar{p}_H. This proves that $r(p_H, p_L)$ is continuous at $p_H$.

The optimal $p^*_H$ lies in $\Gamma_0$ if $\frac{\partial (p_H, p_L)}{\partial p_H} \mid_{p_H} \geq 0$; it equals to $\bar{p}_H$ if $\frac{\partial (p_H, p_L)}{\partial p_H} \mid_{p_H} < 0$ and $\frac{\partial (p_H, p_L)}{\partial p_H} \mid_{p_H} \geq 0$; it lies in $\Gamma_1$ if $\frac{\partial (p_H, p_L)}{\partial p_H} \mid_{p_H} < 0$ and $\frac{\partial (p_H, p_L)}{\partial p_H} \mid_{p_H} = 0$; it lies in $\Gamma_2$ otherwise.

Note that $\frac{\partial (p_H, p_L)}{\partial p_H} \mid_{p_H} \geq 0 \iff x \leq \frac{q_H}{G^{-1}(\frac{q_H}{G(a_H-\nu_H)})} \frac{q_H}{G^{-1}(\frac{q_H}{G(a_H-\nu_H)})} \frac{q_H}{G^{-1}(\frac{q_H}{G(a_H-\nu_H)})}$, and $\frac{\partial (p_H, p_L)}{\partial p_H} \mid_{p_H} \geq 0 \iff x \leq \frac{q_H}{G^{-1}(\frac{q_H}{G(a_H-\nu_H)})} \frac{q_H}{G^{-1}(\frac{q_H}{G(a_H-\nu_H)})} \frac{q_H}{G^{-1}(\frac{q_H}{G(a_H-\nu_H)})} \frac{q_H}{G^{-1}(\frac{q_H}{G(a_H-\nu_H)})}$.
The theorem statement follows after applying first order conditions.

Proof of Corollary 1. The corollary statements are trivially true for \( x \in \Omega_0 \cup \Omega_1 \) because there is no demand for product L in these two regions. For \( x \in \Omega_3 \), \( a_H - p^{*}_H = a_L - p^{*}_L \). Therefore, the corollary statements follow if \( a_H - p^{*}_H = a_L - p^{*}_L \) for \( x \in \Omega_2 \). Note for \( x \in \Omega_2 \), \( p^{*}_H \) satisfies

\[
\frac{G(a_H-p^{*}_H)}{\sqrt{g(a_H-p^{*}_H)}} = \frac{G(a_L-p^{*}_L)}{\sqrt{g(a_L-p^{*}_L)}} \Rightarrow \frac{G(a_L-(p^{*}_H-(a_H-a_L)))}{\sqrt{g(a_L-(p^{*}_H-(a_H-a_L)))}} = \frac{G(a_L-p^{*}_L)}{\sqrt{g(a_L-p^{*}_L)}} \Rightarrow p^{*}_H = p^{*}_L + (a_H - a_L) \text{ is one potential solution. Next we prove that this potential solution is unique. Suppose there exists an alternative solution } p^{*}_H = p^{*}_H + \epsilon, \epsilon \neq 0. \text{ This, however, cannot happen because in region } \Omega_2 \text{ the revenue function } r(p_H,p^{*}_L) \text{ is concave in } p_H. \text{ Therefore, there exists one and only one } p^{*}_H, \text{ which means } \epsilon \text{ must equal to zero and } p^{*}_H \text{ cannot be an optimal solution.}

Proof of Theorem 3. (a) (i) Consider \( a_iH - p_H \geq a_iL - p_L \) and \( a_{iH} - p_H < a_{iL} - p_L \), i.e., class \( i \) prefers H to L and the complement class \( \bar{i} \) prefers L to H, \( i = 1,2 \). In this case, there is no competition for first choice demand because customers have separating preferences. An allocation policy is only relevant for rationing class \( i \)'s spill-over demand and class \( \bar{i} \)'s first choice demand, \( i = 1,2 \). Such instances arise only when class \( i \)'s first choice product is sold out, and class \( i \)'s spill-over demand plus class \( \bar{i} \)'s first choice demand exceeds the firm’s inventory. In these instances, however, the firm is indifferent among all allocation policies because all products are sold regardless of the policy. Thus, when customers have separating preferences, any allocation policy is optimal and priority allocation is one such policy.

(ii) First consider \( a_iH - p_H \geq a_iL - p_L, \) i.e., both customer classes prefer H to L. If \( \sum_{i=1}^{2} x_iG(a_iH-p_H) \leq q_H \), then any allocation is optimal because all first choice demand can be filled. If \( \sum_{i=1}^{2} x_iG(a_iH-p_H) > q_H \), then all first choice demand cannot be filled, and a fraction of customer demand will spill over to product L. The firm’s revenue is maximized when the number of spill-over customers are maximized. The fraction of class \( i \) customers willing to spill over is \( s_{iL}, \) \( i = 1,2 \). Therefore, the optimal allocation policy would spill down class 1 customers if \( s_{1L} > s_{2L} \), spill down class 2 customers if \( s_{1L} < s_{2L} \), and be indifferent otherwise. So, if \( s_{1L} > s_{2L} \), then the firm’s revenue is maximized by first filling demand from class 2 customers, and then filling demand from class 1 customers. Conversely, if \( s_{1L} < s_{2L} \), then the firm’s revenue is maximized by first filling demand from class 1 customers, and then filling demand from class 2 customers. In either case, a priority allocation rule is thus optimal. An analogous argument holds when \( a_iH - p_H < a_iL - p_L \),
i = 1, 2, i.e., both customer classes prefer L to H.

(b) (i) Follows directly from part (a). (ii) From part (a), the optimal priority class depends on
the spill over ratio $s_{iL}, i = 1, 2.$ In advance pricing, because both $a_{iK}, i = 1, 2, k \in \{H, L\},$ and
$p$ are known in advance, the optimal priority class can be determined a priori and therefore the
priority based allocation rule is optimal. \hfill \Box

Proof of Theorem 4. First define $r(q)$ and $r(q, \epsilon)$ as the revenue function with zero and $\epsilon > 0$ units
of product $H$ converted to product $L$, respectively. We prove theorem statement by considering
different regions of price vector $p$. In what follows, we consider an exhaustive and mutually exclusive
list of problem regions.

First consider $p \in \Gamma_1$.

- **A:** $\sum_{i=1}^{2} d_{iH} < q_H.$ Then $r(q) = p_H \sum_{i=1}^{2} d_{iH} \Rightarrow r(q, \epsilon) = r(q) - \epsilon c_D < r(q) \Rightarrow \hat{q}_D = 0.$

- **B:** $\sum_{i=1}^{2} d_{iH} \geq q_H$ and $d_{jH} < q_H.$ Then $r(q) = p_H q_H + p_L \min\{(\sum_{i=1}^{2} d_{iH} - q_H)s_{jL}, q_L\}.$
  - B1: $q_L \geq (\sum_{i=1}^{2} d_{iH} - q_H)s_{jL}.$ Then $r(q) = p_H q_H + p_L (\sum_{i=1}^{2} d_{iH} - q_H)s_{jL} \Rightarrow r(q, \epsilon) = r(q) - \epsilon (p_H - p_L s_{jL}) < r(q) \Rightarrow \hat{q}_D = 0$ because $s_{jL} < 1$ for $p \in \Gamma_2.$
  - B2: $q_L < (\sum_{i=1}^{2} d_{iH} - q_H)s_{jL}.$ Then $r(q) = p_H q_H + p_L q_L \Rightarrow r(q, \epsilon) = r(q) - \epsilon (p_H - p_L) < r(q) \Rightarrow \hat{q}_D = 0.$

- **C:** $\sum_{i=1}^{2} d_{iH} \geq q_H$ and $d_{jH} \geq q_H.$ Then $r(q) = p_H q_H + p_L \min\{(d_{jH} - q_H)s_{jL} + d_{jH} s_{jL}, q_L\}.$
  - C1: $q_L > (d_{jH} - q_H)s_{jL} + d_{jH} s_{jL}.$ Then $r(q) = p_H q_H + p_L ((d_{jH} - q_H)s_{jL} + d_{jH} s_{jL}) \Rightarrow r(q, \epsilon) = r(q) - \epsilon (p_H - p_L s_{jL} + c_D) < r(q) \Rightarrow \hat{q}_D = 0.$
  - C2: $q_L \leq (d_{jH} - q_H)s_{jL} + d_{jH} s_{jL}.$ Then $r(q) = p_H q_H + p_L q_L \Rightarrow r(q, \epsilon) = r(q) - \epsilon (p_H - p_L + c_D) < r(q) \Rightarrow \hat{q}_D = 0.$

Next consider $p \in \Gamma_2$.

- **D:** $d_{jH} \geq q_H, r(q) = p_H q_H + p_L \min\{d_{jL} + (d_{jH} - q_H)s_{jL}, q_L\}.$
  - D1: $q_L > d_{jL} + (d_{jH} - q_H)s_{jL}.$ Then $r(q) = p_H q_H + p_L (d_{jL} + (d_{jH} - q_H)s_{jL}) \Rightarrow r(q, \epsilon) = r(q) - \epsilon (p_H - p_L s_{jL} + c_D) < r(q) \Rightarrow \hat{q}_D = 0.$
The region $p \in \Gamma_3$ can be symmetrically proved as $p \in \Gamma_2$. Finally, consider $p \in \Gamma_4$.

- G: $\sum_{i=1}^{2} d_i < q$. $r(q) = p_L \sum_{i=1}^{2} d_i \Rightarrow r(q, \epsilon) = r(q) - \epsilon c_D < r(q) \Rightarrow \hat{q}_D = 0$.

- H: $d_j \geq q$. $r(q) = p_H \min\{(d_j - q)L_s jH + d_jL s_jH, qH\} + pLqL$.
  
  - H1: $(d_j - q)L_s jH + d_jL s_jH \geq qH$. $r(q) = p_H((d_j - q)L_s jH + d_jL s_jH) + pLqL \Rightarrow r(q, \epsilon) = r(q) - \epsilon(p_H - pL + cD) < r(q) \Rightarrow \hat{q}_D = 0$.
  
  - H2: $(d_j - q)L_s jH + d_jL s_jH < qH$. $r(q) = \partial r(q)/\partial \epsilon > 0$ if $p_L - c_D > p_H s_jH$.

  Therefore, the optimal $\hat{q}_{D1}$ satisfies $(d_j - (q + \hat{q}_{D1}))s_jH + d_jL s_jH = qH - \hat{q}_{D1} \Rightarrow \hat{q}_{D1} = (qH - (d_j - q)L_s jH - d_jL s_jH) / 1 - s_jH$. Now, $r(\hat{q}_{D1}) = p_H \min\{qH - \hat{q}_{D1}, (d_j - qL - \hat{q}_{D1})s_jH + d_jL s_jH\} + pL(qL + \hat{q}_{D1})$. Thus, if $d_jL s_jH < qH - (d_jL - qL - \hat{q}_{D1})s_jH - \hat{q}_{D1}$, one can follow similar logic and show that the optimal $\hat{q}_{D2} = (qH - (d_jL - qL - \hat{q}_{D1})s_jH - \hat{q}_{D1} - d_jL s_jH)/1 - s_jH$.

- I: $d_jL < qL$ and $\sum_{i=1}^{2} d_i \geq qL$. $r(q) = p_H \min\{(\sum_{i=1}^{2} d_i - qL)s_jH, qH\} + pLqL$.
  
  - I1: $(\sum_{i=1}^{2} d_i - qL)s_jH \geq qH$. $r(q) = p_HqH + pLqL \Rightarrow r(q, \epsilon) = r(q) - \epsilon(p_H - pL + cD) < r(q) \Rightarrow \hat{q}_D = 0$. 

APPENDIX
We prove that downconversion can be optimal by constructing a particular example. We use a deterministic demand and yield example, in which case there is no distinction between advance and recourse pricing. Define $x_1$ and $x_2$ as the deterministic market size for class 1 and class 2 customers, and $y$ as the deterministic yield. Let the customer valuations be $a_{1L} = 0$ and $a_{2H} = a_{2L}$. To simplify notation, let $a_1 = a_{1H}$ and $a_2 = a_{2H} = a_{2L}$. Let $\alpha = \frac{a_2 - p_L}{a_2 - p_H}$ and $\beta = \frac{a_2 - p_L}{a_2 - p_H}$.

The profit $\Pi(Q, p_H, p_L, q_D) = -c_P Q + p_H \min\{x_1(a_1 - p_H) + x_2(a_2 - p_L), y Q - q_D\} + p_L \min\{x_2(a_2 - p_H) - [y Q - q_D - x_1(a_1 - p_H)]^+, (1 - y) Q - q_D\} - c_D q_D$ if $p_H \leq p_L$, and $\Pi(Q, p_H, p_L, q_D) = -c_P Q + p_H \min\{x_1(a_1 - p_H) + x_2(a_2 - p_L) - (1 - y) Q - q_D\} + p_L \min\{x_2(a_2 - p_L), (1 - y) Q - q_D\} - c_D q_D$ otherwise. We first note that $p_H \leq p_L$, i.e., class 2 prefers H to L, cannot be optimal (the proof involves showing that $p_H \leq p_L \Rightarrow q_D^* = 0$, and $p_H > p_L$ dominates $p_H \leq p_L$; details available upon request). We can therefore restrict attention to $p_H > p_L$.

If $q_D \leq x_2(a_2 - p_L) - (1 - y) Q$, then $\Pi(Q, p_H, p_L, q_D) = -c_P Q + p_H \min\{x_1(a_1 - p_H) + x_2(a_2 - p_L) - (1 - y) Q - q_D\} + p_L ((1 - y) Q + q_D) - c_D q_D$. In this case one can show that $Q^*$ satisfies $y Q - q_D \geq x_1(a_1 - p_H) + x_2(a_2 - p_L) - (1 - y) Q - q_D$ if $c_D < p_L - p_H \beta$ and decreasing otherwise.

If $x_2(a_2 - p_L) - (1 - y) Q < q_D$, then $\Pi(Q, p_H, p_L, q_D) = -c_P Q + p_H \min\{x_1(a_1 - p_H), y Q - q_D\} + p_L x_2(a_2 - p_L) - c_D q_D \Rightarrow \Pi(Q, p_H, p_L, q_D)$ is strictly decreasing in $q_D$. Combining the above, we have $q_D^*(Q, p_H, p_L) = [x_2(a_2 - p_L) - (1 - y) Q]^+$ if $c_D < p_L - p_H \beta$ and $q_D^*(Q, p_H, p_L) = 0$ otherwise.

Using this $q_D^*(Q, p_H, p_L)$, we can then solve for the optimum $Q$, $p_H$, and $p_L$. One can show that if $c_D \leq \frac{(a_2 - c_P) x_2 y - (a_1 - c_P) x_1 (1 - y)}{x_1 (1 - y)^2 + x_2 y^2}$, then $Q^* = \frac{1}{2}(a_1 x_1 + a_2 x_2 + c_D x_1 - (c_P + y c_D)(x_1 + x_2))$.
By Theorem 2, the optimal price vector always induces customers to prefer product H over L. Downgrading can never be optimal because either all demands are satisfied by product H or there is not leftover of product H. (b) Follows from Theorem 5. (c) (i) and (ii) follows from the fact that, at zero downconversion cost, downgrading is never optimal. We also note that if \( \frac{a_1 - c_P}{a_2 - c_P} > \frac{x_P y}{x_1(1-y)} \) then downconversion is never optimal.

**Proof of Theorem 6.** (a) By Theorem 2, the optimal price vector always induces customers to prefer product H over L. Downgrading can never be optimal because either all demands are satisfied by product H or there is not leftover of product H. (b) Follows from Theorem 5. (c) (i) and (ii) follows from the fact that, at zero downconversion cost, downgrading is a special case of downconversion.

**Proof of Theorem 7.** (i) The case of \( a_H - p_H \geq a_L - p_L \) follows from part (i) of Theorem 1. (ii) We prove that the optimal expected revenue is a unimodal function in \( q_D \). Note that if \( a_H - p_H < a_L - p_L \), then the firm’s expected revenue as a function of downconversion quantity, \( q_D \), is given by

\[
E_X[r(q_D)] = p_L \left( \int_0^{\frac{q_H + q_D}{G(a_L - p_L)}} xG(a_L - p_L) dF_R(x) + \int_{\frac{q_H + q_D}{G(a_L - p_L)}}^{\infty} (q_L + q_D) dF_R(x) \right) + p_H \int_{\frac{q_L + q_D}{G(a_H - p_H)}}^{\frac{q_H + q_D}{G(a_L - p_L)}} (xG(a_L - p_L) - (q_L + q_D)) \frac{G(a_H - p_H)}{G(a_L - p_L)} dF_R(x) + p_H \int_{\frac{q_H + q_D}{G(a_H - p_H)}}^{\infty} (q_H - q_D) dF_R(x) - c_D q_D. \tag{A-1}
\]

By (A-1), we have

\[
\frac{\partial E_X[r(q_D)]}{\partial q_D} = -c_D + p_L F_R \left( \frac{q_L + q_D}{G(a_L - p_L)} \right) - p_H \frac{G(a_H - p_H)}{G(a_L - p_L)} \int_{\frac{q_L + q_D}{G(a_L - p_L)}}^{\frac{q_H + q_D}{G(a_H - p_H)}} dF_R(x) - p_H \int_{\frac{q_H + q_D}{G(a_H - p_H)}}^{\infty} dF_R(x). \tag{A-2}
\]

Note that

\[
\frac{\partial^2 E_X[r(q_D)]}{\partial q_D^2} = -\frac{p_L}{G(a_L - p_L)} f_R \left( \frac{q_L + q_D}{G(a_L - p_L)} \right) + p_H \frac{G(a_H - p_H)}{G^2(a_L - p_L)} f_R \left( \frac{q_L + q_D}{G(a_L - p_L)} \right) - p_H \frac{(G(a_H - p_H) - G(a_L - p_L))^2}{G^2(a_L - p_L)G(a_H - p_H)} f_R \left( \frac{q_L + q_D}{G(a_L - p_L)} \right). \tag{A-3}
\]
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By (A-2), we have

\[
(p_L - p_H \frac{G(a_H - p_H)}{G(a_L - p_L)}) F_R \left( \frac{q_L + q_D}{G(a_L - p_L)} \right) > c_D \Rightarrow p_L > p_H \frac{G(a_H - p_H)}{G(a_L - p_L)}. \tag{A-4}
\]

Substitute (A-4) into (A-3), we have \( \frac{\partial^2 E_X[r(q_D)]}{\partial q_D^2} < 0 \) whenever \( \frac{\partial E_X[r(q_D)]}{\partial q_D} = 0 \). Part (ii)(a) then follows directly from (A-2). (ii)(b) The theorem statement follows by substituting \( q_D = q_H \) into (A-2).

Proof of Theorem A1. See the proof of Theorem 2, which proves a more general case.

Proof of Corollary A2. By Theorem 2 and A1, the optimal recourse prices \( p^*_H|_{RYP1} = p^*|_{CPP1} \) for \( x \in \Omega_0 \cup \Omega_1 \). For \( x \in \Omega_2 \), by definition, \( \frac{q_H}{x} < \sqrt{g(G^{-1}(\frac{q_H}{x}))} \nu_L G(a_L - \nu_L) \Rightarrow \frac{q_H}{x} < \sqrt{g(a_H - p_H^*)p_L^* G(a_L - p_L^*)} = G(a_L - p_L^*) \sqrt{g(a_H - p_H^*)g(a_L - p_L^*)} \Rightarrow p^*_H|_{RYP1} = a - G^{-1}(\frac{q_H}{x}) = a_H - G^{-1}(\frac{q_H}{x}) > a_H - G^{-1}(\frac{q_H + q_L}{x}) = p^*_H|_{CPP1}. \) For \( x \in \Omega_3 \), \( p^*_H|_{RYP1} = a - G^{-1}(\frac{q_H}{x}) \geq a_H - G^{-1}(\frac{q_H + q_L}{x}) = p^*_H|_{CPP1}. \) Thus, \( p^*_H|_{RYP1} \geq p^*_H|_{CPP1} \) for any realization of market size \( x \) and yield \( y \).

Proof of Theorem A2. The theorem statement follows by recognizing that the CPP1 model is a relaxation of RYP1 model by allowing \( a_L \) to take on positive values.

Proof of Theorem A3. We first characterize the revenue function \( r(\infty) = \lim_{q \to \infty} r(q) = p \sum_{i=1}^{2} x_i G_i(a_i - p) \). Partition \( p \) into two regions: \( \Gamma_0 = \{ p : 0 \leq p \leq a_2 \} \) and \( \Gamma_1 = \{ p : a_2 < p \leq a_1 \} \). We next prove \( r(\infty) \) is unimodal in \( p \) in \( \Gamma_0 \) and \( \Gamma_1 \).

For \( p \in \Gamma_0 \), \( r(\infty, p) = p \sum_{i=1}^{2} x_i G_i(a_i - p) \), which is unimodal in \( p \) by assumption (T2) in Lemma A1.

For \( p \in \Gamma_1 \), \( r(\infty, p) = pxG_1(a_1 - p) \), which is also unimodal in \( p \) by assumption (T2) in Lemma A1. Note that \( r(\infty, p) \) is in general not continuous at the \( p = a_2 \). Because \( \frac{\partial r(\infty, p)}{\partial p}|_{p=a_2} = x_1 G_1(a_1 - a_2) - a_2 x_1 g_1(a_1 - a_2) \) and \( \frac{\partial^2 r(\infty, p)}{\partial p^2}|_{p=a_2} = x_1 G_1(a_1 - a_2) - a_2 x_1 g_1(a_1 - a_2) + x_2 g_2(0) \), we have \( \frac{\partial r(\infty, p)}{\partial p}|_{p=a_2} \geq \frac{\partial^2 r(\infty, p)}{\partial p^2}|_{p=a_2} \). If \( \frac{\partial r(\infty, p)}{\partial p}|_{p=a_2} \leq 0 \), then \( p^* \in \Gamma_0 \) and is given by \( p^* = \frac{\sum_{i=1}^{2} x_i G_i(a_i - p^*)}{\sum_{i=1}^{2} x_i g_i(a_i - p^*)} \), which is the definition of \( \nu_1 \). If \( \frac{\partial r(\infty, p)}{\partial p}|_{p=a_2} \geq 0 \), then \( p^* \in \Gamma_1 \) and is given by \( p^* = \frac{G_1(a_1 - p^*)}{g_1(a_1 - p^*)} \), which is the definition of \( \nu_1 \). Finally, if \( \frac{\partial r(\infty, p)}{\partial p}|_{p=a_2} > 0 \) and \( \frac{\partial^2 r(\infty, p)}{\partial p^2}|_{p=a_2} < 0 \), then \( r(\infty, p) \) is separately concave in \( \Gamma_0 \) and \( \Gamma_1 \). In this case, the optimal \( p^* \) is given by \( \arg \max r(\infty, p), p \in \{p_m, p_s\} \).
For a finite realized $q$, $p^*$ is lower bounded by $\bar{p}$, since $r(q, \bar{p} - \epsilon) < r(q, \bar{p}), \forall \epsilon > 0$. Theorem statements then follow directly.

**Proof of Corollary A4.** (a) By Theorem A3, $\nu_j > \nu_j \Rightarrow p^*_2 > p^*_1$ for any realization of $x_i$ and $q$. (b) $x > \frac{a - G^{-1}(\frac{q}{x}) - q}{G^{-1}(\frac{q}{x})} \Rightarrow \bar{p} > p^*_1 \Rightarrow p^*_2 > p^*_1$. (c) Follows from (b). Otherwise, $p^*_2 = \nu_2 \Rightarrow \min\{\bar{p}, \nu_j\} \leq p^*_2 < \nu_j \Rightarrow p^*_2 \leq p^*_1$. The case of $a_1 \leq a_2$ can be analogously proved.

**Proof of Theorem A4.** The theorem statement follows by recognizing that the RYP2 model is a relaxation of RYP1 model by allowing $a_2$ and $x_2$ to take on positive values.

**Proof of Theorem A5.** This theorem can be analogously proved as Theorem 4.

**Proof of Corollary 2, A3, and A1.** Follow directly from Theorems 2, A3, and A1 respectively by setting $G_i(\cdot)$ to $U(0, 1)$ and (without loss of generality) scaling the valuations to between 0 and 1.

### A2 Appendix - A Single-Class, Random Yield Model with Pricing (RYP1)

In this appendix, we consider the special case in which (i) production results in a random quantity of a single product and (ii) there is a single class of customers. We therefore set $a_{1L} = a_{2L} = a_{1H} = 0$. For notational simplicity we let $a = a_{2H}$ and remove the class subscript $i = 1, 2$ from all parameters. Downconversion is not relevant as there is no lower-quality product. Allocation is not relevant as there is only one customer class. The relevant decisions facing the firm are the price $p$ and the production quantity $Q$. Recall that the distribution for the utility of customers’ outside options is $G(\cdot)$, and so for a given price $p$, the fraction of customers who prefer the product to their outside options is $G(a - p)$.

We first consider the quantity-and-price setting problem under recourse-pricing and then consider the case of advance-pricing. Finally we contrast this random-yield model with the co-production model with a single class of customers, i.e., CPP1.
A2.1 Recourse Pricing

In the case of recourse pricing, the firm sets prices after yield and market uncertainties are resolved. The firm chooses its prices to maximize its revenue, which is given by the equation

\[
r(q) = p \min \{xG(a - p), q\}.
\]

**Theorem A1.** For any realization of product quantities \(q\) and market potential \(x\), the optimal recourse price satisfies

\[
p^* = \frac{G(a - p^*)}{g(a - p^*)}, \quad x \leq \frac{q}{(a - G^{-1}\left(\frac{q}{2}\right))g(G^{-1}\left(\frac{q}{2}\right))},
\]

\[
p^* = a - G^{-1}\left(\frac{q}{x}\right), \quad \text{otherwise}.
\]

We note that the optimal prices are increasing in the market potential, reflecting the fact that the firm can charge higher prices when demand is high relative to supply.

**Corollary A1.** Assume \(G(\cdot) \sim U(0, 1)\) and \(a\) is scaled between 0 and 1, the optimal recourse price \(p^*(q, x)\) is given by

\[
p^*(q, x) = \frac{a}{2}, \quad x \leq \frac{2q}{a},
\]

\[
p^*(q, x) = a - \frac{q}{x}, \quad \text{otherwise}.
\]

For the special case of \(G(\cdot) \sim U(0, 1)\), we can use the optimal prices from Corollary A1 to develop expression for the optimal revenue \(r^*_r(q, x)\) as a function of the product quantity and market potential,

\[
r^*_r(Q, y, x) = \left(\frac{a^2}{4}\right)x, \quad x \leq \frac{2yQ}{a},
\]

\[
r^*_r(Q, y, x) = \left(a - \frac{yQ}{x}\right)yQ, \quad \text{otherwise}.
\]

Note that the subscript \(r\) on the price vector is used to indicate that we are considering revenue with recourse pricing and not advance pricing. As one would expect, the optimal revenue is non decreasing in the market potential \(x\), the production quantities \(Q\) and the customer valuation \(a\).
We are now in a position to characterize the firm’s optimal production-quantity \( Q^* \). The firm chooses \( Q \) to maximize its expected profit,

\[ \Pi_r (Q) = -cPQ + E_{Y, X} [r^*_r (Q, y, x)]. \]

The first term is the production cost and the second term is the expected revenue, where the expectation is taken over the yield and market-potential random variables. Using the above expressions for \( r^*_r (Q, y, x) \), we can write the expected profit function as

\[ \Pi_r (Q) = -cPQ + \int_{0}^{1} \left( \int_{0}^{\frac{2aQ}{p}} x dF_X (x) + \int_{\frac{2aQ}{a}}^{\infty} \left( a - \frac{yQ}{x} \right) yQ dF_X (x) \right) dF_Y (y). \]

It is relatively straightforward to show that \( \Pi_r (Q) \) is concave in \( Q \), and so the optimal production quantity \( Q^* \) is given by the first-order condition. Closed form solution to \( Q^* \) will not, however, typically exist.

### A2.2 Advance Pricing

In the case of advance pricing, the firm jointly sets the production quantity \( Q \) and the price \( p \) before yield and market uncertainties are resolved. We can then formulate the firm’s joint quantity-and-price setting problem as

\[ \max_{Q \geq 0, p \geq 0} \Pi_a (Q, p), \]

where

\[ \Pi_a (Q, p) = -cPQ + E_{Y, X} [r_a (Q, p, y, x)], \quad (A-5) \]

\[ r_a (Q, p, y, x) = p \min \{ xG (a - p), yQ \}, \quad (A-6) \]

and the subscript \( a \) is used to indicate that we are considering advance pricing. Substituting equation (A-6) into (A-5), we then obtain,

\[ \Pi_a (Q, p) = -cPQ + \int_{0}^{1} \left( \int_{0}^{\frac{aQ}{G(a-p)}} pG (a - p) x dF_X (x) + \int_{\frac{aQ}{G(a-p)}}^{\infty} pyQ dF_X (x) \right) dF_Y (y). \]
The above expected profit function is in general not concave in price $p$, and therefore is in general not jointly concave in $Q$ and $p$. However, $\Pi_a (Q, p)$ is concave in $Q$ for any given price $p$. The optimal $Q^*$ is implicitly given by

$$\int_0^1 yF_X \left( \frac{yQ^*}{G(a - p)} \right) dF_Y(y) = E[y] - \frac{cP}{p}.$$  

In addition, note that $\frac{\partial^2 \Pi_a(Q,p)}{\partial p^2}$ is given by

$$\int_0^1 \left( \int_0^{\frac{yQ}{G(a-p)}} \left( -2g(a-p) + pg'(a-p) \right) dF_X(x) - \frac{py^2Q^2g^2(a-p)}{G^3(a-p)} f \left( \frac{yQ}{G(a-p)} \right) \right) dF_Y(y).$$

Therefore, if $\frac{pg'(a-p)}{g(a-p)} < 2$ (which is true for a wide class of distributions including the Uniform, Exponential and certain specifications of the Weibull, Gamma and truncated-Normal families), then $\Pi_a (Q, p)$ is concave in $p$ for any given quantity $Q$. It is straightforward to show that $0 < p^* < a$ and $G(a-p)F^{-1}_X \left( 1 - \frac{cP}{pE[y]} \right) \leq Q^* \leq \nu G(a-\nu) \frac{E[X]}{cP}$, where $\nu$ satisfies $\nu = \frac{G(a-\nu)}{g(a-\nu)}$. The optimal solution can therefore be efficiently computed.

**A2.3 Comparison with CPP1 Model**

The RYP1 model is a special case of the CPP1 model in which $a_L = 0$. It is of interest to see how the simultaneous production of a valued "by-product", i.e., product L, influences the optimal price for product H and the expected profit. Because H is the only product in the RYP1 model, we do not subscript it in that model.

**Corollary A2.** For any given realized product quantities and market potential, the optimal recourse price (for product H) under CPP1 is no higher than that under RYP1, i.e., $p_{H|CPP1} \leq p_{H|RYP1}$.

Corollary A2 tells us that with a valued by-product the firm may charge a lower price for the high-quality product. When market potential is small relative to the supply of high-quality product, the price is identical. When market potential is large, however, the price is lower for CPP1 because there is more overall (i.e., including the by-product) supply to sell. The following theorem establishes that, as expected, the simultaneous production of a valued by-product gives the firm a higher expected profit, regardless of whether the firm adopts advance or recourse pricing.
Theorem A2. Regardless of recourse pricing or advance pricing, the optimal expected profit under CPP1 is at least as high as that under RYP1.

Our numeric study\(^8\) shows that the expected profit for CPP1 (averaged over \(a_L = 0.2, 0.4,\) and 0.6) can be significantly higher than that for RYP1. The average expected profit for CPP1 is 29.55% and 15.73% higher than that for RYP1 for advance pricing and recourse pricing, respectively. Note that the value of CPP1 over RYP1 is significantly reduced with recourse pricing. Consistent with Corollary A2, the average expected price for product H under CPP1 is 4.35% and 3.43% lower than that under RYP1.

A3 Appendix - A Two-Class, Random Yield Model with Pricing (RYP2)

In this appendix, we consider the special case in which (i) production results in a random quantity of a single product and (ii) there are two classes of customers. We therefore set \(a_{1L} = a_{2L} = 0\). For notational simplicity we let \(a_1 = a_{1H}\) and \(a_2 = a_{2H}\). Without loss of generality, we assume \(a_1 \geq a_2\). Downconversion is not relevant as there is no lower-quality product. We note that the firm is indifferent between allocation rules because both class of customers pay the same price and there is no spill over to another product. Therefore, the relevant decisions facing the firm are the price \(p\), the production quantity \(Q\), and the allocation policy. Recall that for a given price \(p\), the fraction of customers who prefer the product to their outside options is \(G_i(a_i - p), i = 1, 2\). We consider the quantity-and-price setting problem under both recourse- and advance-pricing.

In the case of recourse pricing, the firm sets prices after yield and market uncertainties are resolved. The firm chooses its prices to maximize its revenue, which is given by the equation

\[
r(q) = p \min \left\{ \sum_{i=1}^{2} x_i G_i(a_i - p), q \right\},
\]

where \(q = Qy\). In what follows we assume that the distribution functions \(G_i(\cdot)\) satisfy condition T2 (see Lemma A1 in Appendix A4.) We note that a sufficient condition for T2 to hold is that the

\(^8\)Similar to that described in §4.3, except that uniform distributions were used for the outside utility, market potential and yield. Details available upon request.
Assume (T2) holds. For any realization of product quantities \( q \) and market potentials \( x = (x_1, x_2) \), define \( \bar{p} = 0 \) if \( \sum_{i=1}^{2} x_i G_i(a_i) \leq q \). Otherwise define \( \bar{p} \) as the unique solution to \( \sum_{i=1}^{2} x_i G_i((a_i - \bar{p})^+) = q \). In addition, define \( \nu \) as the solution to \( \nu = \frac{G_1(a_1 - \nu)}{g_1(a_1 - \nu)} \), and \( \nu_{12} \) as the solution to \( \nu = \frac{\sum_{i=1}^{2} x_i G_i(a_i - \nu)}{\sum_{i=1}^{2} x_i G_i(a_i - \nu)} \). The optimal recourse price is given by

\[
\begin{align*}
p^* &= \bar{\nu}_{12}, \quad \frac{G_1(a_1 - a_2)}{g_1(a_1 - a_2)} \leq a_2, \\
p^* &= \bar{\nu}_{12}, \quad \frac{G_1(a_1 - a_2)}{g_1(a_1 - a_2)} > a_2, \quad \frac{x_2}{x_1} > \frac{G_1(a_1 - a_2) - a_2 g_1(a_1 - a_2)}{a_2 g_2(0)}, \quad r(q, \bar{\nu}_{12}) \geq r(q, \bar{\nu}), \\
p^* &= \bar{\nu}_1, \quad \frac{G_1(a_1 - a_2)}{g_1(a_1 - a_2)} > a_2, \quad \frac{x_2}{x_1} > \frac{G_1(a_1 - a_2) - a_2 g_1(a_1 - a_2)}{a_2 g_2(0)}, \quad r(q, \bar{\nu}_{12}) < r(q, \bar{\nu}), \\
p^* &= \bar{\nu}_1, \quad \frac{x_2}{x_1} \leq \frac{G_1(a_1 - a_2) - a_2 g_1(a_1 - a_2)}{a_2 g_2(0)},
\end{align*}
\]

where \( \bar{\nu}_1 = \max\{\bar{p}, \nu_1\} \) and \( \bar{\nu}_{12} = \max\{\bar{p}, \nu_{12}\} \).

We note that the condition \( r(q, \bar{\nu}_{12}) \geq r(q, \bar{\nu}) \) can be specified by model primitives but is omitted here for the sake of brevity. Note that the above theorem collapses to the RYP1 model when the market potential \( x_2 \) and/or the valuation \( a_2 \) are set to zero. In fact, this theorem generalizes Theorem A1. Theorem A3 tells us that, depending on customer valuations \( a_i, i = 1, 2 \), realized market potentials \( x_i, i = 1, 2 \), and the realized production quantity \( q \), the firm may not serve both class of customers. If the customer valuations are sufficiently close, then it is optimal to serve both customer classes unless the realized quantity \( q \) is too small. On the other hand, if realized market sizes are sufficiently different, then it is optimal to serve only the high-valuation class.

For the special case of \( G(\cdot) \sim U(0, 1) \) (and, without loss of generality, valuations scaled between 0 and 1.), we can derive explicit expressions for the optimal recourse price.

**Corollary A3.** Assume \( G_i(\cdot) \sim U(0, 1) \ i = 1, 2 \). For any realization of product quantities \( q = (q_H, q_L) \) and market potential \( x = (x_1, x_2) \), (a) If \( \frac{a_1}{a_2} \leq 2 \), then the optimal recourse price is given
APPENDIX

\[ p^*(q, x) = a_1 - \frac{q}{x_1}, \quad q \leq x_1(a_1 - a_2), \]

\[ p^*(q, x) = \frac{\sum_{i=1}^{2} x_i a_i - q}{\sum_{i=1}^{2} x_i}, \quad x_1(a_1 - a_2) < q < \frac{\sum_{i=1}^{2} x_i a_i}{2}, \]

\[ p^*(q, x) = \frac{\sum_{i=1}^{2} x_i a_i}{2 \sum_{i=1}^{2} x_i}, \quad q \geq \frac{\sum_{i=1}^{2} x_i a_i}{2}. \]

(b) If \( \frac{a_1}{a_2} > 2 \), then the optimal recourse price is given by

\[ p^*(q, x) = a_1 - \frac{q}{x_1}, \quad q \leq \frac{x_1 a_1}{2}, \]

\[ p^*(q, x) = \frac{a_1}{2}, \quad \frac{x_1 a_1}{2} < q, \quad \frac{x_2}{x_1} \leq \frac{a_1}{a_2} - 2, \]

\[ p^*(q, x) = \frac{a_1}{2}, \quad \frac{x_1 a_1}{2} < q < \frac{M}{2}, \quad \frac{a_1}{a_2} - 2 < \frac{x_2}{x_1}, \]

\[ p^*(q, x) = \frac{\sum_{i=1}^{2} x_i a_i - q}{\sum_{i=1}^{2} x_i}, \quad \frac{M}{2} < q < \frac{\sum_{i=1}^{2} x_i a_i}{2}, \quad \frac{a_1}{a_2} - 2 < \frac{x_2}{x_1}, \]

\[ p^*(q, x) = \frac{\sum_{i=1}^{2} x_i a_i}{2}, \quad \frac{\sum_{i=1}^{2} x_i a_i}{2} \leq q, \quad \frac{a_1}{a_2} - 2 < \frac{x_2}{x_1}, \]

\[ p^*(q, x) = \frac{\sum_{i=1}^{2} x_i a_i}{2}, \quad \frac{\sum_{i=1}^{2} x_i a_i}{2} \leq q, \quad \frac{a_1}{a_2} \left( \frac{a_1}{a_2} - 2 \right) \leq \frac{x_2}{x_1}, \]

where \( M = \sum_{i=1}^{2} x_i a_i - \sqrt{(\sum_{i=1}^{2} x_i a_i)^2 - a_1^2 x_1 \sum_{i=1}^{2} x_i}. \)

We note that if \( a_1 \leq 2a_2 \), the optimal recourse price \( p^* \) is non-increasing in the realized production quantity \( q \) for any realized market potentials \( x_i, i = 1, 2 \), that is, the larger the supply the lower the price. However, if \( a_1 > 2a_2 \), that is, the classes differ greatly in their valuation, then the optimal recourse price \( p^* \) is not necessarily monotonic in \( q \). However, one can show that the optimal revenue is non decreasing in the market potentials \( x_i, i = 1, 2 \), the launched quantity \( Q \) and the customer valuations \( a_i, i = 1, 2 \).

Using the optimal prices from Corollary A3, we can develop an expression for the optimal
revenue \( r^*_r(q, x) \) as a function of the product quantity and market potential. For \( a_1 \leq 2a_2 \),

\[
\begin{align*}
    r^*_r(Q, y, x) &= \left( a_1 - \frac{yQ}{x_1} \right) yQ, & yQ \leq x_1(a_1 - a_2), \\
    r^*_r(Q, y, x) &= \left( \frac{\sum_{i=1}^2 x_i a_i - yQ}{\sum_{i=1}^2 x_i} \right) yQ, & x_1(a_1 - a_2) < yQ < \frac{\sum_{i=1}^2 x_i a_i}{2}, \\
    r^*_r(Q, y, x) &= \frac{\left( \sum_{i=1}^2 x_i a_i \right)^2}{4 \sum_{i=1}^2 x_i}, & yQ \geq \frac{\sum_{i=1}^2 x_i a_i}{2}.
\end{align*}
\]

For \( a_1 > 2a_2 \),

\[
\begin{align*}
    r^*_r(Q, y, x) &= \left( a_1 - \frac{yQ}{x_1} \right) yQ, & (Q, y, x) \in \Lambda_0, \\
    r^*_r(Q, y, x) &= \frac{a_1^2 x_1}{4}, & (Q, y, x) \in \Lambda_1, \\
    r^*_r(Q, y, x) &= \left( \frac{\sum_{i=1}^2 x_i a_i - yQ}{\sum_{i=1}^2 x_i} \right) yQ, & (Q, y, x) \in \Lambda_2, \\
    r^*_r(Q, y, x) &= \frac{\left( \sum_{i=1}^2 x_i a_i \right)^2}{4 \sum_{i=1}^2 x_i}, & (Q, y, x) \in \Lambda_3.
\end{align*}
\]

As one would expect, the optimal revenue is non-decreasing in the market potentials \( x_i, i = 1, 2 \), the production quantity \( Q \) and the customer valuations \( a_i, i = 1, 2 \).

In the case of advance pricing, the firm jointly sets the production quantity \( Q \) and the price \( p \) before yield and market uncertainties are resolved. We can then formulate the firm’s joint quantity- and-price setting problem as

\[
\max_{Q \geq 0, p \geq 0} \Pi_a(Q, p),
\]

where

\[
\Pi_a(Q, p) = -c_p Q + E_{Y|X} \left[ r_a(Q, p, y, x) \right], \tag{A-7}
\]

\[
r_a(Q, p, y, x) = p \min \left\{ \sum_{i=1}^2 x_i G_i (a_i - p), yQ \right\}, \tag{A-8}
\]

and the subscript \( a \) is used to indicate that we are considering advance pricing. Substituting
In addition, note that \( \Pi \) is in general not concave in price \( p \), and therefore is in general not jointly concave in \( Q \) and \( p \). However, \( \Pi (Q, p) \) is concave in \( Q \) for any given price \( p \). The optimal \( Q^* \) is implicitly given by

\[
\int_0^1 y \left( \int_0^{G_1 (a_1 - p)/G_2 (a_2 - p)} \left( \int_0^{G_1 (a_1 - p)/G_2 (a_2 - p)} F_{X_2} \left( \frac{yQ - x_1 G_1 (a_1 - p)}{G_2 (a_2 - p)} \right) dF_{X_1} (x_1) + F_{X_1} \left( \frac{yQ^*}{G_1 (a_1 - p)} \right) \right) dF_Y (y) \right) = \frac{cP}{p}.
\]

In addition, note that \( \partial^2 \Pi (Q, p) / \partial p^2 \) is \( \int_0^1 V'' dF_Y (y) \), where

\[
V'' = \int_0^1 \int_0^{G_1 (a_1 - p)/G_2 (a_2 - p)} \int_0^{G_1 (a_1 - p)/G_2 (a_2 - p)} \left( \sum_{i=1}^2 (-2x_i g_i (a_i - p) + px_i g_i' (a_i - p)) \right) dF_{X_2} (x_2) dF_{X_1} (x_1)
- \frac{p}{G_2 (a_2 - p)} \left( x_1 g_1 (a_1 - p) + \frac{yQ - x_1 G_1 (a_1 - p)}{G_2 (a_2 - p)} g_2 (a_2 - p) \right)^2.
\]

\[
f_{X_2} \left( \frac{yQ - x_1 G_1 (a_1 - p)}{G_2 (a_2 - p)} \right) dF_{X_1} (x_1).
\]

Therefore, if \( \frac{g' (a-p)}{g (a-p)} < 2 \) (which is true for a wide class of distributions including the Uniform, Exponential and certain specifications of the Weibull, Gamma and truncated-Normal families), then \( \Pi (Q, p) \) is concave in \( p \) for any given quantity \( Q \). Because both \( p \) and \( Q \) are bounded, the optimal solution can be efficiently computed.
A3.1 Comparison with RYP1 Model

We now investigate how the optimal recourse price in RYP2 compares to that in RYP1, i.e., how is the price influenced by the presence of a second class of customers. As the following corollary demonstrates, the optimal recourse price under RYP2 can be higher or lower than that under RYP1.

Corollary A4. Define \( p_1^* \) and \( p_2^* \) as the optimal recourse prices in the RYP1 and RYP2 models, respectively. In addition, define \( x \) as the realized market potential in RYP1. Let \( j \) denote the additional customer class in RYP2 and \( \bar{j} \) the original class. Let \( x_{\bar{j}} = x \), i.e., the original class has the same realized potential in RYP2. Let \( \nu_j \) be the unique solution to \( \nu = \frac{G_j(a_j - \nu)}{g_j(a_j - \nu)} \). Then, for any realized product quantities, \( p_1^* \leq p_2^* \) if (a) \( \nu_j > \nu_{\bar{j}} \), or (b) \( x > \frac{a_j}{a_{\bar{j}} - G_{\bar{j}}(a_{\bar{j}}/x)} g_{\bar{j}}(G_{\bar{j}}(a_{\bar{j}}/x)) \), or (c) \( \bar{p} = \nu_j \), where \( \bar{p} \) is defined in Theorem A3. Otherwise, \( p_1^* \geq p_2^* \) if \( p_2^* = \nu_1 \) and \( p_1^* \leq p_2^* \) if \( p_2^* = \nu_1 \), where \( \nu_1 \) and \( \nu_{\bar{j}} \) are defined in Theorem A3 (note \( \nu_1 \) and the optimality condition in Theorem A3 can be conversely defined if \( a_j > a_{\bar{j}} \)).

Corollary A4 tells us that the optimal price in RYP2 is at least as high as that in RYP1 if, in absence of the existing class of customers, it is optimal to induce the additional class of customers to pay a higher price; or if the realized market potential is relatively large. Otherwise, if it is optimal to serve both classes under RYP2, then the optimal price in NVRP2 can be less than that in RYP1. However, as expected, the optimal expected profit under RYP2 is always higher than or equals that under RYP1.

Theorem A4. Regardless of recourse pricing or advance pricing, the optimal expected profit under RYP2 is at least as high as that under RYP1.

Our numeric study\(^9\) shows that the expected profit for RYP2 (averaged over \( a_H = 0.3, 0.4, 0.5, \) and 0.6) is 8.62% and 10.47% higher than that for RYP1 for advance pricing and recourse pricing, respectively. In contrast to CPP1, whose value over RYP1 is dampened by recourse pricing, the value of RYP2 over RYP1 is strengthened by recourse pricing. In another words, the firm accrues more benefits by adopting recourse pricing if there are two class of customers, one possible reason being that there is more uncertainty in the two class case. As expected, the expected profit

\(^9\)Similar to that described in §5.4, except that uniform distributions were used for the outside utility, market potentials and yield, and the market correlation was fixed at -0.5. Details available upon request.
under RYP2 is increasing in $a_{2H}$, i.e., as product $H$ becomes more valuable to the second class of customers.

A4 Appendix - A Single-Class, Co-Production Model with Advance Pricing

In the case of advance pricing, the firm jointly sets the production quantity $Q$ and the price vector $p = (p_H, p_L)$ before yield and market uncertainties are resolved, but downconversion occurs after uncertainties are resolved. For a given price vector, market-size and yield realization, the optimal downconversion quantity is given by Theorem 1. Let $\pi(Q, p, y, x)$ denote the resulting revenue less the downconversion cost. We can then formulate the firm’s joint quantity-and-price setting problem as

$$\Pi_a(Q, p) = \max_{Q \geq 0, p \geq 0} \{-c_F Q + E_{Y, X} [\pi^*(Q, p, y, x)]\}, \quad (A-9)$$

where using Theorem 1, if $a_H - p_H \geq a_L - p_L$

$$\pi^*(Q, p, y, x) = p_H \min \{xG(a_H - p_H), yQ\}$$
$$+ p_L \min \left\{[xG(a_H - p_H) - yQ]^+ \frac{G(a_L - p_L)}{G(a_H - p_H)}, (1 - y) Q \right\},$$

and if $a_H - p_H < a_L - p_L$

$$\pi^*(Q, p, y, x) = p_L \min \{xG(a_L - p_L), (1 - y)Q\}$$
$$+ p_H \min \left\{[xG(a_L - p_L) - (1 - y)Q]^+ \frac{G(a_H - p_H)}{G(a_L - p_L)}, yQ \right\},$$

if $c_D \geq p_L - p_H \frac{G(a_H - p_H)}{G(a_L - p_L)}$,

$$\pi^*(Q, p, y, x) = -c_D q_D^* + p_L \min \{xG(a_L - p_L), (1 - y)Q + q_D^*\}$$
$$+ p_H \min \left\{[xG(a_L - p_L) - (1 - y)Q - q_D^*]^+ \frac{G(a_H - p_H)}{G(a_L - p_L)}, yQ - q_D^* \right\},$$

if $c_D < p_L - p_H \frac{G(a_H - p_H)}{G(a_L - p_L)}$. 

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where \( q^*_D = \min \left\{ z, \left( yQ - z \frac{G(a_H - p_H)}{G(a_L - p_L)} \right) / \left( 1 - \frac{G(a_H - p_H)}{G(a_L - p_L)} \right) \right\} \) and \( z = (xG(a_L - p_L) - (1 - y)Q^+) \).

Substituting above equations into (A-9), we then obtain, if \( a_H - p_H \geq a_L - p_L \),

\[
\Pi_a(Q, p) = -c_pQ + \int_0^1 \int_0^{\frac{\alpha Q}{G(a_H - p_H) + \frac{(1-y)Q}{G(a_L - p_L)}}} p_HG(a_H - p_H) \, xdF_X(x) \, dF_Y(y)
\]

\[+ \int_0^1 \int_0^{\infty} \left( p_HyQ + p_L(xG(a_H - p_H) - yQ) \frac{G(a_L - p_L)}{G(a_H - p_H)} \right) \, dF_X(x) \, dF_Y(y), \]

and if \( a_H - p_H < a_L - p_L \),

\[
\Pi_a(Q, p) = -c_pQ + \int_0^1 \int_0^{\frac{(1-y)Q}{G(a_L - p_L) + \frac{(1-y)Q}{G(a_H - p_H)}}} p_LG(a_L - p_L) \, xdF_X(x) \, dF_Y(y)
\]

\[+ \int_0^1 \int_0^{\infty} \left( p_L(1 - y)Q + p_H(xG(a_L - p_L) - (1 - y)Q) \frac{G(a_H - p_H)}{G(a_L - p_L)} \right) \, dF_X(x) \, dF_Y(y), \]

\[+ \int_0^1 \int_0^{\infty} (p_Hy + p_L(1 - y))QdF_X(x) \, dF_Y(y), \text{ if } c_D \geq p_L - p_H \frac{G(a_H - p_H)}{G(a_L - p_L)}, \]
\[
\Pi_a (Q, p) = -cpQ + \frac{1}{\alpha(a_L-p_L)} \int_0^Q \int_0^{\min(Q, pL)} pLG(a_L - p_L) x dF_X(x) dF_Y(y)
\]
\[
+ \frac{1}{\alpha(a_H-p_H)} \int_0^{\min(Q, pL)} \int_0^Q \left( pL \frac{xG(a_H - p_H)}{G(a_L - p_L) - G(a_H - p_H)} G(a_L - p_L) + pH \frac{xG(a_L - p_L) - Q}{G(a_L - p_L) - G(a_H - p_H)} G(a_H - p_H) \right) dF_X(x) dF_Y(y)
\]
\[
+ \frac{1}{\alpha(a_H-p_H)} \int_0^{\min(Q, pL)} \int_0^Q \left( pHy + pL (1 - y) \right) Q dF_X(x) dF_Y(y)
\]
\[
- cD \int_0^{\min(Q, pL)} \frac{1}{\alpha(a_H-p_H)} \int_0^{\min(Q, pL)} \left( xG(a_L - p_L) - (1 - y)Q \right) dF_X(x) dF_Y(y)
\]
\[
- cD \int_0^{\min(Q, pL)} \frac{1}{\alpha(a_H-p_H)} \int_0^{\min(Q, pL)} \frac{yQ - (xG(a_L - p_L) - (1 - y)Q) G(a_H - p_H)}{1 - G(a_H - p_H) G(a_L - p_L)} dF_X(x) dF_Y(y),
\]
\[
\text{if } cD < pL - pH \frac{G(a_H - p_H)}{G(a_L - p_L)}.
\]

For the recourse-pricing case, we were able to obtain implicit solutions for the optimal price vector (and closed form solutions in the case of a uniform utility distribution), and show that a first-order condition was sufficient for optimality for the production quantity \(Q\). Not surprisingly, closed form solutions to the optimal price vector and optimal production quantity do not exist in the advance-pricing case. The function is in general not jointly concave in \(Q\) and \(p\). However, the revenue function is concave in \(Q\) for any given price vector \(p\) and the price vector is bounded. Therefore, an optimal solution to the joint quantity-and-price problem can be found efficiently.
A5 Appendix - Two Customer Classes (CPP2) with Randomized Allocation Policy

In this appendix, we analyze the CPP2 model under a randomized allocation policy. For a price vector \( p = (p_H, p_L) \), realized quantities \( q = (q_H, q_L) \) and market-potential realizations \( x = (x_1, x_2) \), the firm’s revenue as a function of the downconversion quantity \( q_D \) is

\[
\begin{align*}
  r(q_D) &= p_H \min \left\{ \sum_{i=1}^{2} d_{iH}, (q_H - q_D) \right\} + p_L \min \left\{ \sum_{i=1}^{2} d_{iL}, (q_L - q_D) \right\} + \frac{\sum_{i=1}^{2} d_{iH}}{\sum_{i=1}^{2} d_{iH}} (q_L + q_D), \quad p \in \Gamma_1, \\
  r(q_D) &= p_H \min \left\{ d_{jH} + (d_{jL} - (q_L + q_D))^+ s_{jH}, (q_H - q_D) \right\} + p_L \min \left\{ d_{jL} + (d_{jH} - (q_H - q_D))^+ s_{jL}, (q_L + q_D) \right\}, \quad p \in \Gamma_2, \\
  r(q_D) &= p_H \min \left\{ d_{jH} + (d_{jL} - (q_L + q_D))^+ s_{jH}, (q_H - q_D) \right\} + p_L \min \left\{ d_{jL} + (d_{jH} - (q_H - q_D))^+ s_{jL}, (q_L + q_D) \right\}, \quad p \in \Gamma_3, \\
  r(q_D) &= p_H \min \left\{ \left( \sum_{i=1}^{2} d_{iL}, (q_L + q_D) \right) + \frac{\sum_{i=1}^{2} d_{iH}}{\sum_{i=1}^{2} d_{iL}} (q_H - q_D) \right\} + p_L \min \left\{ \sum_{i=1}^{2} d_{iL}, (q_L + q_D) \right\}, \quad p \in \Gamma_4,
\end{align*}
\]

where \( \Gamma_1, \Gamma_2, \Gamma_3, \) and \( \Gamma_4 \) partition the pricing space and are given by

\[
\begin{align*}
  \Gamma_1 : & \quad p_H - p_L \leq \min_i \{a_{iH} - a_{iL}\}, \\
  \Gamma_2 : & \quad a_{jH} - a_{jL} < p_H - p_L \leq a_{jH} - a_{jL}, \\
  \Gamma_3 : & \quad a_{jH} - a_{jL} < p_H - p_L \leq a_{jH} - a_{jL}, \\
  \Gamma_4 : & \quad p_H - p_L > \max_i \{a_{iH} - a_{iL}\}.
\end{align*}
\]

Note that \( \Gamma_2 \) and \( \Gamma_3 \) cannot exist simultaneously.

**Theorem A5.** For a price vector \( p = (p_H, p_L) \), realized quantities \( q = (q_H, q_L) \) and market-potential realizations \( x = (x_1, x_2) \), the optimal downconversion quantity \( q_D^* = 0 \) if (a) both class of customers prefer product \( H \) to \( L \); or (b) class \( j \) customers prefer product \( L \) and \( c_D \geq p_L - p_H s_{jH}; \) or (c) both class customers prefer product \( L \) and \( c_D \geq p_L - p_H \frac{\sum_{i=1}^{2} d_{iH}}{\sum_{i=1}^{2} d_{iL}}. \) Otherwise, the optimal
downconversion quantity is given by $q_D^* = \min\{z, \hat{q}_D\}$, where

\[
\hat{q}_D = \frac{(q_H - d_{jH}) - (d_{jL} - q_L)s_{jH}}{1 - s_{jH}}, \quad p \in \Gamma_2, \quad d_{jH} \geq q_L \cap (d_{jL} - q_L)s_{jH} < q_H - d_{jH},
\]

\[
\hat{q}_D = \frac{(q_H - d_{jH}) - (d_{jL} - q_L)s_{jH}}{1 - s_{jH}}, \quad p \in \Gamma_3, \quad d_{jL} \geq q_L \cap (d_{jL} - q_L)s_{jH} < q_H - d_{jH},
\]

\[
\hat{q}_D = \frac{q_H - \left(\sum_{i=1}^{2} d_{iL} - q_L\right) \left(\sum_{i=1}^{2} d_{iH} \sum_{i=1}^{2} d_{iL}\right)}{1 - \sum_{i=1}^{2} d_{iH} \sum_{i=1}^{2} d_{iL}} \quad p \in \Gamma_4, \quad \sum_{i=1}^{2} d_{iL} \geq q_L \cap \left(\sum_{i=1}^{2} d_{iL} - q_L\right) \left(\sum_{i=1}^{2} d_{iH} \sum_{i=1}^{2} d_{iL}\right) < q_H,
\]

\[
\hat{q}_D = 0, \quad \text{otherwise},
\]

and

\[
z = \max\{0, d_{jL} - q_L\}, \quad p \in \Gamma_2,
\]

\[
z = \max\{0, d_{jL} - q_L\}, \quad p \in \Gamma_3,
\]

\[
z = \max\left\{0, \sum_{i=1}^{2} d_{iL} - q_L\right\}, \quad p \in \Gamma_4.
\]

We conducted a numeric investigation\(^{10}\) to compare the expected profits under randomized and prioritized allocation. Our numeric investigation shows that, relative to recourse prioritization, randomized allocation results in a decrease in the expected profit of 0.58% on average under advance pricing, and 1.21% on average under recourse pricing. The maximum decrease in profit is even more significant: 8.76% under advance pricing and 7.21% under recourse pricing. This indicates that knowledge of customers’ identity can be of significant value to the firm.

\(^{10}\)Comprising 1920 problem instances. Details available upon request.