Managing the Customer Mix and Crowding for Shared Services via Pricing and Capacity Allocation

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We study the pricing and capacity allocation problem of a service provider who serves two distinct customer classes. Customers within each class are inherently heterogeneous in their willingness to pay for service, but their utilities are also affected by the presence of other customers in the system. Specifically, customer utilities depend on how many customers are in the system at the time of service as well as who these other customers are. If the service provider can price discriminate between customer classes, pricing out a class, i.e., operating an exclusive system, can sometimes be optimal and that depends only on classes’ perceptions about each other. If the provider must charge a single price, an exclusive system is even more likely. We extend our analysis to a service provider who can prevent class interaction by allocating separate capacity segments to different customer classes. Under price discrimination, allocating capacity is optimal if our measure of net appreciation between classes is negative. However, under a single-price policy, allocating capacity can be optimal even if this measure is positive. Depending on the nature of asymmetry between customer classes, capacity allocation under price discrimination might be more or less likely than under a single-price policy.

Key words: customer interaction, crowding, pricing, capacity management

1. Introduction

In many service systems, service is simultaneously delivered to many customers who share the same physical environment. For example, members of a gym share the same space and equipment with other members, passengers on a cruise ship share the common areas on the ship, and customers of a nightclub share the same physical space with all the other customers. In such systems, an individual customer’s perception of the service quality is highly influenced by the presence or the absence of others in the service environment and thus the “others” is an important component of each individual customer’s service experience.

There are mainly two different ways the others influence individual perceptions about service quality. One is simply through the crowding effect. In most systems, customers do not like overcrowded establishments, but in many cases they do not enjoy undercrowding either. For example, few people would enjoy being one of the fifteen passengers on a large cruise ship sailing to Alaska
and not many would like to cheer for their favorite rock band in a concert hall filled with only a handful of people. But, in many cases, a more important question than whether or not there are others or how many of them there are, is who exactly these other customers are. Thus, the customer mix effect is the second way the others influence perceptions about service quality in parallel to the simple crowding effect based on sheer numbers. For example, in some service establishments, genders of the others is an important factor. Some female gym members do not enjoy sharing the same facility with males. In nightclubs and bars, customers have strong preferences regarding who the other customers should be. One study found that whereas for males the number one reason for going to a bar is the opposite sex and the number of female customers is an important criterion when choosing a location, for female customers, a mix of male and female clientele is more popular than one that is predominantly male or female (Skinner et al. 2005, Kubacki et al. 2007). Other service settings where customer satisfaction is influenced by the others’ characteristics (such as age, economic class, intellectual capabilities, etc.) include social clubs, health clubs, schools (Buchanan 1965, Basu 1989, Sandler and Tschirhart 1997), beauty salons (Moore et al. 2005), amusement and recreation parks, adventure sports (Thakor et al. 2008), restaurants (Huang 2008), and professional conferences (Gruen et al. 2007).

The influence of the others’ characteristics in customer satisfaction has long been recognized in the marketing literature and in relevant industries such as tourism, with the management of customer–to–customer interactions (CCI) getting increasingly more attention (e.g., see Wu 2007 and Huang and Hsu 2010). For example, in the cruise industry, companies provide a variety of options to their customers for their cruise experience. In addition to cruises that cater to the general population, there are also theme cruises (e.g., church cruises, clothing–optional cruises, cruises for singles with or without age restrictions, Star Trek fans, former spies), which target a relatively homogeneous group of customers with the idea of gathering together individuals who are likely to enjoy each other’s company and thus creating a more enjoyable social atmosphere for the passengers (http://www.onboard-adventures.com/church-cruises.html, http://www.singlescruise.com, http://www.travelandleisure.com/articles/worlds-wackiest-theme-cruises).

For the types of service establishments where crowding and CCI effects are present, there are mainly two challenges, which are closely interconnected: the effective management/choice of the customer clientele and the effective management of the service capacity. The goal is not simply to get to the ideal crowding level with an appropriate pricing and/or capacity allocation strategy but to get there with the “right” customer mix. In some cases, firms restrict access to certain customer segments. This is what gyms and health clubs do when they choose to become women–only establishments or allocate certain times of the week for the exclusive use of women. Some firms design the service experience so as to appeal to a particular segment and let the customers self–select.
This is the idea behind cruise ships offering various themes and nightclubs catering to different types of clientele on different floors of the venue or at different nights of the week by choosing the music and decoration. When such direct capacity allocation options are not available or as a complementary tool, firms also use pricing as a means to manage their capacity and composition of their clientele, and maximize their profits. For example, nightclubs use various pricing promotions (e.g., “ladies’ nights”), which can help in attracting the “right” mix of customers.

The main objective of this paper is to investigate these pricing and capacity allocation decisions and provide insights into what kind of a strategy firms should adopt given various restrictions they might face regarding the policies they can use. Among others, we answer the following questions. If the firm can charge different prices to different customers, is it better to try to appeal to all potential customers or perhaps exclude certain segments so as to have a more homogeneous customer clientele? How do the perceptions across customer classes, the asymmetry in their willingness to pay, and the system capacity affect the optimal customer mix and the resulting crowding levels? How does the structure of the optimal policy change depending on whether or not the perceptions across customer classes are aligned with their respective willingness to pay? How does the firm’s optimal policy change if it cannot price discriminate? Finally, when it is practically feasible to do so, under what conditions would the firm prefer allocating different portions of its capacity for the exclusive use of certain classes over the policy of letting all customers share the whole capacity?

In order to provide insights into these questions, we investigate a service provider’s optimal pricing and capacity allocation decisions when faced with two classes of customers who have rational expectations about the equilibrium quantities when they choose whether or not to join the system. In Section 4 of the paper, we focus on the pricing decision alone. We consider two different settings, one in which the firm has the flexibility to charge different prices to different classes, and one in which the firm has to charge the same price to all customers. We find that if customer classes are symmetric in their inherent willingness to pay for service and the firm can price discriminate, whether or not it chooses to exclude a particular class from service depends only on classes’ perceptions about each other, i.e., only on customer mix effects, not on the system capacity or the crowding effects. However, if customer classes differ in their inherent willingness to pay, whether or not classes’ perceptions of each other call for an exclusive system depends on the system capacity. If the firm has to charge a single price and serve both customer classes, then it might choose not to operate at all. Interestingly, this happens when there is strong asymmetry in how the two classes feel about each other, but not necessarily when there is mutual dislike between them. We also find that an exclusive system is more likely to be the firm’s choice if she must use a single-price policy. This suggests that attempts to achieve price “fairness” might lead the service provider to deny service to a particular class when possible.
In Section 5 of the paper, we consider the capacity allocation decision together with pricing. More specifically, if the firm can allocate capacity to different customer classes, we identify the optimal allocation and pricing policy and investigate when separating different classes would be the firm’s preferred option. We find that if the firm can price discriminate, whether or not the firm chooses to allocate capacity depends purely on classes’ perceptions about each other, not on any potential willingness-to-pay asymmetry between classes. However, this choice is more complicated if the firm has to charge the same price to both classes. In most cases, we again find that a firm that cannot price discriminate is more likely to prefer capacity allocation compared to a firm that can charge different prices; however, this is not always true if customer classes are asymmetric in their inherent willingness to pay for service.

2. Literature Review

Our work touches upon three disciplines: economics, marketing, and operations. Although many articles from the economics and marketing literatures have investigated customer interaction effects and the operations literature is rich with articles that deal with pricing and capacity decisions under congestion effects, this paper is the first to incorporate CCI and investigate their influence on the firms’ pricing and capacity allocation decisions.

A significant portion of the relevant articles from the economics literature belong to a stream of work on “club theory,” which originated from the seminal papers by Tiebout (1956) and Buchanan (1965). (For an extensive review of this literature, see Cornes and Sandler 1996 and Sandler and Tschirhart 1997.) However, this literature typically investigates questions that are completely different from ours. Specifically, except for a few papers (Hearne 1988, Basu 1989), the traditional club theory has not focused on pricing and/or capacity allocation considerations of a profit-maximizing firm. Moreover, again except for a few papers (e.g., Basu 1989, Brueckner and Lee 1989, Scotchmer 1997), the club theory literature has typically assumed that customers are homogeneous and their utilities do not depend on the characteristics of the individuals they share the service facility with.

The four papers we cite above as exceptions (Hearne 1988, Basu 1989, Brueckner and Lee 1989, Scotchmer 1997) need further elaboration to clarify in what sense they are different from this paper. Hearne (1988) considers the profit-maximization problem of a monopolistic club and shows that a two-part tariff mechanism (membership and per usage fees) will lead to a Pareto-optimal solution. Apart from the focus, the paper is different from ours in that the customers are assumed to be homogeneous. Basu (1989) is generally interested in developing a theory of association. He considers a number of different formulations that are relevant in different application contexts but one in particular, which is motivated by schools, is worthwhile to mention in a little more detail. In his model, Basu considers four types of students in the population: clever and rich, clever and
poor, mediocre and rich, and mediocre and poor. Rich students are willing to pay more than poor students and (rich or poor) students’ willingness to pay depends on what fraction of the school population is clever. This work differs from ours in mainly two respects. First, even though Basu (1989) also considers the capacity of the establishment, (i.e., the school capacity), customer utilities do not depend on how crowded the system is. Second, Basu is purely interested in the question of whether the schools should be allowed to charge different prices to different types of students. Brueckner and Lee (1989) are also motivated by schools. They assume that there are two groups in the population and the utility of the customers from one of the groups depends on the proportion of customers in the club who belong to the other group. The paper characterizes the Pareto–efficient club configurations and carries out an equilibrium analysis for a competition model. Scotchmer (1997) considers a formulation in which the utility of each customer type depends on the number of customers from each type. She defines a new notion of approximate competitive equilibrium and shows that there exists such an equilibrium when the economy is sufficiently large. Note that neither Brueckner and Lee (1989) nor Scotchmer (1997) develop insights on the optimal pricing and capacity allocation decisions from an individual club’s perspective.

Outside the club theory literature, another stream of articles within the economics literature deal with systems whose customers experience positive network effects. (The term “network effect” is used to refer to the effect of the existence of the other users in the network on the utility of an individual user.) For example, see Oren and Smith (1981), and more recently Candogan et al. (2011). As Johari and Kumar (2009) indicate, these articles typically ignore congestion effects. They also ignore the possibility that network effects across different groups within the population could be different. To the best of our knowledge, the only exception to this is Katz and Spiegel (1996). However, even though this paper uses a demand formulation that has some similarities with ours, it ignores capacity considerations. There is also a large body of work that ignores positive network externalities but focuses on congestion effects. For examples of such work, we refer the reader to MacKie-Mason and Varian (1995), Wang and Schulzrinne (2006), and references therein.

In the operations literature, many articles have considered pricing and capacity decisions in the presence of congestion effects. Most of these articles use a queueing framework with delay–sensitive customers (e.g., Naor 1969, Mendelson 1985, Mendelson and Whang 1990); in some papers, queue lengths provide signals regarding the service quality (e.g., see Debo and Veeraraghavan 2009, Veeraraghavan and Debo 2009, 2011). This is unlike our formulation, which does not incorporate customer waiting in any explicit manner. For the service settings we are interested in, formulation of the system while the service is being delivered (specifically, how many customers are present during a single service period and who these other customers are)—as opposed to delays in access to service—is far more relevant. In relation to this, it might also be useful for the reader to note that
in our model, capacity refers to the maximum number of customers that can be accommodated in one service period. This is slightly different from the traditional queueing setup, where a change in service capacity is typically equivalent to a change in service speed.

One paper that is relatively closer to our work is Johari and Kumar (2009), which considers positive network effects in addition to congestion effects. Because the authors are mainly motivated by online services, the way these two effects are formulated is more general than our approach in that they do not only depend on the number of active users in the system but also on the load these users generate. However, unlike the case in our model, Johari and Kumar assume that all customers have the same utilities under the same set of conditions. Thus, their model ignores possible asymmetry in how customers from different segments feel about each other. Furthermore, their focus is completely different from ours. The authors are not interested in pricing and capacity allocation decisions for a profit–maximizing firm but rather focus on the comparison of the subscriber base, i.e., the optimal club size, under two different assumptions: one in which individuals act in their own interest and the other in which the service provider maximizes total welfare.

Finally, there are many articles in the marketing literature that investigate CCI in services (see Nicholls 2010 for an extensive review). A number of articles empirically study CCI in various service environments including nightclubs (Skinner et al. 2005, Kubacki et al. 2007), professional conferences (Gruen et al. 2007), adventure sports (Thakor et al. 2008), beauty salons (Moore et al. 2005), cruise ships (Huang and Hsu 2010), and organized tours (Wu 2007), and find that customers can have strong preferences for the others with whom they share their service experience. On the other hand, some articles discuss the importance of the management of CCI in the service industry in general and point to various strategies the providers might employ. Among these, Martin (1996) and Grove and Fisk (1997) are important to mention because these articles also discuss operational issues including the effective use of capacity, which we also address in this paper. In particular, Martin (1996) investigates customers’ perceptions of and reactions to the others’ behavior. He suggests capacity allocation either through physical separation or by designating different times of day for the use of different segments in order to improve the service experience of customers who might not enjoy each other’s company. This is a suggestion that is widely practiced and one we also investigate in this paper. On the other hand, Grove and Fisk (1997) are also interested in the effect of presence and behavior of others in customers’ service satisfaction. They find that it might not be ideal to operate systems at their maximum capacity and call for more research into identifying the optimal capacity for systems that serve many customers simultaneously. In this paper, we provide some insights into this question by investigating how the optimal crowding level depends on various model parameters.
3. Model

We consider a service system with capacity $K > 0$ that serves two distinct customer classes, each one with finite size $\Lambda > 0$. Class membership of a customer is observable to the service provider and all the other customers. Customer utilities of being in the service system consist of three different components and depend on $\lambda_1$ and $\lambda_2$, the number of customers in the system belonging to class 1 and 2, respectively. First, in the absence of other customers, the utilities of class–1 customers are uniformly distributed on the line segment $[0, 1]$. Likewise, (in the absence of other customers) the utilities of class–2 customers are uniformly distributed on the line segment $[a, 1+a]$, $a \geq 0$. Thus, on average, class–2 customers have the same or larger inherent willingness to pay for service than class–1 customers. Second, the customer mix might have an effect on the utility of an individual customer. This is because customers of a particular class might like or dislike sharing the same service environment with the other class. Third, customer utilities might be affected by the crowding level—the ratio of the total number of customers ($\lambda_1 + \lambda_2$) to the system’s capacity ($K$). For example, customers might dislike close–to–empty systems as well as very crowded systems. In this paper, we call a system full if its crowding level is equal to one; we call a system not full if its crowding level is strictly less than one. In mathematical terms, the gross utility ($U_i$) of customer $x$ in class $i$ is given by

$$U_i(x, \lambda_i, \lambda_j) = x + b_i \lambda_j + c[(\lambda_i + \lambda_j)/K], \quad i = 1, 2, \quad j = 3 - i,$$  

(1)

where $0 \leq x \leq 1$ if $i = 1$, $a \leq x \leq 1 + a$ if $i = 2$. The term $b_i \lambda_j$ in (1) captures the customer mix effect on class–$i$ customer utilities and parameter $b_i$ in particular should be interpreted in a relative manner: it represents how customers of class $i$ perceive class–$j$ customers relative to their own class. If $b_i > 0$ ($b_i < 0$), customers in class $i$ like customers in class $j$ more (less) than customers in their own class. We also define $b \equiv b_1 + b_2$, which will be useful when presenting our results, and can be seen as the net “appreciation” between the two customer classes. Continuously differentiable function $c : [0, 1] \rightarrow \mathbb{R}$ captures the crowding effect on customer utilities. We assume that $c''(\cdot) < 0$, thereby guaranteeing a uniquely optimal crowding level for an arbitrary customer. We impose no further restrictions on $c(\cdot)$; it can take positive or negative values, it can be a monotone or a unimodal function. Overall, our model allows for nonlinear CCI effects: in equation (1), the utility of a class–$i$ customer is, in general, a nonlinear concave function of the number of customers from class $i$ and a nonlinear concave function of the number of customers from class $j$.

Notice in equation (1) that the two customer classes are possibly different in two dimensions: their perceptions of each other and their inherent willingness to pay for service. Hereafter, by slightly abusing terminology, we refer to the case $a = 0$ as symmetric classes and to the case $a > 0$
as asymmetric classes\textsuperscript{1}. On the other hand, crowding effects are symmetric across classes. This simplification, which is partially necessitated to keep the model analytically tractable, helps us isolate the effect of classes’ perceptions of each other on the optimal decisions and thereby provide crisper insights on this relationship.

The service provider (she) charges price $p_i$ to a customer in class $i$ who joins the system; as a result, class-$i$ customer $x$ joins the system if and only if $U_i(x) - p_i \geq 0$. Because customer utilities depend on $\lambda_1$ and $\lambda_2$, which are equilibrium quantities, a potential customer must construct beliefs about the equilibrium values $\lambda_1$ and $\lambda_2$ when deciding whether or not to join the system. In turn, these beliefs must be confirmed in equilibrium, that is, customers should be able to correctly predict the equilibrium values $\lambda_1$ and $\lambda_2$. What we just described is the notion of a rational expectations equilibrium (REE), which we employ in this paper. We refer the reader to Sheffrin (1996) for a detailed treatment of rational expectations in Economics. In addition to perfect predictions about the equilibrium, a REE requires a) individual rationality, and b) consistency. In our setting, individually rational customer $x$ in class $i$ joins the system if and only if $U_i(x) \geq p_i$. Consistency implies that equilibrium values $(\lambda_1, \lambda_2)$ must be consistent with the customers’ decisions to join or not, i.e.,

$$\frac{\lambda_i}{\Lambda} = P(U_i(X_i) \geq p_i) \iff \frac{\lambda_i}{\Lambda} = 1 - P(U_i(X_i) \leq p_i),$$

where $X_1 \sim \text{Uniform}(0, 1)$ and $X_2 \sim \text{Uniform}(a, 1 + a)$. Using equation (1), we can rewrite the last condition as

$$\frac{\lambda_i}{\Lambda} = 1 - P(X_i \leq p_i - b_i \lambda_j - c[(\lambda_i + \lambda_j)/K]), \quad i = 1, 2, \quad j = 3 - i.$$  

Because $X_1 \sim \text{Uniform}(0, 1)$ and $X_2 \sim \text{Uniform}(a, 1 + a)$, equilibrium prices are as follows.

$$p_1(\lambda_1, \lambda_2) = 1 - \lambda_1/\Lambda + b_1 \lambda_2 + c[(\lambda_1 + \lambda_2)/K],$$  \hspace{1cm} (2)

$$p_2(\lambda_1, \lambda_2) = 1 + a - \lambda_2/\Lambda + b_2 \lambda_1 + c[(\lambda_1 + \lambda_2)/K].$$  \hspace{1cm} (3)

Finally, an individually rational provider maximizes revenue by solving the following problem.

$$\max_{\lambda_1, \lambda_2} R(\lambda_1, \lambda_2) = \sum_{i=1}^{2} \lambda_i p_i(\lambda_i, \lambda_{3-i})$$

subject to

$$\lambda_1 + \lambda_2 \leq K, \quad \lambda_1 \geq 0, \quad \lambda_2 \geq 0. \quad \text{(P1)}$$

Before studying problem (P1) in its generality, we briefly comment on the base case, in which classes are symmetric and customer mix effects do not exist or are simply ignored, i.e., if $a = 0$ and $b_1 = b_2 = 0$. In that case, it is easy to show that—as one would expect—the service provider 1) always prefers to have both classes in the system in order to sustain higher prices; 2) charges both

\textsuperscript{1} Classes are truly symmetric only if $a = 0$ and $b_1 = b_2$. 
classes the same price. Therefore, if classes are symmetric and the customer mix does not affect customer utilities, neither capacity allocation nor price discrimination are of any value to a service provider. As we demonstrate in this paper, asymmetry in the willingness to pay for service and/or customer mix effects make both price discrimination and capacity allocation important for service providers, and explain to a great extent what we observe in practice.

In Section 4 we study the provider’s revenue maximization problem when she makes the entire system capacity available to both customer classes. We study two settings: one in which the provider can price discriminate, and one in which the provider must charge both classes the same price. In Section 5 we study the provider’s pricing problem together with the capacity allocation problem. In that case, in addition to finding the optimal \((\lambda_1, \lambda_2)\), the service provider must determine the optimal capacity allocation between the two customer classes. We then derive conditions under which the provider prefers to keep classes separate through capacity allocation.

4. Optimal pricing without capacity allocation

In this section, we model a service provider who gives both customer classes access to the entire capacity if they pay the admission price charged to them. The provider might or might not be allowed to price discriminate between classes. We consider and compare these two scenarios (price discrimination and single-price policy), which allows us to study the effects of the adopted pricing policy on the customer mix and the crowding level.

4.1. Price discrimination

If customers can access the entire capacity and the provider can price discriminate, she must optimally choose \((\lambda_1, \lambda_2)\) in problem (P1). We first provide the following result, which establishes the conditions for uniqueness of the optimal solution to problem (P1) in the general case.

**Lemma 1.**

1. If \(a > 0\), there exists a unique optimal solution to (P1).
2. If \(a = 0\), a solution to (P1) at which \(\lambda_1\lambda_2 > 0\) is optimal only if \(\lambda_1 = \lambda_2\).
3. If \(a = 0\), a feasible solution to (P1) at which \(\lambda_1 = \lambda > 0\), \(\lambda_2 = 0\), is revenue-equivalent to a feasible solution to (P1) at which \(\lambda_1 = 0\), \(\lambda_2 = \lambda > 0\).

According to Lemma 1, the REE is always unique if classes are asymmetric. If classes are symmetric, the REE is unique if both classes are in the system. Otherwise, a REE with only class 1 present in the system is—in terms of revenue—indistinguishable from a REE with only class 2 in the system. The next proposition fully characterizes the REE, i.e., the optimal solution to (P1).

**Proposition 1.** *If customers from different classes are allowed to share the same space and the service provider can price discriminate, the optimal solution to the revenue maximization problem is as follows.*
Figure 1  System type under price discrimination without capacity allocation when $\Lambda = 100$, $a = 0$, $c(u) = -1 + 2u - u^2$.

1. If $b \leq a/K - 2/\Lambda$, then
   
   (a) If $K \leq \Lambda[1 + a + c(1) + c'(1)]/2$, then $\lambda_1^* = 0$, $\lambda_2^* = K$.
   
   (b) If $K > \Lambda[1 + a + c(1) + c'(1)]/2$, then $\lambda_1^* = 0$, $\lambda_2^* = \lambda^*$.

2. If $a/K - 2/\Lambda$ and $K \leq \Lambda[1 + a + c(1) + c'(1)]/2$, then $\lambda_1^* = K/2 - a/[2(b + 2/\Lambda)]$, $\lambda_2^* = K/2 + a/[2(b + 2/\Lambda)]$.

3. If $K > \Lambda[1 + a + c(1) + c'(1)]/2$ and $a/\lambda^* - 2/\Lambda > b > a/K - 2/\Lambda$, then $\lambda_1^* = 0$, $\lambda_2^* = \lambda^*$.

4. If $K > \Lambda[1 + a + c(1) + c'(1)]/2$ and $2/\Lambda - a/K - 2[1 + c(1) + c'(1)]/K > b \geq a/\lambda^* - 2/\Lambda$, then $\lambda_1^* = \lambda^* - a/(b + 2/\Lambda) > 0$, $\lambda_2^* = \lambda^* > 0$.

5. If $K > \Lambda[1 + a + c(1) + c'(1)]/2$ and $b \geq 2/\Lambda - a/K - 2[1 + c(1) + c'(1)]/K$, then $\lambda_1^* = K/2 - a/[2(b + 2/\Lambda)]$, $\lambda_2^* = K/2 + a/[2(b + 2/\Lambda)]$.

Note that $\lambda^*$ uniquely satisfies the equation $1 + a - 2\lambda^*/\Lambda + c(\lambda^*/K) + \lambda^*c'(\lambda^*/K)/K = 0$, and that $\lambda^*$ uniquely satisfies the equation

$$1 - 2[\lambda^* - a/(b + 2/\Lambda)]/\Lambda + b\lambda^* + c\{2\lambda^* - a/(b + 2/\Lambda)/K\}$$

$$+ [2\lambda^* - a/(b + 2/\Lambda)]c'\{2\lambda^* - a/(b + 2/\Lambda)/K\}/K = 0.$$ 

Proposition 1 completely characterizes the REE. Figure 1 illustrates the different system types that arise in equilibrium if classes are symmetric. Figure 2 does the same for the case of asymmetric classes. In both figures, we label a system as exclusive if only one class is in the system in equilibrium, otherwise we label a system as inclusive. In what follows, we first discuss the case of symmetric classes and then we highlight the main differences that arise if classes are asymmetric.

**Symmetric classes:** First, notice that if $a = 0$, both Figure 1 and Proposition 1 demonstrate that the net appreciation term $b$ fully determines whether the system is exclusive or inclusive.
Second, the size of capacity $K$ (relative to the class size $\Lambda$) determines, to a large extent, whether or not the system is full. Third, as the explicit bounds in Proposition 1 indicate, customer sensitivity to crowding (i.e., parameters $c(1)$ and $c'(1)$) determines how much capacity is sufficiently small to result in a full system. Thus, the structure of the REE nicely (albeit not perfectly) decomposes into three parts, which are regulated by the values of $b$, $K/\Lambda$, $c(1)$, and $c'(1)$.

If classes are symmetric, the critical value of $b$ that determines a system’s exclusivity is $-2/\Lambda$. If $b \leq -2/\Lambda$, customer mix effects are so damaging to customer utilities that the provider is better off leaving one class out regardless of the capacity. Although it is possible that one class likes the other (e.g., $b_1 > 0$), if the feelings of the other class are opposite and more intense (i.e., $b_2 \leq -b_1 - 2/\Lambda$), then an exclusive system arises. For example, in a cruise ship, couples with children would likely not be bothered by couples with no kids but couples with no kids might be unhappy about the presence of children running around. Similarly, some female customers of gyms and health clubs are not willing to share the same physical space with male customers. In either case, if this disutility is strong, no matter what the ship or gym capacity is, it is best to run an exclusive system. On the other hand, overall positive class interaction is not necessary for an inclusive system. Even if $b < 0$, it can still be optimal to run an inclusive system. As long as the net appreciation effect is not too negative, it is more profitable to admit customers from each class rather than from only one. This is due to customer heterogeneity within each class and its reflection on prices: if $b \geq -2/\Lambda$, the average admission price is higher if both classes are in the system.

Whereas customer mix effects determine a system’s exclusivity, their impact on the crowding level is somewhat weaker. As long as the net appreciation between classes is sufficiently negative, i.e., $b \leq -2/\Lambda$, there is no class interaction at all. In this case, the sole determinant of the resulting
crowding level is the system capacity. If the capacity is small, the service provider finds it profitable to operate a full system because she can find enough customers who would be willing to pay a “good” price. This does not happen if the capacity is large, because, in that case, running a full system would necessitate charging unjustifiably low prices. However, when there is class interaction (i.e., \( b > -2/\Lambda \)), even if the capacity is large, customers’ willingness to pay increases as the net appreciation gets stronger and this makes it more profitable for the provider to admit more customers and possibly run a full system. In short, for a system to be full in equilibrium, either its capacity must be sufficiently small or its customers must like each other (or not dislike each other too much). At nightclubs, even though women might not share men’s enthusiasm about sharing the same physical space, in many cases it would be unlikely for the net appreciation between men and women to be highly negative. Therefore, for such establishments, a full system is likely to be the service provider’s choice.

**Asymmetric classes**: Next we discuss the most important differences that arise if classes are asymmetric, which we can observe by comparing Figures 1 and 2. First, if class 2 is, on average, willing to pay more for service than class 1, the net appreciation between classes needs to be less negative or even positive (i.e., \( b > a/K - 2/\Lambda \)) to lead to an inclusive system. In other words, the bar for an inclusive system is now higher because the provider can find more customers in class 2 than in class 1 to pay a good price for service, and thus she is more inclined to run an exclusive system with only class–2 customers. Second, the capacity threshold below which the provider runs a full system is now larger. Because \( a > 0 \), at any capacity level, the utility of class–2 customers is now higher compared to the case \( a = 0 \). As a result, the provider is more likely to run a full system as \( a \) increases.

The third difference between Figures 1 and 2 is the most interesting one. If classes are asymmetric, Figure 2 illustrates that larger capacity leads to inclusive systems. This is unlike in the case with symmetric classes, where capacity does not have any effect on exclusivity. The intuition is as follows. If classes are symmetric, only the net appreciation between classes matters and neither class is more profitable than the other. As a result, it is optimal for the provider to either run an exclusive system (if class interaction does not help revenue) or admit the same number of customers from each class (see parts 2 and 3 of Lemma 1). In other words, at any capacity level, both classes are equally profitable; therefore, capacity does not result in more or less exclusivity if classes are symmetric. However, classes are not equally profitable if they are asymmetric. Specifically, class–2 customers whose inherent willingness to pay \( p > 1 \) do not have equally profitable counterparts in class 1. This leads to an exclusive system, which serves only class–2 customers (see part 1a of Proposition 1), but only up to a certain capacity level. Once that level is crossed and the “best”
class–2 customers are already in the system, the next unit of available capacity will go to the best
class–1 customer, i.e., the provider switches to an inclusive system.

We conclude our discussion in this section with a comparison of the prices that classes pay to be
in an inclusive system. Equations (2) and (3) imply that
\[ p^*_2 - p^*_1 = a + (b_2 + 1/\Lambda)\lambda^*_1 - (b_1 + 1/\Lambda)\lambda^*_2. \]
If \( a = 0 \) and \( \lambda^*_1\lambda^*_2 > 0 \), we know that \( \lambda^*_1 = \lambda^*_2 \); therefore,
\[ p^*_2 - p^*_1 = (b_2 - b_1)\lambda^*_1. \] In other words, as long
as classes are symmetric and the provider runs an inclusive system, the class who likes (dislikes)
the other the most (the least) pays a higher price for service. This explains why “ladies’ night” is
a common promotional event at nightclubs or why some colleges offer reduced tuition to students
of high caliber. “Ladies” are offered discounts to compensate for their relatively stronger disutility
(or weaker utility) of having “gentlemen” around.

With asymmetric classes \( (a > 0) \), the price comparison is not as straightforward. Because \( \lambda^*_1 < \lambda^*_2 \)
in this case, class–2 customers might end up paying less than class–1 customers if \( b_1 >> b_2 \). In the
absence of customer mix effects, class 2 is “wealthier” than class 1 and would pay a higher price
for service. However, if class–1 customers value the presence of class 2 much more than class–2
customers value them in response, the former will end up paying more than the latter although they
are not as wealthy on average. This result perfectly explains why famous and wealthy individuals
enjoy a free ride at certain social events. It is the strong desire of less wealthy and less famous
people to be around them that gives rise to this outcome.

4.2. Single price
In many service industries, it is possible to implement price discrimination through promotional
mechanisms that target certain customer segments. In some cases, however, the service provider
might be unable to identify individual customers’ class identities or unwilling to discriminate
among classes. For example, education level can be an important factor in predicting how any
two customers would get along in a social club but there could be no way for the club provider
to identify individual customers’ education level and charge prices accordingly. Similarly, it would
be unethical—if not illegal—to charge different prices depending on customers’ race, ethnicity,
religion, or sexual orientation.

In this section, we turn our attention to settings in which the provider must charge the same
price to both customer classes. Using equations (2) and (3), the constraint \( p_1 = p_2 \) implies
\[ (b_1 + 1/\Lambda)\lambda_2 = (b_2 + 1/\Lambda)\lambda_1 + a. \]
Because \( a \geq 0 \), a REE in which the provider charges a single price and \( \lambda^*_1\lambda^*_2 > 0 \) is possible only if
\( b_1 < -1/\Lambda \) and \( b_2 < -a/K - 1/\Lambda \), or if \( b_2 > -1/\Lambda \) and \( b_1 > a/K - 1/\Lambda \). Hence, without proof, the
following lemma.
Lemma 2. The service provider can charge a single price in a REE in which \( \lambda_1^* \lambda_2^* > 0 \) only if \( b_1 < -1/\Lambda \) and \( b_2 < -a/K - 1/\Lambda \), or if \( b_2 > -1/\Lambda \) and \( b_1 > a/K - 1/\Lambda \).

The necessary conditions of Lemma 2 essentially say that there is a limit to how differently the two customer classes can feel about each other and still allow a profitable single-price policy in a REE. If the asymmetry is strong, then \( \lambda_1^* \lambda_2^* = 0 \), i.e., there is no way to make profit by charging the same price to both classes. In that case, the service provider should not be in business unless she is willing to deny service to one of the customer classes. Interestingly, if the dislike between customer classes is mutual, this is not a reason for the provider not to be in business. In that case, there exist a price that makes the service establishment profitable no matter how customers from different classes feel about each other. Because of the heterogeneity within customer classes, there are always customers who are willing to pay the asking price and bear with the customers from the other segment. The fact that customer feelings are mutual keep the customer clientele in balance. However, if there is strong asymmetry, it is not possible to keep the same balance using a single price, because customer reactions to a given price are widely different depending on which class the customer is from.

Although Lemma 2 identifies conditions under which it is possible to run an inclusive system and make a profit while using a single-price policy, it does not mean it is necessarily optimal to do so. Even if these conditions hold, it might actually be better for the provider to run an exclusive system, e.g., establish a women-only gym and not worry about charging the same price to both men and women. For example, if parameter values are such that the conditions of Lemma 2 hold and the provider prefers an exclusive system under price discrimination, she will certainly prefer the same under a single-price policy as well. Because the provider will choose to run an exclusive system whenever the conditions of Lemma 2 are violated, we can conclude that the single-price clause leads to more exclusivity compared to the case of price discrimination.

Next, we solve problem (P2), which is (P1) with the addition of the single-price constraint (4).

\[
\begin{align*}
\max_{\lambda_1, \lambda_2} & \quad R(\lambda_1, \lambda_2) = \sum_{i=1}^{2} \lambda_i p_i(\lambda_i, \lambda_{3-i}) \\
\text{s.t.} & \quad \lambda_1 + \lambda_2 \leq K \\
& \quad (b_1 + 1/\Lambda)\lambda_2 = (b_2 + 1/\Lambda)\lambda_1 + a \\
& \quad \lambda_i \geq 0, \lambda_2 \geq 0.
\end{align*}
\]

(P2)

The following proposition characterizes the REE.

Proposition 2. If customers from both classes are allowed to share the same space and the service provider cannot price discriminate, the optimal solution to the revenue maximization problem is as follows.

1. If the conditions of Lemma 2 are violated or \( b \leq a/K - 2/\Lambda \), then
   
   (a) If \( K \leq \Lambda[1 + a + c(1) + c'(1)]/2 \), then \( \lambda_1^* = 0, \lambda_2^* = K \).
(b) If $K > \Lambda[1 + a + c(1) + c'(1)]/2$, then $\lambda_1^* = 0$, $\lambda_2^* = \lambda^*$.

2. If $b_2 > -1/\Lambda$, $b_1 > a/K - 1/\Lambda$ and $K \leq \Lambda[1 + a + c(1) + c'(1)]/2$, then
   (a) If $a < a^*$, then $\lambda_1^* = [\lambda_2^*(b_1 + 1/\Lambda) - a]/(b_2 + 1/\Lambda)$, $\lambda_2^* = \lambda_{SP}^*$.
   (b) If $a \geq a^*$, then $\lambda_1^* = (b_1 + 1/\Lambda)/(b + 2/\Lambda) - a/(b + 2/\Lambda)$, $\lambda_2^* = (b_2 + 1/\Lambda)/(b + 2/\Lambda) + a/(b + 2/\Lambda)$.

3. If $b_2 > -1/\Lambda$, $b_1 > a/K - 1/\Lambda$, $K > \Lambda[1 + a + c(1) + c'(1)]/2$ and $a/\lambda^* - 2/\Lambda > b > a/K - 2/\Lambda$, then $\lambda_1 = 0$, $\lambda_2 = \lambda^*$.

4. If $b_2 > -1/\Lambda$, $b_1 > a/K - 1/\Lambda$, $K > \Lambda[1 + a + c(1) + c'(1)]/2$, $b \geq a/\lambda^* - 2/\Lambda$ and $a < a^*$, then $\lambda_1^* = [\lambda_{SP}^*(b_1 + 1/\Lambda) - a]/(b_2 + 1/\Lambda)$, $\lambda_2^* = \lambda_{SP}^*$.

5. If $b_2 > -1/\Lambda$, $b_1 > a/K - 1/\Lambda$, $K > \Lambda[1 + a + c(1) + c'(1)]/2$, $b \geq a/\lambda^* - 2/\Lambda$ and $a \geq a^*$, then $\lambda_1^* = (b_1 + 1/\Lambda)/(b + 2/\Lambda) - a/(b + 2/\Lambda)$, $\lambda_2^* = (b_2 + 1/\Lambda)/(b + 2/\Lambda) + a/(b + 2/\Lambda)$.

Note that $a^* = -((b + 2/\Lambda)[1 + c(1) + c'(1)] + 2K(b_1 b_2 - 1/\Lambda^2))/[b_1 + 1/\Lambda]$, the implicit value of $\lambda^*$ is given in Proposition 1 and that $\lambda_{SP}^*$ uniquely satisfies the equation

\[
1 - 2[\lambda_{SP}^*(b_1 + 1/\Lambda) - a]/(b_2 + 1/\Lambda) + b\lambda_{SP}^* + c\{[(\lambda_{SP}^*(b_2 + 2/\Lambda) - a)]/(b_2 + 1/\Lambda)\}/K
\]

\[
+[(\lambda_{SP}^*(b_1 + 1/\Lambda) - a)/(b_2 + 1/\Lambda) + c^2\{[(\lambda_{SP}^*(b_2 + 2/\Lambda) - a)]/(b_2 + 1/\Lambda)\}/K
\]

\[-[\lambda_{SP}^*(b_2 - b_1) + a(b_1 + 1/\Lambda)]/(b + 2/\Lambda) = 0.
\]

Whereas all the statements of Proposition 1 depend only on the net appreciation term $b$, Proposition 2 depends on both $b$ ($\equiv b_1 + b_2$) and the individual values $b_1$, $b_2$. If the provider can price discriminate, the revenue coming from class–1 customers due to their interaction with class–2 customers is simply added to the revenue coming from class–2 customers due to the same interaction. Therefore, as long as terms $b_1$ and $b_2$ result in the same $b$, the provider is going to make the same money. However, when the provider charges a single price, revenue comes with conditions, more precisely, the single–price condition. As we explained in the discussion of Lemma 2, this condition critically depends on how different terms $b_1$ and $b_2$ are from each other. Thus, when the provider charges a single price, not only the net appreciation term, but also the individual terms $b_1$ and $b_2$ are important. In other words, the customer mix effects on revenue, which are symmetric across classes under price discrimination, become asymmetric under the single–price clause. This is evident in Proposition 2: even if classes are symmetric, the provider must admit more customers from the class that appreciates the other more (i.e., if $a = 0$, $\lambda_1^* \geq \lambda_2^*$ $\iff b_1 \geq b_2$) in order to comply with the single–price clause.

We conclude this section by summarizing the key insights when the provider’s only managerial lever is price. First, if the provider can price discriminate, the net appreciation between classes fully determines exclusivity. Second, if the provider cannot price discriminate, each class’s perception of
the other class—as opposed to only the net appreciation between them—determines exclusivity; large asymmetry in these perceptions is prohibitive for an inclusive system. Third, capacity has no effect on a system’s exclusivity unless classes are asymmetric, in which case more capacity leads to less exclusivity. Finally, having to charge the same price to both classes forces the provider toward more exclusivity. An important practical implication of this result is that regulations that attempt to achieve “price fairness” may force the service provider to exclude an entire class of customers from service when that is practically feasible. We will return to this finding in the following section, in which the provider can also establish exclusive systems through capacity allocation.

5. Optimal pricing and capacity allocation

In the previous section we saw that sometimes the service provider is better off running an exclusive system—both when she can and when she cannot price discriminate. This observation hints toward the possibility of further increasing revenue by allocating different capacity segments to different customer classes. In this section, we study this problem.

There are many service systems where capacity can be allocated to different segments of the population. The service provider might be able to divert customers to the “right” location depending on their class identities or she can design the service and the service environment for different segments in a way that will induce customers to self-select. A nightclub manager can allocate each floor of a building to a different class of customers. Nightclubs and bars also host theme nights on different days of the week so as to appeal to customers with particular tastes and interests. Theme cruises typically do not fill up the whole ship. There could be multiple theme cruises going on simultaneously or there could be theme cruise passengers plus other regular passengers on a single trip. Some gyms, health clubs, and public swimming pools allocate their capacity to different segments either through physical separation or by allocating their space to a particular segment depending on the time of the day.

We will first analyze the service provider’s problem while assuming that she exercises her option to allocate capacity and prevent class interaction. Then we will compare the resulting revenue with her revenue if capacity allocation is not possible (see section 4) to determine the conditions that make capacity allocation beneficial to the provider. If the service provider can allocate capacity, she needs to decide the capacity to allocate to each class in addition to determining the optimal \((\lambda_1, \lambda_2)\). In the subsequent analysis, we assume (without loss of generality) that \((1 - x)K\) is the fraction of capacity allocated to class–1 customers; \(xK\) is the fraction of capacity allocated to class–2 customers. As a result, equilibrium prices are modified as follows.

\[
p_1(\lambda_1) = 1 - \frac{\lambda_1}{\Lambda} + c\left(\frac{\lambda_1}{(1 - x)K}\right), \tag{5}
\]

\[
p_2(\lambda_2) = 1 - \frac{\lambda_2}{\Lambda} + a + c\left(\frac{\lambda_2}{xK}\right). \tag{6}
\]

As previously, we will distinguish between price discrimination and a single-price policy.
5.1. Capacity allocation with price discrimination

Given the choice of capacity allocation and price discrimination, the service provider is faced with
the following revenue maximization problem.

\[
\max_{\lambda_1, \lambda_2, x} \quad R(\lambda_1, \lambda_2, x) = \lambda_1 p_1(\lambda_1) + \lambda_2 p_2(\lambda_2)
\]

s.t.  
\[
0 \leq \lambda_1 \leq (1 - x)K \\
0 \leq \lambda_2 \leq xK
\]

(P3)

We first provide the following results, which establish the uniqueness of the optimal solution to
problem (P3), as well as some important properties of the optimal allocation \(x^*\).

**Lemma 3.** 1. There exists a unique optimal solution to \((P3)\).

2. In a REE with \(\lambda_1^* \lambda_2^* > 0\), crowding levels are the same in both capacity segments, i.e., \(\lambda_1^*/[(1 - x^*)K] = \lambda_2^*/(x^*K)\).

3. The optimal allocation \(x^*\) is \(\lambda_2^*/(\lambda_1^* + \lambda_2^*)\).

4. If \(a = 0\), \(x^* = 1/2\). In addition, the unique optimal allocation is increasing in \(a\).

Lemma 3 will turn out to be a key result for the remainder of our analysis. Part 1 of the lemma
guarantees a unique solution to the joint pricing and capacity allocation problem while part 2
states that the provider can maximize revenue only by maintaining the same crowding level in
both capacity segments. To understand why both classes should observe the same crowding level
in the REE, first recall that classes are symmetric in how they feel about crowding, and function \(c\),
which describes the relevant utility component, is (strictly) concave. The concavity of the crowding
function \(c\) is critical for the result: any crowding level that is a convex combination of two different
crowding levels yields higher gross utility to all customers and thus higher revenue to the provider
than the two different crowding levels do. This is the underlying fact behind part 2 of Lemma
3. Furthermore, crowding levels are inextricably linked to each other because the two capacity
segments share the same capacity. As a result, there is a unique capacity allocation that makes
crowding levels in the two segments equal to each other, which is stated as part 3 of Lemma 3.
Lastly, part 4 of Lemma 3 says that as class 2 becomes more willing to pay for service, the provider
naturally allocates more capacity to that class. If classes are symmetric, the capacity split is 50−50.

The next corollary exploits the results of Lemma 3, especially the optimal allocation and its
uniqueness, to establish a useful link between the optimal solution to problem (P1) and the optimal
solution to problem (P3).

**Corollary 1.** If \(b = 0\), the optimal revenue and customer mix are the same with or without
capacity allocation.

Corollary 1 essentially says that if the net appreciation term is zero, the ability to allocate
capacity does not change anything: the provider makes the same revenue with or without capacity
allocation, and the resulting customer mix is the same. To appreciate the equivalence between the two settings, recall that if the net appreciation term is zero, the customer mix does not affect revenue; in that case, only crowding effects matter. If the provider could increase revenue through capacity allocation, this could happen only if crowding levels in the two capacity segments differed, which, as we already explained, cannot be optimal. At the same time, the provider does not lose revenue by keeping classes separate in this case; note that the optimal allocation \( (x^* = \frac{\lambda_2^*}{(\lambda_1^* + \lambda_2^*)} ) \) results in the same crowding level that classes experience when they share the same space.

If the net appreciation term \( b \) is zero, letting classes share the same space results in the same revenue and customer mix as capacity allocation does. This result might leave one with the impression that prices in the two settings are the same. This is not true in general! Unless \( b_1 = b_2 = 0 \), a simple pairwise comparison of equations (2)–(3) and (5)–(6) reveals that the provider charges different prices when she allocates capacity and when she does not. For example, if \( b_1 > 0, b_2 < 0, b_1 + b_2 = 0 \), class–1 customers pay a lower price when the provider allocates capacity than when she does not. The opposite is true for class–2 customers. Of course, this price difference does not affect the net customer utility, which is the same in both settings, but it does have operational implications, e.g., changing posted prices when adopting a capacity allocation strategy.

We now proceed to compare the revenue when the provider allows different classes to share the entire capacity to the revenue when she allocates capacity to each class for its exclusive use. We know that the two strategies yield equivalent outcomes if \( b = 0 \) and we are interested in the general case. The next theorem provides sufficient conditions for the optimality of each strategy.

**Theorem 1.** If the service provider can allocate capacity and can price discriminate, the overall optimal solution to the revenue maximization problem is as follows:

1. If \( b \leq 0 \), it is optimal to allocate capacity and the optimal solution \((\lambda_1^*, \lambda_2^*)\) is given in Proposition 1 for \( b = 0 \).

2. If \( b \geq 0 \), it is optimal to let classes share the same space and the optimal solution \((\lambda_1^*, \lambda_2^*)\) is given in Proposition 1.

Theorem 1 confirms the reality of many service systems, in which providers allocate capacity to mitigate or eliminate negative interactions between different customer classes. The theorem prescribes a simple policy for the service provider: keep classes separate if their net appreciation \( b \) is not positive. An interesting observation in Theorem 1 is that any asymmetry in the classes’ inherent willingness to pay for service (i.e., the value of \( a \)) does not affect the sufficient conditions.

Although one might expect larger asymmetry to favor capacity allocation, one must realize that the existence of a class that is more willing to pay for service than the other is neither a reason to separate classes nor a reason to keep them together. The aim of capacity allocation is only to
prevent customer interactions that hurt revenue, which, of course, have nothing to do with the inherent willingness to pay for service. The provider does take into account any asymmetry in the willingness to pay for service by letting more class–2 customers in the optimal customer mix through pricing (when classes share the same space) or by allocating more capacity to them (when classes use different capacity segments).

Theorem 1 also has an important practical implication for the service provider, if she knows or can guess that the net appreciation between classes will not be negative. In that case, our last result means there is no reason to make any capacity allocation arrangements, because letting classes share the same space would maximize revenue. In general, if the provider can price discriminate, there is relatively little that she should know about the customer mix effects in order to decide in favor of or against capacity allocation. It is the ability to price discriminate that affords the provider a great deal of simplicity in choosing between allocating capacity and letting classes share the entire capacity. On the contrary, we will see that the same choice is significantly more complicated if price discrimination is not possible.

5.2. Capacity allocation with single price
In this section, we investigate the following question. If the service provider must charge the same price to both customer classes, is it more profitable to reserve capacity for each class or have the customers share the whole capacity? Using equations (5) and (6) and enforcing \( p_1 = p_2 \) yields the following condition.

\[
\lambda_2 / \Lambda - c(\lambda_2 / (xK)) = \lambda_1 / \Lambda - c(\lambda_1 / [(1 - x)K]) + a. \tag{7}
\]

As a result, the service provider is faced with the following revenue maximization problem.

\[
\begin{align*}
\max_{\lambda_1, \lambda_2, x} \quad & R(\lambda_1, \lambda_2, x) = \lambda_1 p_1(\lambda_1) + \lambda_2 p_2(\lambda_2) \\
\text{s.t.} \quad & 0 \leq \lambda_1 \leq (1 - x)K \\
& 0 \leq \lambda_2 \leq xK \\
& \lambda_2 / \Lambda - c[\lambda_2 / (xK)] = \lambda_1 / \Lambda - c(\lambda_1 / [(1 - x)K]) + a.
\end{align*} \tag{P4}
\]

Notice that the only difference between problems (P3) and (P4) is constraint (7). Moreover, recall that in the case of price discrimination, we showed that for any \((\lambda_1, \lambda_2)\) the unique optimal allocation is \(x^* = \lambda_2 / (\lambda_1 + \lambda_2)\). Therefore, the unique optimal allocation is the same for any solution \((\lambda_1, \lambda_2)\) that also satisfies constraint (7). This observation allows us to use the optimal solution to problem (P2) for \(b_1 = b_2 = 0\) to generate the optimal solution to problem (P4).

Before proceeding to compare the provider’s optimal revenue without capacity allocation to the revenue with capacity allocation, a note is in order. Unlike in the case of price discrimination, we will see that the single–price constraint makes asymmetry a a significant driver of the provider’s decision to allocate capacity. This is not necessarily obvious; in the case of price discrimination we argued
that the inherent willingness to pay for service has nothing to do with customer interactions and the provider’s decision to allocate capacity. Why does the single-price constraint change this insight? As equations (4) and (7) indicate, asymmetry $a$ is always part of the single-price constraint, but the constraint as a whole is quite different depending on whether the provider allocates capacity. More specifically, classes’ perceptions of each other are part of the constraint when classes share the same space, but these perceptions do not matter when classes use separate capacity segments. In other words, the single-price constraint involves different dynamics in each case; therefore, asymmetry in the inherent willingness to pay for service, which is part of these dynamics, plays a role in the provider’s decision between the two strategies. To facilitate intuition, we first state and discuss the results for symmetric classes ($a = 0$).

**Theorem 2.** If $a = 0$ and the service provider can allocate capacity but cannot price discriminate, the overall optimal solution to the revenue maximization problem is as follows:

1. If $b_1 \leq 0$, $b_2 \leq 0$, it is optimal to allocate capacity and the optimal solution $(\lambda_1^*, \lambda_2^*)$ is given in Proposition 2 for $b_1 = b_2 = 0$.

2. If $b_2 > 0 > b_1$,
   
   (a) If $b_1 \leq -1/\Lambda$, it is optimal to allocate capacity and the optimal solution $(\lambda_1^*, \lambda_2^*)$ is given in Proposition 2 for $b_1 = b_2 = 0$.

   (b) If $b_1 > -1/\Lambda$ and $b \leq b^*$, $b^* \geq 0$, it is optimal to allocate capacity and the optimal solution $(\lambda_1^*, \lambda_2^*)$ is given in Proposition 2 for $b_1 = b_2 = 0$.

   (c) If $b_1 > -1/\Lambda$ and $b \geq b^*$, $b^* \geq 0$, it is optimal to let classes share the same space and the optimal solution $(\lambda_1^*, \lambda_2^*)$ is given in Proposition 2.

3. If $b_1 > 0 > b_2$,
   
   (a) If $b_2 \leq -1/\Lambda$, it is optimal to allocate capacity and the optimal solution $(\lambda_1^*, \lambda_2^*)$ is given in Proposition 2 for $b_1 = b_2 = 0$.

   (b) If $b_2 > -1/\Lambda$ and $b \leq b^*$, $b^* \geq 0$, it is optimal to allocate capacity and the optimal solution $(\lambda_1^*, \lambda_2^*)$ is given in Proposition 2 for $b_1 = b_2 = 0$.

   (c) If $b_2 > -1/\Lambda$ and $b \geq b^*$, $b^* \geq 0$, it is optimal to let classes share the same space and the optimal solution $(\lambda_1^*, \lambda_2^*)$ is given in Proposition 2.

4. If $b_1 \geq 0$, $b_2 \geq 0$, it is optimal to let classes share the same space and the optimal solution $(\lambda_1^*, \lambda_2^*)$ is given in Proposition 2.

Note that at $b = b^*$, allocating capacity and letting classes share the same space yield the same revenue to the provider.

As Theorem 2 shows, the provider’s decision between allocating capacity and letting classes share the same space is more complicated if she cannot price discriminate. Nonetheless, notice that parts
1 and 4 of Theorem 2 are essentially the equivalent of Theorem 1. If classes dislike each other, it is better to keep them separate. On the contrary, it is more profitable to let classes share the same space if there is mutual appeal. Therefore, as long as class feelings are mutual, price discrimination and a single-price policy yield the same insights on the issue of capacity allocation.

The provider’s choice is less straightforward if class perceptions go in opposite directions, i.e., if either \( b_1 > 0 > b_2 \) or \( b_2 > 0 > b_1 \). Notice that these two cases are symmetric and parts 2 and 3 of Theorem 2 are essentially identical statements. Therefore, here we discuss only part 3. If \( b_1 > 0 > b_2 \), class–1 customers want to be around class–2 customers but not vice versa. If this asymmetry in perceptions is so large that the conditions of Lemma 2 are violated and the provider cannot allocate capacity, she is better off running an exclusive system. Naturally, allocating capacity is the best choice in that case, as part 3a of Theorem 2 indicates. Moreover, notice that if \( b_2 \leq -1/\Lambda \), capacity allocation is optimal even if \( b > 0 \). In other words, if classes feel sufficiently different about each other, their net appreciation—no matter how strong it is—does not have any effect on the capacity allocation decision. This result means that a single-price policy leads to capacity allocation in more cases than price discrimination does and further supports the point we made in section 4.2: a single-price policy leads to more exclusivity than price discrimination does.

If, on the other hand, class 2 does not feel that negatively about sharing the same space with class 1 (i.e., if \( b_2 > -1/\Lambda \)), the provider should let them share the same space if the net appreciation term is sufficiently positive (part 3c of Theorem 2). We see that under a single-price policy, it might be optimal for the provider to allocate capacity even if the net appreciation term is positive (part 3b of Theorem 2). In general, if classes are symmetric and the provider has to decide whether to allocate capacity, the two substantial differences between a price–discriminating and a single–price policy is that a) a single–price policy requires that the provider know or guess how each class feels about the other rather than just the sign of their net appreciation; b) a single–price policy leads to capacity allocation in more cases than price discrimination does. We will see next to what extent these insights hold if classes are asymmetric.

**Theorem 3.** If \( a > 0 \) and the service provider can allocate capacity but cannot price discriminate, the overall optimal solution to the revenue maximization problem is as follows:

1. If \( b_1 \leq 0, b_2 \leq 0 \), it is optimal to allocate capacity and the optimal solution \((\lambda_1^*, \lambda_2^*)\) is given in Proposition 2 for \( b_1 = b_2 = 0 \).

2. If \( b_2 > 0 > b_1 \),
   
   (a) If \( b_1 \leq a/K - 1/\Lambda \), it is optimal to allocate capacity and the optimal solution \((\lambda_1^*, \lambda_2^*)\) is given in Proposition 2 for \( b_1 = b_2 = 0 \).

   (b) If \( b_1 > a/K - 1/\Lambda \) and \( b \leq b^*(a), b^*(a) \geq 0 \), it is optimal to allocate capacity and the optimal solution \((\lambda_1^*, \lambda_2^*)\) is given in Proposition 2 for \( b_1 = b_2 = 0 \).
(c) If \( b_1 > a/K - 1/\Lambda \) and \( b = b^*(a) \), \( b^*(a) > 0 \), it is optimal to let classes share the same space and the optimal solution \( (\lambda_1^*, \lambda_2^*) \) is given in Proposition 2.

3. If \( b_1 > 0 > b_2 \).

   (a) If \( b_2 \leq -1/\Lambda \), or \( b_1 \leq a/K - 1/\Lambda \) and \( b_2 > -1/\Lambda \), it is optimal to allocate capacity and the optimal solution \( (\lambda_1^*, \lambda_2^*) \) is given in Proposition 2 for \( b_1 = b_2 = 0 \).

   (b) If \( b_1 > a/K - 1/\Lambda \), \( b_2 > -1/\Lambda \), and \( \lambda_1^* = 0 \) in Proposition 2, it is optimal to allocate capacity and the optimal solution \( (\lambda_1^*, \lambda_2^*) \) is given in Proposition 2 for \( b_1 = b_2 = 0 \).

   (c) If \( b_1 > a/K - 1/\Lambda \), \( b_2 > -1/\Lambda \), \( b_1 - b_2 \leq \Delta b^*(a) \), \( \Delta b^*(a) > 0 \), \( b \geq b^{**}(a) \), \( b^{**}(a) < 0 \), and \( \lambda_1^* > 0 \) in Proposition 2, it is optimal to let classes share the same space and the optimal solution \( (\lambda_1^*, \lambda_2^*) \) is given in Proposition 2.

4. If \( b_1 \geq 0 \), \( b_2 \geq 0 \), it is optimal to let classes share the same space and the optimal solution \( (\lambda_1^*, \lambda_2^*) \) is given in Proposition 2.

Note that at \( b = b^*(a) \), allocating capacity and letting classes share the same space yield the same revenue to the provider. At \( b_1 - b_2 = \Delta b^*(a) \) and \( b = b^{**}(a) \), letting classes share the same space yields strictly higher revenue than allocating capacity.

A swift comparison of Theorems 2 and 3 reveals that parts 1 and 4 are exactly the same in both theorems. Thus, if class feelings are mutual, neither the pricing policy nor the asymmetry in the inherent willingness to pay for service have an impact on the capacity allocation decision. If \( b_2 > 0 > b_1 \), on the other hand, the sufficient conditions of part 2 of Theorem 3 are qualitatively the same as the conditions of part 2 of Theorem 2 but quantitatively differ because they account for class asymmetry \( a \). Nevertheless, part 2 of Theorem 3 still confirms that the single–price constraint results in capacity allocation in more cases than price discrimination does.

Part 3 of Theorem 3 considers the case \( b_1 > 0 > b_2 \), which is not symmetric to the case \( b_2 > 0 > b_1 \) if \( a > 0 \). Of course, if classes’ feelings toward each other are so different that the provider’s best choice without capacity allocation is an exclusive system, then she is better off allocating capacity (parts 3a and 3b of Theorem 3). Now what if the provider optimally runs an inclusive system while charging a single price? This can certainly happen if \( b_1 = b_2 = 0 \), as we established in Proposition 2; in that case, letting classes share the same space and allocating capacity yield the same revenue. Suppose now that \( b_1 \) increases by a small amount while \( b_2 \) decreases by the same amount so that \( b_1 > 0 > b_2 \), \( b = 0 \), and the system without capacity allocation remains inclusive. These changes do not affect the revenue under capacity allocation (because different classes do not interact) but they change the revenue of an inclusive system. How do they change the revenue of an inclusive system if classes are ex ante asymmetric? There are two effects in play here. First, if classes are ex ante symmetric \( (a = 0) \), any asymmetry in customer mix effects hurts revenue due to the single–price
constraint. Second, ex ante asymmetry \( (a > 0) \)—as opposed to ex ante symmetry—helps revenue because it means that class–2 customers, whom class–1 customers like, are willing to pay more for service. The fine point is that these two effects are intertwined, as we explain next.

If classes are ex ante asymmetric and \( b_1 = b_2 = 0 \), class 2 would pay more for service than class 1 if the provider could price discriminate. However, the single–price constraint requires that class–1 customers pay more (and that class–2 customers pay less) than what they would under price discrimination, thereby resulting in inefficient pricing. Suppose now that \( b = 0 \) but \( b_1 > 0 > b_2 \). In that case (and all else being equal), class–1 customers are willing to pay more than class–2 customers in order to be around the customers of the other type; in other words, the effect of (small) asymmetry in class feelings is in line with the single–price mandate. Therefore, this particular asymmetry in class feelings helps mitigate the pricing imbalance that the single–price mandate creates without changing the net appreciation between classes. What does this mean for the revenue of an inclusive system? Compared with the case \( b_1 = b_2 = 0, b = 0 \), increasing difference \( b_1 - b_2 \) by a small amount while maintaining \( b = 0 \) strictly increases revenue. Of course, if the difference \( b_1 - b_2 \) is large, it does not facilitate the implementation of a single–price policy. On the contrary, it becomes the primary difficulty in implementing a single–price policy and eventually forces the provider toward an exclusive system.

If \( b_1 > 0 > b_2 \) and \( \lambda_1^* > 0 \), it is not possible to provide collectively exhaustive sufficient conditions that guarantee the (sub)optimality of capacity allocation. Nonetheless, part 3c of Theorem 3 states conditions that guarantee the optimality of an inclusive system, and these conditions are particularly interesting. As we just explained, a small asymmetry in classes’ feelings about each other increases the revenue of an inclusive system as long as the net appreciation term remains at zero. Because it is strictly better to run an inclusive system (than allocate capacity) if \( b_1 - b_2 \) is sufficiently small and \( b = 0 \), the provider would also be better off doing so for small yet negative values of \( b \). This means that in some cases, a single–price policy makes capacity allocation less likely than price discrimination does! This might appear to be the opposite of one of the insights we have obtained so far, i.e., that single–price policies lead to more exclusivity in general. An explanation is in order. If the provider’s choice is only between an inclusive and an exclusive system, a single–price mandate always leads to more exclusivity because that very mandate disappears in an exclusive system. On the other hand, capacity allocation leads to a different kind of exclusivity: if the provider can allocate capacity, choosing to do so does not always make the single–price constraint disappear. She still has to charge a single price if capacity allocation results in two exclusive systems. In that case, there might be some benefit from keeping ex ante asymmetric classes together and mitigating the pricing inefficiency that a single price causes, even if these classes feel differently about being around each other and their net appreciation is negative.
6. Conclusions

This paper deals with a particular type of service setting, where service takes an extended period of time and is shared by others so that what happens during service or more specifically who else is there during service is a very important determinant of customers’ utility. Despite the prevalence of such services in practice, they have received almost no attention in the operations literature. One of the important contributions of this paper is the development of a framework that can be helpful in building new models to investigate various research questions (e.g., effects of competition) regarding shared service systems.

We used the framework we developed to provide insights into the use of pricing and capacity allocation as leverages to control the customer mix and crowding. Some of our findings conform to what we observe in practice and our intuition as to what a profit–maximizing service provider should be doing (for example, the use of discounts if there is asymmetry between how different classes feel about each other), whereas others are either counter-intuitive or help us gain a deeper understanding of some of the issues for which intuition does not provide much help. For example, we find that if the service provider is restricted to charge the same price to both customer classes, she would choose to be in business even if the two customer classes do not like each other but interestingly, this is not the case if this dislike is not mutual, i.e., if one class likes the other but not vice versa, or if classes are highly asymmetric in their inherent willingness to pay for service. When faced with such asymmetric customer population, the only way for the service provider to survive is by restricting access to a particular class of customers or by allocating different portions of its capacity for the exclusive use of different customer classes. Thus, strong asymmetry requires some sort of discrimination or capacity allocation for the survival of the firm.

For a service provider who can use price discrimination, the choice between allocating capacity for the exclusive use of different classes and making the whole capacity available to all its customers depends purely on the net appreciation between classes, not on crowding effects, capacity, or the degree of asymmetry in the two customer classes’ willingness to pay—if there is any. Specifically, capacity allocation is desirable if the net appreciation is negative. If price discrimination is not an option, however, capacity allocation could be desirable even if the net appreciation is positive. Thus, in many cases, disallowing price discrimination makes it more likely for firms to choose capacity allocation to different customer classes for their exclusive use. It is, however, possible that restriction to a single price might lead the provider to switch and embrace an inclusive policy with no capacity allocation. This can only happen if the class with the lower willingness to pay likes the other class but not vice versa.

Our results highlight the importance of having a deeper understanding of customer mix effects on the utilities of different customer classes because they are highly relevant in choosing the pricing
and capacity allocation policies to be employed. Many articles in the marketing literature have established the presence and importance of these effects but we are not aware of any work that has aimed to quantify them. In many service systems, it might be possible to have a good idea about or at least quickly determine the sign of the effect (positive or negative) and which effect would dominate the other but clearly this is not always the case. Thus, one avenue for future research is to develop a framework that can be utilized in measuring customer mix effects empirically in different service settings.

Appendix. Proofs

Proof of Lemma 1 For part 1, note that because \( a > 0 \), \( R(\lambda, 0) < R(0, \lambda) \) if \( \lambda > 0 \). Thus, it remains to show that an optimal solution at which \( \lambda_1^* \lambda_2^* > 0 \) or an optimal solution at which \( \lambda_1^* = 0, \lambda_2^* > 0 \), are uniquely optimal. To that end, it suffices to show that the first–order conditions yield a unique solution. Here we show the proof for the case \( \lambda_1^* \lambda_2^* > 0 \). The proof for the case \( \lambda_1^* = 0, \lambda_2^* > 0 \), is identical in spirit and thus omitted. Note that \( R(\lambda_1, \lambda_2) = \lambda_1 \{1 - \lambda_1 / \Lambda + b \lambda_2 + c(\lambda_1 + \lambda_2) / K\} + \lambda_2 \{1 - \lambda_2 / \Lambda + b_2 \lambda_1 + c(\lambda_1 + \lambda_2) / K\} + a \}, \) and thus

\[
\frac{\partial R(\lambda_1, \lambda_2)}{\partial \lambda_1} = 0 \Leftrightarrow 1 - 2 \lambda_1 / \Lambda + b \lambda_2 + c(\lambda_1 + \lambda_2) / K + (\lambda_1 + \lambda_2) c'(\lambda_1 + \lambda_2) / K = 0, \tag{8}
\]

\[
\frac{\partial R(\lambda_1, \lambda_2)}{\partial \lambda_2} = 0 \Leftrightarrow 1 + a - 2 \lambda_2 / \Lambda + b_1 \lambda_1 + c(\lambda_1 + \lambda_2) / K + (\lambda_1 + \lambda_2) c'(\lambda_1 + \lambda_2) / K = 0. \tag{9}
\]

Subtracting the equations above from each other yields \( (\lambda_2 - \lambda_1)(b + 2 / \Lambda) = a \). If \( b = -2 / \Lambda \), then \( \lambda_1 \lambda_2 > 0 \) cannot be optimal. Otherwise, an optimal solution at which \( \lambda_1 \lambda_2 > 0 \) must satisfy

\[
\lambda_2 = \lambda_1 + a / (b + 2 / \Lambda). \tag{10}
\]

Adding up equations (8) and (9) yields

\[
F(\lambda_1 + \lambda_2) / K = a + (b - 2 / \Lambda)(\lambda_1 + \lambda_2) + 2c(\lambda_1 + \lambda_2) / K + 2(\lambda_1 + \lambda_2) c'(\lambda_1 + \lambda_2) / K = 0. \tag{11}
\]

Let \( (\lambda_1 + \lambda_2) / K \equiv u \). We show next that \( F(u) \) cannot have two roots in \( [0, 1] \). Note that \( \partial F(u) / \partial u = (b - 2 / \Lambda) K + 4c'(u) + 2uc''(u) \). For \( F(u) = 0 \) to have two roots in \( [0, 1] \), \( \partial F(u) / \partial u \) must change sign twice in \( [0, 1] \). Because \( c''(u) < 0 \) and thus \( uc''(u) < 0 \), that would be possible only if \( c'(u) \uparrow u \) in some interval within \( [0, 1] \). However, \( c'(u) \downarrow u \) because \( c''(u) < 0 \). Thus, \( F(u) = 0 \) can have at most one root in \( [0, 1] \). Combining this result with (10), there exists unique optimal solution to (P1) such that \( \lambda_1^* \lambda_2^* > 0 \).

Part 2 follows directly from (10) by letting \( a = 0 \).

Part 3 follows directly from the fact that \( R(\lambda_1, \lambda_2) = R(\lambda_2, \lambda_1) \) if \( a = 0 \).

Proof of Proposition 1 Consider problem (P1) and let \( \{\mu_i \geq 0 : i = 1, 2, 3, (\mu_1, \mu_2, \mu_3) \neq (0, 0, 0)\} \) be KKT multipliers for constraints \(- (\lambda_1 + \lambda_2) + K \geq 0, \lambda_1 \geq 0, \lambda_2 \geq 0 \), respectively. A candidate optimal solution must satisfy \( \mu_1 (-\lambda_1 - \lambda_2 + K) = 0, \mu_2 \lambda_1 = 0, \mu_3 \lambda_2 = 0 \), and the following stationarity conditions.

\[
1 - 2 \lambda_1 / \Lambda + b \lambda_2 + c(\lambda_1 + \lambda_2) / K + (\lambda_1 + \lambda_2) c'(\lambda_1 + \lambda_2) / K - \mu_1 + \mu_2 = 0, \tag{12}
\]

\[
1 + a - 2 \lambda_2 / \Lambda + b \lambda_1 + c(\lambda_1 + \lambda_2) / K + (\lambda_1 + \lambda_2) c'(\lambda_1 + \lambda_2) / K - \mu_1 + \mu_3 = 0. \tag{13}
\]
An exhaustive analysis of all the possible values of \((\mu_1, \mu_2, \mu_3)\) yields the following candidate optimal solutions. Note that because \(a \geq 0\), in the ensuing analysis we do not consider the solutions \((K, 0)\) and \((\lambda^*, 0)\), as they are clearly dominated by solutions \((0, K)\) and \((0, \lambda^*)\), respectively.

1. \((0, K)\) is a candidate optimal solution if \(K \leq \Lambda[1 + a + c(1) + c'(1)]/2\) and either \(b \leq -2/\Lambda\), or \(b > -2/\Lambda\) and \(b \leq a/K - 2/\Lambda\). It yields revenue \(R(0, K) = K[1 + a - K/\Lambda + c(1)]\).

2. \((K/2 - a/[2(b + 2/\Lambda)], K/2 + a/[2(b + 2/\Lambda)])\) is a candidate optimal solution if \(b > -2/\Lambda - a/K - 2[1 + c(1) + c'(1)]/K\) and either \(b \leq -a/K - 2/\Lambda\) or \(b \geq a/K - 2/\Lambda\). It yields revenue \(R(K/2 - a/[2(b + 2/\Lambda)], K/2 + a/[2(b + 2/\Lambda)]) = K[1 + c(1) + K(b - 2/\Lambda)/4 + a(K + a/[2(b + 2/\Lambda)])]/2\).

3. \((0, \Lambda^*)\) is a candidate optimal solution if \(b \leq -2/\Lambda\) and \(K > \Lambda[1 + a + c(1) + c'(1)]/2\), or if \(b \geq -2/\Lambda\), \(K > \Lambda[1 + a + c(1) + c'(1)]/2\) and \(b < a/\Lambda^* - 2/\Lambda\). It yields revenue \(R(0, \Lambda^*) = \Lambda^*[1 - \Lambda^*/\Lambda + (\Lambda^*/K) + a]\). In addition, \(\Lambda^*\) uniquely satisfies the equation \(1 + a - 2\Lambda^*/\Lambda + c(\Lambda^*/K) + \Lambda^* c'(\Lambda^*/K)/K = 0\).

We next determine the optimal solution in each of the following cases.

Case \(b \leq -a/K - 2/\Lambda\) and \(K \leq \Lambda[1 + a + c(1) + c'(1)]/2\): Note that \(R(0, K) = R(K/2 - a/[2(b + 2/\Lambda)], K/2 + a/[2(b + 2/\Lambda)]) = -(b\Lambda + 2)K^2/4(\Lambda) + a[K/2 - a/[2(b + 2/\Lambda)] \geq 0\), thus \((0, K)\) is optimal.

Case \(b \leq -a/K - 2/\Lambda\) and \(K > \Lambda[1 + a + c(1) + c'(1)]/2\): Note that—although \((0, K)\) is not a candidate optimal solution in this case—it remains true that \(R(K/2 - a/[2(b + 2/\Lambda)], K/2 + a/[2(b + 2/\Lambda)]) \leq R(0, K)\), thus \((K/2 - a/[2(b + 2/\Lambda)], K/2 + a/[2(b + 2/\Lambda)])\) cannot be optimal. Instead, \((0, \Lambda^*)\) is optimal.

Case \(-2/\Lambda \geq b > -a/K - 2/\Lambda\) and \(K \leq \Lambda[1 + a + c(1) + c'(1)]/2\): \((0, K)\) is the only candidate optimal solution from the above three, thus it is optimal.

Case \(-2/\Lambda \geq b > -a/K - 2/\Lambda\) and \(K > \Lambda[1 + a + c(1) + c'(1)]/2\): \((0, \Lambda^*)\) is the only candidate optimal solution from the above three, thus it is optimal.

Case \(a/K - 2/\Lambda \geq b > -2/\Lambda\) and \(K \leq \Lambda[1 + a + c(1) + c'(1)]/2\): \((0, K)\) is the only candidate optimal solution from the above three, thus it is optimal.

Case \(a/K - 2/\Lambda \geq b > -2/\Lambda\) and \(K > \Lambda[1 + a + c(1) + c'(1)]/2\): \((0, \Lambda^*)\) is the only candidate optimal solution from the above three, thus it is optimal.

Case \(b > a/K - 2/\Lambda\) and \(K \leq \Lambda[1 + a + c(1) + c'(1)]/2\): Note that, in this case, \(b \geq 2/\Lambda - a/K - 2[1 + c(1) + c'(1)]/K\). \((K/2 - a/[2(b + 2/\Lambda)], K/2 + a/[2(b + 2/\Lambda)])\) is the only candidate optimal solution from the above three, thus it is optimal.

Case \(K > \Lambda[1 + a + c(1) + c'(1)]/2\) and \(a/\Lambda^* - 2/\Lambda > b > a/K - 2/\Lambda\): \((0, \Lambda^*)\) is the only candidate optimal solution, thus it is optimal.

Case \(K > \Lambda[1 + a + c(1) + c'(1)]/2\) and \(2/\Lambda - a/K - 2[1 + c(1) + c'(1)]/K > b > a/\Lambda^* - 2/\Lambda\): None of the three solutions above are possible optimal solutions, thus the optimal solution is the unique solution to (12) and (13) when \(\mu_1 = \mu_2 = \mu_3 = 0\). In that case, \(\lambda_1^* = \lambda_2^* - a/(b + 2/\Lambda)\), as we showed in the proof of Lemma 1. Using the last equation and (12) implies that \(\lambda_2^* = \lambda^*\) must uniquely satisfy the equation

\[
1 - 2[\lambda^* - a/(b + 2/\Lambda)]/\Lambda + b\lambda^* + c[2\lambda^* - a/(b + 2/\Lambda)]/K \\
+ 2[\lambda^* - a/(b + 2/\Lambda)]c[2\lambda^* - a/(b + 2/\Lambda)]/K = 0.
\]

Case \(K > \Lambda[1 + a + c(1) + c'(1)]/2\) and \(b \geq 2/\Lambda - a/K - 2[1 + c(1) + c'(1)]/K\): \((K/2 - a/[2(b + 2/\Lambda)], K/2 + a/[2(b + 2/\Lambda)])\) is the only candidate optimal solution from the above three, thus it is optimal. □
Proof of Proposition 2  
Part 1 follows from Lemma 2 and part 1 of Proposition 1. Part 3 follows from part 3 of Proposition 1. Next we show the remaining parts. Let \( g \in \mathbb{R} \) be the Lagrange multiplier for constraint

\[-(b_2 + 1/\lambda)\lambda_1 + (b_1 + 1/\lambda)\lambda_2 - a = 0. \]

Let \( \{ \mu_i \geq 0 : i = 1, 2, 3, (\mu_1, \mu_2, \mu_3) \neq (0, 0, 0) \} \) be KKT multipliers for constraints \(-(\lambda_1 + \lambda_2) + K \geq 0, \lambda_1 \geq 0, \lambda_2 \geq 0, \) respectively. A candidate optimal solution must satisfy

\[
\mu_1 (-\lambda_1 - \lambda_2 + K) = 0, \mu_2 \lambda_1 = 0, \mu_3 \lambda_2 = 0, -(b_2 + 1/\lambda)\lambda_1 + (b_1 + 1/\lambda)\lambda_2 - a = 0, \text{ and the following stationarity conditions.}
\]

\[
1 - 2\lambda_1/\Lambda + b\lambda_2 + c((\lambda_1 + \lambda_2)/K) + (\lambda_1 + \lambda_2)c'(\lambda_1 + \lambda_2)/K - \mu_1 + \mu_2 - g(b_2 + 1/\lambda) = 0, \\
1 + a - 2\lambda_2/\Lambda + b\lambda_1 + c((\lambda_1 + \lambda_2)/K) + (\lambda_1 + \lambda_2)c'(\lambda_1 + \lambda_2)/K - \mu_1 + \mu_3 + g(b_1 + 1/\lambda) = 0.
\]

In what follows, we exploit the proof of Proposition 1. Compared to the set of candidate optimal solutions under price discrimination, the only new candidate optimal solution under the single-price policy is \((K(b_1 + 1/\lambda))/(b_2 + 2/\Lambda) + a/(b_2 + 2/\Lambda)/, K(b_1 + 1/\lambda)/(b_2 + 2/\Lambda) + a/(b_2 + 2/\Lambda)/,\) which is not feasible any more. Solution \((K(b_1 + 1/\lambda)/(b_2 + 2/\Lambda) - a/(b_2 + 2/\Lambda), K(b_1 + 1/\lambda)/(b_2 + 2/\Lambda) + a/(b_2 + 2/\Lambda)/\) is a candidate optimal solution if \( a \geq a^* \equiv -((b_2 + 2/\Lambda)[1 + c(1) + c'(1)] + 2K(b_1b_2 - 1/\Lambda^2))/b_1 + 1/\Lambda)/ and either \( b_1 < -1/\Lambda, b_2 < -a/K - 1/\Lambda, \) or \( b_2 > -1/\Lambda, b_1 > a/K - 1/\Lambda. \)

It yields revenue \( K[1 - K/\Lambda + c(1)] + K(b_1 + 1/\Lambda)[(b_2 + 1/\Lambda)K + a]/(b_2 + 2/\Lambda).\)

To derive the optimal solution in each case, first note that in all cases of Proposition 1 in which either \((0, K) \) or \((0, \lambda^*) \) is optimal, the same solutions are optimal under the single-price constraint. Second, for the remainder of the analysis, we note that \( b_2 > -1/\Lambda, b_1 > a/K - 1/\Lambda \Rightarrow b > a/K - 2/\Lambda. \)

Recall the two cases in Proposition 1 in which \((K/2 - a/[2(b + 2/\Lambda)], K/2 + a/[2(b + 2/\Lambda)]\) is the optimal solution to (P1). The first is \( b > a/K - 2/\Lambda \) and \( K \leq K[1 + a + c(1) + c'(1)]/2, \) in which it is the only candidate optimal solution. Under the single-price constraint, solution \((K(b_1 + 1/\lambda)/(b_2 + 2/\Lambda) - a/(b_2 + 2/\Lambda), K(b_1 + 1/\lambda)/(b_2 + 2/\Lambda) + a/(b_2 + 2/\Lambda)/\) is optimal in the same region only if \( a \geq a^*. \) If \( a < a^*, \) the optimal solution is the unique solution to (14) and (15) when \( \mu_1 = \mu_2 = \mu_3 = 0. \) Note that the single-price constraint implies \( \lambda_1 = [\lambda_2(b_1 + 1/\lambda) - a]/(b_2 + 1/\Lambda). \) In addition, subtracting (14) from (15) yields \( g = \lambda_2 - \lambda_1 - a/(b_2 + 2/\Lambda). \)

The last two equations and (14) jointly imply that \( \lambda^*_2 = \lambda^*_2 P, \) must uniquely satisfy the equation

\[
1 - 2[\lambda^*_2 P(b_1 + 1/\lambda) - a]/(b_2 + 1/\Lambda)/\Lambda + b\lambda^*_2 P + c[((\lambda^*_2 P(b_2 + 2/\Lambda) - a)/(b_2 + 1/\Lambda)]/K) \\
+ [(\lambda^*_2 P(b_2 + 2/\Lambda) - a)/(b_2 + 1/\Lambda)]c'[((\lambda^*_2 P(b_2 + 2/\Lambda) - a)/(b_2 + 1/\Lambda)]/K) \\
- [\lambda^*_2 P(b_2 + 2/\Lambda) - a]/(b_1 + 1/\Lambda)/(b_2 + 2/\Lambda) = 0.
\]

The second case in which \((K/2 - a/[2(b + 2/\Lambda)], K/2 + a/[2(b + 2/\Lambda)]\) can be optimal in (P1) is if \( b \geq a/K - 2/\Lambda \) and \( K > K[1 + a + c(1) + c'(1)]/2. \) In (P2), \((K(b_1 + 1/\lambda)/(b_2 + 2/\Lambda) - a/(b_2 + 2/\Lambda), K(b_1 + 1/\lambda)/(b_2 + 2/\Lambda) + a/(b_2 + 2/\Lambda)/\) is the optimal solution only if \( a \geq a^*. \) Otherwise, if \( a < a^*, \) the optimal solution under the single-price policy is again the unique solution to (14) and (15) when \( \mu_1 = \mu_2 = \mu_3 = 0. \) □

Proof of Lemma 3  Here we show the proof for the case \( \lambda_1^* \lambda_2^* > 0. \) The proof for the case \( \lambda_1^* = 0, \lambda_2^* > 0, \) is identical in spirit and thus omitted. The first–order conditions are as follows.

\[
\partial R(\lambda_1, \lambda_2, x)/\partial \lambda_1 = 0 \Leftrightarrow 1 - 2\lambda_1/\Lambda + c[\lambda_1/((1-x)K)] + \lambda_1 c'[\lambda_1/((1-x)K)]/((1-x)K) = 0, \\
\partial R(\lambda_1, \lambda_2, x)/\partial \lambda_2 = 0 \Leftrightarrow 1 + a - 2\lambda_2/\Lambda + c[\lambda_2/(xK)] + \lambda_2 c'[\lambda_2/(xK)]/(xK) = 0, \\
\partial R(\lambda_1, \lambda_2, x)/\partial x = 0 \Leftrightarrow \lambda_1^2 c'\{\lambda_1/((1-x)K)]/((1-x)^2K) - \lambda_2^2 c'[\lambda_2/(xK)]/(x^2K) = 0.
\]
We first show that for any allocation \( x \in (0, 1) \), there exists unique solution \((\lambda_1^*(x), \lambda_2^*(x))\) satisfying (16) and (17). Note that it suffices to show that there exists unique \( \lambda_2^*(x) \) satisfying (17), as the two equations differ only by constant \( a \) once \( 1-x \) is replaced by \( x \) in (16). To that end, let \( u \equiv \lambda_2/(xK) \) and \( G(u) \equiv 1+a-2uxK/\Lambda + c(u) + uc'(u) \). It suffices to show that \( G(u) = 0 \) cannot have two roots in \([0, 1]\). Note that \( \partial G(u)/\partial u = -2xK/\Lambda + 2c'(u) + uc''(u) \). For \( G(u) = 0 \) to have two roots in \([0, 1]\), \( \partial G(u)/\partial u \) must change sign twice in \([0, 1]\). Because \( c''(u) < 0 \) and thus \( uc''(u) < 0 \), that would be possible only if \( c'(u) \uparrow u \) in some interval within \([0, 1]\). However, \( c'(u) \downarrow u \) because \( c''(u) < 0 \). Thus, \( G(u) = 0 \) can have at most one root in \([0, 1]\).

Therefore, for any allocation \( x \in (0, 1) \), there exists unique optimal solution to (P3) such that \( \lambda_1^*(x) \lambda_2^*(x) > 0 \).

To complete the proof for part 1, we note that the allocation \( x = \lambda_2/(\lambda_1 + \lambda_2) \) satisfies (18) and invoke parts 2 and 3 of the lemma, which we show next.

For part 2, we show that when \( \lambda_1^* \lambda_2^* > 0 \), the uniquely optimal allocation satisfies \( \lambda_1^*/[(1-x^*)K] = \lambda_2^*/(x^*K) \) \( \Leftrightarrow x^* = \lambda_2^*/(\lambda_1^* + \lambda_2^*) \), i.e., the two capacity segments have the same crowding level at the optimal solution. To that end, let \( \lambda_1/[(1-x)K] \equiv u_1 \) and \( \lambda_2/(xK) \equiv u_2 \). The objective function in (P3) in terms of \( u_1, u_2 \) is

\[
R(u_1, u_2) = \lambda_1 + \lambda_2 - (\lambda_1^2 + \lambda_2^2)/\Lambda + \lambda_1 c(u_1) + \lambda_2 c(u_2) + \lambda_2 a.
\]

In the revenue function above, fix \( \lambda_1, \lambda_2 \), where \( \lambda_1^* \lambda_2 > 0 \), and notice that only the term \( \lambda_1 c(u_1) + \lambda_2 c(u_2) \) involves allocations \( u_1, u_2 \). Suppose \( u_1 \neq u_2 \). Because \( c'' < 0 \),

\[
\begin{align*}
[\lambda_1/(\lambda_1 + \lambda_2)]c(u_1) + [\lambda_2/(\lambda_1 + \lambda_2)]c(u_2) & < c[u_1 \lambda_1/(\lambda_1 + \lambda_2) + u_2 \lambda_2/(\lambda_1 + \lambda_2)] \\
\Leftrightarrow \lambda_1 c(u_1) + \lambda_2 c(u_2) & < \lambda_1 c[u_1 \lambda_1/(\lambda_1 + \lambda_2) + u_2 \lambda_2/(\lambda_1 + \lambda_2)] + \lambda_2 c[u_1 \lambda_1/(\lambda_1 + \lambda_2) + u_2 \lambda_2/(\lambda_1 + \lambda_2)].
\end{align*}
\]

Therefore, crowding levels \( u_1', u_2' \) such that \( u_1' = u_2' = u_1 \lambda_1/(\lambda_1 + \lambda_2) + u_2 \lambda_2/(\lambda_1 + \lambda_2) \) yield strictly higher revenue than crowding levels \( u_1, u_2 \). As a result, \( u_1 = u_2 \) at optimality.

Part 3 follows directly from part 2 if \( \lambda_1^* \lambda_2^* > 0 \). If \( \lambda_1^* = 0 \), notice that an allocation \( x^* < 1 \) cannot satisfy (18), thus it cannot be optimal.

For part 4, the optimal allocation if \( a = 0 \) follows directly from the fact that \( x^* = \lambda_2/(\lambda_1 + \lambda_2) \) and equations (16) and (17). Next we show that \( x^*(a) \uparrow a \). Recall that \( u = \lambda_2/(xK) \) and that (17) is equivalent to \( G(u) = 1+a-2uxK/\Lambda + c(u) + uc'(u) = 0 \). Taking the total derivative of both sides of the equation \( G(u) = 0 \) with respect to \( a \) yields

\[
1-2K[x(\partial u/\partial x)(\partial x/\partial a) + u(\partial x/\partial a)]/\Lambda + 2c'(u)(\partial u/\partial x)(\partial x/\partial a) + uc''(u)(\partial u/\partial x)(\partial x/\partial a) = 0,
\]

\[
\Leftrightarrow 1 + (\partial u/\partial x)(\partial x/\partial a)[-2xK/\Lambda + 2c'(u) + uc''(u)] - 2uK(\partial x/\partial a)/\Lambda = 0,
\]

\[
\Leftrightarrow 1 + (\partial u/\partial x)(\partial x/\partial a) \partial G(u)/\partial u - 2uK(\partial x/\partial a)/\Lambda = 0,
\]

\[
\Leftrightarrow \partial x/\partial a = 1/[2uK/\Lambda - (\partial u/\partial x)(\partial G(u)/\partial u)].
\]

In the last equation, \( u > 0 \), and note that \( \partial u/\partial x = -\lambda_2/(Kx^2) < 0 \). In addition, \( \partial G(u)/\partial u \mid_{u=\lambda_2/(x^*K)} \leq 0 \) because \( x^* \) is an optimal allocation. Hence, the result. \( \square \)
Proof of Corollary 1  If \( b = 0 \), the objective of (P1) is
\[
R^{P1}(\lambda_1, \lambda_2) = \lambda_1 + \lambda_2 - (\lambda_1^2 + \lambda_2^2)/\Lambda + (\lambda_1 + \lambda_2)c(\lambda_1 + \lambda_2)/K + \lambda_2 a,
\]
whereas the objective of (P3) is
\[
R^{P3}(\lambda_1, \lambda_2, x) = \lambda_1 + \lambda_2 - (\lambda_1^2 + \lambda_2^2)/\Lambda + \lambda_1 c(\lambda_1/[(1-x)K]) + \lambda_2 c(\lambda_1/(xK)) + \lambda_2 a.
\]
Consider now an optimal solution to (P3) \( \{\lambda_1^*, \lambda_2^*, x^* = \lambda_2^*/(\lambda_1^* + \lambda_2^*)\} \) and notice that \( (\lambda_1^*, \lambda_2^*) \) is a feasible solution to (P1) and yields the same revenue. Thus, \( \max R^{P1}(\lambda_1, \lambda_2) \geq \max R^{P3}(\lambda_1, \lambda_2, x) \). Likewise, consider an optimal solution to (P1) \( (\xi_1^*, \xi_2^*) \) and notice that solution \( \{\xi_1^*, \xi_2^*, x^* = \xi_2^*/(\xi_1^* + \xi_2^*)\} \) is a feasible solution to (P3) and yields the same revenue. Thus, \( \max R^{P1}(\lambda_1, \lambda_2) \leq \max R^{P3}(\lambda_1, \lambda_2, x) \). The last condition along with \( \max R^{P1}(\lambda_1, \lambda_2) \geq \max R^{P3}(\lambda_1, \lambda_2, x) \) jointly imply that \( \max R^{P1}(\lambda_1, \lambda_2) = \max R^{P3}(\lambda_1, \lambda_2, x) \). Because both (P1) and (P3) have unique optimal solutions, they must have the same optimal solution. \( \square \)

Proof of Theorem 1  Note that if \( b = 0 \), the optimal solutions and the revenues are the same with or without capacity allocation, as Corollary 1 suggests. Further, by the Envelope Theorem, when classes do not interact, \( \partial[\max R(\lambda_1, \lambda_2, x)]/\partial b = 0 \). On the other hand, when classes interact, \( \partial[\max R(\lambda_1, \lambda_2)]/\partial b = \lambda_1^* \lambda_2^* > 0 \). Hence, the result. \( \square \)

Proof of Theorems 2 and 3  In the entire proof we make (implicit) use of the fact that if \( b_1 = b_2 = 0 \), the optimal solutions and the revenues are the same with or without capacity allocation. Note that if \( \lambda_1^* = 0 \) in some region of the \( b_1 \times b_2 \) space, revenue is invariant of \( b_1, b_2 \) in that region. Thus, throughout the proof it suffices to focus on the case \( \lambda_1^* \lambda_2^* > 0 \). Because \( \lambda_1^* \lambda_2^* > 0 \), Lemma 2 and part 1 of Proposition 2 jointly imply that \( b_1 > -1/\Lambda, b_2 > -1/\Lambda \Rightarrow b > -2/\Lambda \). In addition, the single–price constraint implies \( \lambda_1^* = \lambda_2^* (b_1 + 1/\Lambda - a)/(b_2 + 1/\Lambda) > 0 \Rightarrow \lambda_2^* > a/(b_1 + 1/\Lambda) > a/(b + 2/\Lambda) \). Consider now stationarity conditions (14) and (15). Because \( \lambda_1^* \lambda_2^* > 0 \), \( \mu_2 = \mu_3 = 0 \); subtracting (14) from (15) yields the optimal Lagrange multiplier \( g^* = \lambda_2^* - \lambda_1^* - a/(b + 2/\Lambda) \). To apply the Envelope Theorem, we need the term in the Lagrange function that depends on \( b_1, b_2 \). That term is \( L(b_1, b_2) \equiv b_1 \lambda_1 \lambda_2 + g[-(b_2 + 1/\Lambda) \lambda_1 + (b_1 + 1/\Lambda) \lambda_2 - a] \).

To show parts 1 and 4 of the theorems, note that \( \partial L(b_1, b_2)/\partial b_1 = \lambda_1 \lambda_2 + g \lambda_2 \Rightarrow \partial L(b_1, b_2)/\partial b_1 |_{\lambda_1 = \lambda_1^*, \lambda_2 = \lambda_2^*, g = g^*} = \lambda_2^* [\lambda_1^* - a/(b + 2/\Lambda)] > 0 \). Likewise, \( \partial L(b_1, b_2)/\partial b_2 |_{\lambda_1 = \lambda_1^*, \lambda_2 = \lambda_2^*, g = g^*} = \lambda_1^* [\lambda_2^* - a/(b + 2/\Lambda)] > 0 \). Hence, parts 1 and 4 of both theorems.

Next we show parts 2 of both theorems, which require \( b_2 > 0 > b_1 \). If \( b_1 \leq a/\Lambda - 1/\Lambda \), we know by Lemma 2 and part 1 of Proposition 2 that without capacity allocation, an exclusive system is optimal. Because the revenue of an exclusive system can be replicated by allocating capacity \( x = 1 \), allocating capacity can only improve revenue. Hence, part 2a of both theorems. To show parts 2b and 2c of both theorems, suppose \( b_1 > a/\Lambda - 1/\Lambda \) and \( b_1 = -\Delta b \Rightarrow b_2 = \Delta b, \Delta b \geq 0 \). We must have \( \Delta b < 1/\Lambda - a/\Lambda \) for an inclusive system. Note that
\[
\partial L(-\Delta b, \Delta b)/\partial \Delta b |_{\lambda_1 = \lambda_1^*, \lambda_2 = \lambda_2^*, g = g^*} = -g^* (\lambda_1^* + \lambda_2^*).
\]
To determine the sign of the last expression, it suffices to determine the sign of \( g^* \). To that end, recall that \( g^* = \lambda_2^* - \lambda_1^* - a/(b + 2/\Lambda) = \lambda_2^* - \lambda_1^* - a\Lambda/2 \), because \( b = \Delta b - \Delta b = 0 \). We continue this part of the proof assuming that \( \lambda_1^* + \lambda_2^* < K \). (The proof for the case \( \lambda_1^* + \lambda_2^* = K \) is identical in spirit.) When
\[ \lambda_1^* + \lambda_2^* < K, \lambda_1^* = \lambda_2^*(b_1 + 1/\lambda)/(b_2 + 1/\lambda) - a/(b_2 + 1/\lambda), \] by Proposition 2. Because \( b_1 = -\Delta b \) and \( b_2 = \Delta b \),

\[ \lambda_1^* = \lambda_2^*(-\Delta b + 1/\lambda)/(\Delta b + 1/\lambda) - a/(\Delta b + 1/\lambda). \]

Therefore,

\[ g^* = \lambda_2^* - \lambda_2^*(-\Delta b + 1/\lambda)/(\Delta b + 1/\lambda) + a/(\Delta b + 1/\lambda) - a\Lambda/2 \]

\[ = [2\lambda_2^*\Delta b + a - a\Lambda(\Delta b + 1/\lambda)/2]/(\Delta b + 1/\lambda) = [a\lambda_2^*\Delta b + a(1 - \Lambda\Delta b)]/[2(\Delta b + 1/\lambda)] \]

\[ = [4\lambda_2^*\Delta b + a\Lambda(1/\lambda - \Delta b)]/[2(\Delta b + 1/\lambda)]. \]

Because \( \Delta b \geq 0 \) and \( \Delta b < 1/\Lambda - a/K \leq 1/\Lambda, \) \( g^* \geq 0 \) and \( \partial L(-\Delta b, \Delta b)/\partial \Delta b |_{\lambda_1^* = \lambda_1^*, \lambda_2^* = \lambda_2^*, g^* = g^*} \leq 0. \) As a result, if \( b = 0 \) and \( b > 0 > b_1, \) allocating capacity yields (weakly) higher revenue than letting classes share the same space. Recall now that \( \partial L(b_1, b_2)/\partial b_1 |_{\lambda_1^* = \lambda_1^*, \lambda_2^* = \lambda_2^*, g^* = g^*} > 0 \) and \( \partial L(b_1, b_2)/\partial b_2 |_{\lambda_1^* = \lambda_1^*, \lambda_2^* = \lambda_2^*, g^* = g^*} > 0. \) If the two strategies yield the same revenue for some \( \Delta b > 0, \) increasing \( b_1 \) or \( b_2 \) (thus making \( b > 0 \)) would make letting classes share the same space weakly more profitable than capacity allocation. Otherwise, if capacity allocation is strictly more profitable for some \( \Delta b > 0, \) increasing \( b_1 \) or \( b_2 \) (thus making \( b > 0 \)) by a sufficiently small amount would not change that. This is due to the continuous differentiability of \( R(\lambda_1^*, \lambda_2^*, b_1, b_2) \) in \( \lambda_1^*, \lambda_2^*, b_1, b_2. \) As a result, there exists \( b^*(a) \geq 0 \) such that capacity allocation is (weakly) better if \( b \leq b^*(a) \) and letting classes share the same space is weakly better if \( b > b^*(a) \). At \( b = b^*(a) \), the two strategies yield the same revenue.

Note that parts 2 and 3 of Theorem 2 are symmetric, thus we omit the proof for part 3 of Theorem 2. We next show part 3 of Theorem 3, which requires \( b_1 > 0 > b_2. \) If \( b_2 \leq -1/\Lambda, \) or \( b_1 \leq a/K - 1/\Lambda \) and \( b_2 > -1/\Lambda, \) we know by Lemma 2 that without capacity allocation, an exclusive system is optimal. In that case, as we argued earlier, allocating capacity yields higher revenue. Hence, part 3a. Suppose now that \( b_1 > a/K - 1/\Lambda \) and \( b_2 > -1/\Lambda. \) According to Proposition 2, it is possible that \( \lambda_1^* = 0, \) i.e., an exclusive system is optimal. In that case, the proof for part 3a of the theorem applies.

To show part 3c of Theorem 3, suppose \( b_1 > a/K - 1/\Lambda, \) \( b_2 > -1/\Lambda, \) and \( \lambda_1^* > 0; \) let \( b_1 = \Delta b, b_2 = -\Delta b, \)

\[ \Delta b \geq 0. \]

We must have \( \Delta b < 1/\Lambda \) for an inclusive system. Moreover, notice in the conditions of part 3 of Proposition 2 that if \( \lambda_1^* = 0 \) at \( \Delta b = 0, \) then \( \lambda_1^* = 0 \) for \( \Delta b > 0. \) Therefore, because we assume that \( \lambda_1^* > 0, \)

\[ \lambda_1^* > 0 \] at \( \Delta b = 0. \] Note that

\[ \partial L(\Delta b, -\Delta b)/\partial \Delta b |_{\lambda_1^* = \lambda_1^*, \lambda_2^* = \lambda_2^*, g^* = g^*} = g^*(\lambda_1^* + \lambda_2^*). \]

As previously, it suffices to determine the sign of \( g^*. \) To that end, recall that \( g^* = \lambda_2^* - \lambda_1^* - a\Lambda/2. \) We continue the proof while assuming that \( \lambda_1^* + \lambda_2^* < K, \) thus \( \lambda_1^* = \lambda_2^*(b_1 + 1/\lambda)/(b_2 + 1/\lambda) - a/(b_2 + 1/\lambda), \) by Proposition 2. In this case,

\[ \lambda_1^* = \lambda_2^*(\Delta b + 1/\lambda)/(-\Delta b + 1/\lambda) - a/(-\Delta b + 1/\lambda). \] Therefore,

\[ g^* = \lambda_2^* - \lambda_2^*(\Delta b + 1/\lambda)/(-\Delta b + 1/\lambda) + a/(-\Delta b + 1/\lambda) - a\Lambda/2 \]

\[ = [-2\lambda_2^*\Delta b + a - a\Lambda(\Delta b + 1/\lambda)/2]/(-\Delta b + 1/\lambda) \]

\[ = [-4\lambda_2^*\Delta b + a + a\Lambda\Delta b]/[2(-\Delta b + 1/\lambda)]. \]

In the last expression, \(-\Delta b + 1/\Lambda > 0; \) in addition, \( g^* = a > 0 \) at \( \Delta b = 0. \) Because both the objective and the constraints in (P2) are continuously differentiable everywhere, \( \lambda_1^*(\Delta b) \) and \( \lambda_2^*(\Delta b) \) are absolutely continuous in \( \Delta b; \) thus, there exists (sufficiently small) \( \Delta b^{**}(a) > 0 \) such that \( \lambda_1^* > 0 \) and \( g^* > 0 \) if \( \Delta b \leq \Delta b^{**}(a). \)
Therefore, if $b_1 > 0 > b_2$, $b = 0$, $\lambda_1^* > 0$, letting classes share the same space yields strictly higher revenue than allocating capacity if $\Delta b \leq \Delta b^{**}(a)$, i.e., if $b_1 - b_2 \leq \Delta b^*(a) \equiv 2\Delta b^{**}(a)$. Decreasing $b_1$ or $b_2$ (thus making $b < 0$) by a sufficiently small amount would not change the last result due to the continuous differentiability of $R(\lambda_1, \lambda_2, b_1, b_2)$ in $\lambda_1, \lambda_2, b_1, b_2$. Therefore, there exists $b^{**}(a) < 0$ such that letting classes share the same space is strictly more profitable than capacity allocation if $b \geq b^{**}(a)$.  

□

References


