Choosing the Customer Signal for Personalized Dynamic Pricing of Limited Inventories

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We consider a firm which sells a product with a limited inventory over a finite sales horizon. In addition to the inventory level and time, the firm also sets its prices based on customer signals, which are quantified scores of relevant information about the customers and are correlated with customers' reservation prices. We investigate the properties of “ideal” customer signals, i.e., signals that lead to simple, easy-to-implement policies, and higher expected profits.

1. Introduction

Perfect price discrimination, charging each customer a price that is equal to his/her willingness-to-pay and thereby extracting the whole consumer surplus, has been a theoretical ideal for firms. However, even if we leave aside any potential customer backlash against such a practice, firms face serious practical challenges in realization of such an ideal. Most importantly, how can the firms possibly know each customer’s willingness-to-pay, identify the customers as they visit their store, and charge the corresponding price? Although it is not likely that this challenge will ever be overcome fully, in recent years, there have been significant advances due to the increasing popularity of Internet commerce and decreasing cost and increasing speed of storing and processing information. Online companies can now track surfing and purchasing habits of their customers (through cookies or asking them to sign-in), which they can use to come up with personalized estimates for reservation prices and determine the “right” price for each customer. Similarly, traditional stores offer personalized discount coupons to their customers, whom they track and identify through their loyalty cards. For example, a recent New York Times article discusses how Sam’s Club uses its customers’ purchase histories to personalize price offers (Martin 2010). The article also cites CVS and Kroger as two other firms that use price personalization. For a number of other examples from
practice, see Johnson et al. (2000), Bridis (2005), Desjardins (2007), and Clifford (2009).

Firms might have the means to personalize prices to a certain extent, but that does not mean that customers are ready to embrace the practice. In fact, there are examples that clearly show that there could be serious customer backlash if customers do not perceive the practice to be fair. As Philips (2005) argues, it is crucial for firms to carefully manage customers’ perceptions. For example, it is known that customers normally react negatively to the idea that firms can charge different prices to different customers, but they are generally more accepting if these price differences are handled through discount coupons. (See Aydin and Ziya 2009 for more on customers’ reactions to different forms of price personalization.) In some industries (e.g. airline, hospitality, fashion), customers are already accustomed to paying different prices for the same products purchased at different times. Therefore, in these industries, implementation of price personalization, using customer specific information to set prices at the individual customer level, can be relatively easier. In this paper, we consider such a firm whose objective is to maximize its profits from the sale of a product with a limited inventory over a finite sales horizon by setting prices depending on the inventory level, time, as well as some relevant customer specific information that correlates with customers’ reservation prices.

Price personalization has received significant attention in the economics and marketing literature. For a review of classical work on price discrimination in economics literature, we refer the reader to Varian (1989). Fudenberg and Villas-Boas (2006) provide a survey of behavior-based price personalization research within marketing literature while Murthi and Sarkar (2003) provide a review of marketing work on personalization in general. Within the operations literature, price personalization does not seem to have received much attention. A number of articles consider some specific forms of price personalization. For example, Netessine, Savin, and Xiao (2006) and Aydin and Ziya (2008) consider models where prices are set depending on the most recent purchase of the customers. On the other hand, Kuo, Ahn, and Aydin (2009) consider pricing decisions through negotiation. The paper that is closest to ours is Aydin and Ziya (2009). The authors consider a model where arriving customers provide a signal to the firm. The signal is essentially a quantified
summary of the relevant information available about the customer and it gives some indication about the customer’s reservation price. More precisely, the authors assume that the customer population can be divided into two segments and the firm knows the reservation price distribution for both of these segments. The signal of a customer does not reveal the segment identity of the customer with certainty, but it provides some clues as to which segment the customer is more likely to belong to. In fact, the firm can compute the probability of the customer belonging to each segment given the signal. With this probability, the firm estimates the reservation price distribution for each customer and charges the corresponding “optimal” price.

The primary contribution of Aydin and Ziya (2009) is that the authors identify the conditions the customer signal must satisfy so that the optimal pricing policies are structurally simple and satisfy some intuitive properties, thereby making price personalization easier-to-implement in practice. However, the conditions determined by Aydin and Ziya (2009) are only relevant within the specific customer signal formulation the authors consider in their model. Therefore, it is of interest to identify more general conditions that lead to similar properties for the optimal pricing policies under a larger class of customer signal formulations. This is one of the main contributions of this paper. More specifically, using a model that is a significant generalization of the model of Aydin and Ziya (2009), we identify a condition that leads to simple and intuitive optimal pricing policies and demonstrate the generality of the model and general appeal of the condition via several example signal formulations.

One of the important decisions a firm needs to make for price personalization concerns the selection of the piece of customer information to be used when personalizing prices. Using different pieces of the available data on customers or processing the same data differently, firms might have alternative ways of coming up with estimates of reservation prices at the individual customer level. For example, they might have the option of using some demographic data such as zipcode or age, or something from customers’ purchase history (e.g. total purchase amount in the last 3 months, 6 months, or 12 months). The question is, “which piece of information the firm should use?” More specifically, given several alternative customer signals, which one would lead to largest profits for
the firm? Are there any common characteristics that “good” signals possess? To the best of our knowledge, this paper is the first to provide some insights into these questions. In particular, using a model that is a generalization of the model of Aydin and Ziya (2009), we find that signals that are more variable across customers in the population lead to higher expected profits, and thus are more preferable.

The rest of this paper is organized as follows. In Section 2, we describe our model with a general customer signal formulation. In Section 3, we introduce a condition and show that the condition leads to simple structures for the optimal policy under both full personalization and partial personalization. Section 4 demonstrates the generality of the formulation and the results of Section 3 with a number of example signal formulations. Section 5 presents our results on the comparison of different customer signals. Finally, we provide our concluding remarks in Section 6. Proofs for all the results are given in the Appendix.

2. Model Description

Consider a firm that is selling a product with a finite selling season. The initial inventory level is finite and there are no replenishment opportunities throughout the season. We assume that the selling horizon consists of $T$ discrete-time periods and these periods are small enough that during each period, at most one customer arrives. We use $\lambda$ to denote the probability of a single customer arrival in a period.

Each arriving customer provides a signal $x \in S$, which is a quantified summary of the relevant information about the customer and gives some indication about the customer’s tendency to purchase. More specifically, the firm knows $q(x,p)$, which is the probability that a customer with a signal $x$ would be willing to purchase the product at a price $p$. As in Aydin and Ziya (2009), the firm employs one of two pricing policies throughout the whole selling season: full personalization or partial personalization.

2.1. Full personalization

In full personalization, the firm first observes the customer signal and then determines the price to be charged to the customer depending on the signal as well as the inventory level and time. Let
$V(n,t)$ denote the optimal expected revenue under the full personalization policy starting with an inventory level of $n$ at time $t$. Then, the optimality equations can be written as follows:

$$V(n,t) = \lambda E_S \left[ \max_p \{ q(S,p) (p + V(n-1,t-1)) + (1-q(S,p)) V(n,t-1) \} \right] + (1-\lambda) V(n,t-1),$$

for $n > 0, t = 1, \ldots, T$,

$$V(0,t) = 0, t = 1, \ldots, T, \text{ and } V(\cdot,0) = 0.$$

### 2.2. Partial personalization

In partial personalization, at the beginning of each period $t$, the firm first determines $K$ different price levels, choosing prices $p_1, p_2, \ldots, p_K$ from respective sets $P_{1t}, P_{2t}, \ldots, P_{Kt}$. Then, if a customer arrives during that period, the firm observes the customer’s signal and charges one of these $K$ prices. Let $V^P(n,t)$ denote the optimal expected revenue under the partial personalization policy starting with an inventory level of $n$ at time $t$. Then, the optimality equations can be written as follows:

$$V^P(n,t) = \max_{p_i \in P_{it}} E_S \left[ \lambda \max_{i=1,\ldots,K} \beta(S,p_i,n,t) + (1-\lambda) V^P(n,t-1) \right],$$

for $n > 0, t = 1, \ldots, T$,

$$V^P(0,t) = 0, t = 1, \ldots, T, \text{ and } V^P(t,0) = 0, n > 0.$$  

where

$$\beta(x,p_i,n,t) = p_i q(x,p_i) + q(x,p_i) V^P(n-1,t-1) + (1-q(x,p_i)) V^P(n,t-1).$$

If the sets $P_{1t}, P_{2t}, \ldots, P_{Kt}$ all have single elements which do not change with $t$, the partial personalization model reduces to a setting where the firm determines $K$ price levels at the beginning of the whole horizon and charges one of those prices in each period throughout the season.

Clearly, expected profits under full personalization are always larger than those under partial personalization. However, full personalization policies might not always be preferable since it is more difficult to manage the customers’ perception of the practice when price discrimination is deeper. Thus, the hidden costs of personalizing prices could be higher in the case of full personalization. Partial personalization policies, particularly those with two or three price levels, might also be easier to implement.
3. Signals that Lead to Optimal Policies with Simple Structures

In this section, our objective is to identify the conditions that the customer signal must satisfy so that the optimal pricing policies under both full personalization and partial personalization exhibit some simple and intuitive properties. Specifically, assume the following holds:

Assumption 1. The ratio \( q(x, p_1)/q(x, p_2) \) is non-decreasing in \( x \) for any \( p_1 > p_2 \).

In order to intuitively understand what this assumption means, let us also make the reasonable assumption that \( q(x, p_1) < q(x, p_2) \), i.e., for any given signal, purchase probability is smaller when price is higher. (Note that this second assumption is not necessary for our results to go through.) Then, the ratio \( q(x, p_1)/q(x, p_2) \) is less than 1 for any \( x \) and therefore under Assumption 1, \( q(x, p_1) \) gets closer to \( q(x, p_2) \) as \( x \) increases. Thus, we can say that, under Assumption 1, the effect of price changes on the purchase probability is smaller when the signal is higher.

First, we consider the full personalization policy. It turns out that Assumption 1 ensures that the optimal price to be charged to a customer increases with the customer signal for any fixed level of inventory and time. To be precise, define \( p^*(x, n, t) \) to be the optimal price to be charged to a customer with a signal \( x \) when the inventory level is \( n \) and time remaining until the end of the selling season is \( t \). In the case of multiple optima, we define \( p^*(x, n, t) \) to be the smallest. Then, we can show the following.

Theorem 1. Suppose that Assumption 1 holds. Then, for any fixed \( n > 0 \) and \( t > 0 \), \( p^*(x_1, n, t) \geq p^*(x_2, n, t) \) for any \( x_1, x_2 \in S \) such that \( x_1 > x_2 \).

Theorem 1 essentially says that if the customer signal is chosen in such a way that Assumption 1 holds, then the firm’s optimal pricing policy has a simple monotonic structure with respect to the customer signal. Using induction arguments, it is also straightforward to establish that the optimal price \( p^*(x, n, t) \) decreases with the inventory level \( n \) for any fixed customer signal \( x \) and time \( t \) and increases with time \( t \) for any fixed customer signal \( x \) and inventory level \( n \). We skip the formal statements and proofs of these results for the sake of brevity.
Next, we consider the partial personalization policy. For a fixed inventory level $n$ and time $t$, let $p_1^*, p_2^*, \ldots, p_K^*$ denote the $K$ optimal price levels that are determined before the arrival of a potential customer. (Note that the rest of the discussion and Theorem 2 hold even though these prices are not determined optimally.) Without loss of generality assume that $p_1^* > p_2^* > \cdots > p_K^*$. When a customer arrives and provides a signal, the firm chooses one of these $K$ prices and offers to the customer. Thus, the “price-offering” policy is defined by a mapping from the signal space $\mathcal{S}$ to the price set $\{p_1^*, p_2^*, \ldots, p_K^*\}$. Equivalently, for each $i \in \{1, 2, \ldots, K\}$, there is a corresponding set $S_i^* \subseteq \mathcal{S}$ so that if the signal of a customer falls in the set $S_i^*$, the customer is charged a price of $p_i^*$. Note that by definition, we must have $\bigcup_{i=1}^{N} S_i^* = \mathcal{S}$ and $S_i^* \cap S_j^* = \emptyset$ for any $i \neq j$.

In general, the optimal price-offering policy can have a very complex structure. The sets $S_1^*, S_2^*, \ldots, S_K^*$ can be quite dispersed. For example, for some $x_1 < x_2 < x_3$, it is possible that $x_1, x_3 \in S_j$ and $x_2 \in S_k$ where $j \neq k$. However, if Assumption 1 holds, the optimal price offering policy has a simple structure, described by $K-1$ threshold levels each separating one class from another. The following theorem formally states this result.

**Theorem 2.** In the partial personalization model, for a fixed inventory level $n$ and time $t$, let $p_1^* > p_2^* > \cdots > p_K^*$ denote the $K$ (optimal) price levels that are determined before the arrival of a potential customer. Then, if Assumption 1 holds, there exists an optimal price offering policy according to which the optimal signal sets $S_1^*, S_2^*, \ldots, S_K^*$ (corresponding to prices $p_1^*, p_2^*, \ldots, p_K^*$, respectively) can be described by $K$ threshold levels $\tau_1 \geq \tau_2 \geq \cdots \geq \tau_{K-1}$ so that $S_1^* = [\tau_1, \infty), S_2^* = [\tau_2, \tau_1), \ldots, S_i^* = [\tau_i, \tau_{i-1}), \ldots, S_K^* = (-\infty, \tau_{K-1})$.

Theorem 2 states that if the customer signal is chosen in such a way that Assumption 1 holds, then the optimal price-offering policy is of threshold-type, i.e., $K-1$ threshold levels are sufficient to describe the policy and furthermore customers whose signals fall into sets with larger signals are charged higher prices. In particular, if the firm can charge one of two different prices, one regular price the other discounted price, a single threshold value determines who is charged which price. Those above the threshold are charged the regular price while the others are offered a discount
price.

We found that Assumption 1 is sufficient to ensure that the optimal policies under both full and partial personalization have simple and intuitive properties. Knowing that optimal policies have such simple structures is important since they provide useful guidance in setting and calibrating prices in practice. For example, knowing that the optimal price is increasing in the customer signal, the firm would at least keep a certain order among prices for different signal values. If the firm uses partial personalization with two prices, it would simply need to worry about where to set the threshold value, which will determine who receives a discount and who does not.

4. Example Signal Formulations

The signal formulation we considered so far is quite general as it simply assumes the existence of a function \( q(x, p) \) that gives the probability of purchase for a given signal \( x \) and price \( p \). To demonstrate this generality, in this section, we give a number of example signal formulations, and for each one, we determine conditions that are equivalent to Assumption 1, which ensures simple structures for the optimal policies.

4.1. Reservation prices with Weibull distribution

Suppose that the reservation prices for customers with signal \( x \) have Weibull distribution with shape parameter \( k \) and scale parameter \( \lambda(x) \). Then, the expected reservation price for a customer with signal \( x \) is \( \lambda(x)\Gamma(1 + \frac{1}{k}) \) (where \( \Gamma(\cdot) \) is the gamma function) and the purchase probability function is given by

\[
q(x, p) = e^{\left(-\frac{p}{\lambda(x)}\right)^k}.
\]

It is then straightforward to show that Assumption 1 holds if and only if \( \lambda(x) \) is non-decreasing in \( x \), which is equivalent to mean reservation price being non-decreasing in the customer signal given a fixed shape parameter \( k \). Thus, for this particular example, having the mean reservation price being increasing in the customer signal is sufficient for optimal policies to have simple structures, e.g., for the optimal price to be increasing in the customer signal under the full personalization model.
4.2. Customer segments with ordered price elasticity functions

Suppose that $S$ is a discrete set and let $F_x(\cdot)$ denote the cumulative distribution function for the reservation price of customers with signal $x$ for $x \in S$. To make things more concrete, suppose that there are three observable characteristics $A$, $B$, and $C$ for the customers and as shown in Figure 1, the signal can take one of eight different values depending on which of these three characteristics the customer possesses. For instance, the customer is said to have $A$ if she is a female, $B$ if she is below the age of 30, and $C$ if she is a frequent shopper with the firm. Then, this particular customer's signal is 8 and her reservation price distribution is given by $F_8(\cdot)$. On the other hand, the signal for a customer possessing $B$ and $C$, but not $A$ is 3.

For any $x \in S$, the purchase probability function is given by

$$q(x, p) = 1 - F_x(p).$$

Now, the failure rate ordering for random variables is defined as follows (see Shaked and Shanthikumar 2007):
Definition 1. **Failure Rate Ordering:** Suppose that \( F_X(\cdot) \) and \( F_Y(\cdot) \) are absolutely continuous with failure rate functions \( r_X(\cdot) \) and \( r_Y(\cdot) \), respectively. If \( r_X(x) \leq r_Y(x) \) (or equivalently \( \frac{1-F_X(x)}{1-F_Y(x)} \) is increasing) over the common support of \( X \) and \( Y \), then we say that \( F_X \) is greater than \( F_Y \) in failure rate ordering (denoted by \( F_X \geq_f r F_Y \)).

We can then show that Assumption 1 holds if and only if \( F_x \geq_f r F_y \) for any \( x > y \). Note that since numbering of the signals are arbitrary, as long as reservation price distributions can be ordered from largest to smallest in failure rate sense, Assumption 1 can be satisfied by appropriately renumbering the signals.

If \( F_x \geq_f r F_y \), then one can show that the price elasticity of demand for customers with signal \( y \) is larger than the price elasticity of demand for customers with signal \( x \) (see Ziya et al. 2004). In other words, customers with signal \( y \) are more sensitive to changes in price than customers with signal \( x \). This means that when Assumption 1 holds, the optimal price to be charged is less for those customers who are more price sensitive in either full personalization or partial personalization.

As this particular example shows, the question of what customer signal to use is very much related to the question of how to segment the customer population or how to pick the customer characteristics that will determine the customer segments. As it turns out, if the firm determines customer segments in such a way that the reservation price distributions corresponding to different customer segments can be ordered in failure rate sense, then the optimal pricing policies will have some desirable features as described in Section 3.

### 4.3. Linear, exponential, constant-elasticity, and logit demand functions

Talluri and van Ryzin (2005) list four different deterministic demand functions as being the most common. These are linear, exponential, constant-elasticity, and logit demand functions. Here, we consider four different forms for the purchase probability function, each leading to one of these four functions for the expected demand for each fixed value of the signal. Specifically, assume that \( q(x, p) \) either leads to a linear demand function with

\[
q(x, p) = \frac{a - b(x)p}{a} \text{ for } p \in [0, a/b(x)],
\]
an exponential demand function with

\[ q(x, p) = \frac{e^{a-b(x)p}}{e^a} \quad \text{for } p \in [0, \infty), \]

a constant-elasticity demand function with

\[ q(x, p) = p^{-b(x)} \quad \text{for } p \in [0, \infty), \]

or a logit demand function with

\[ q(x, p) = \frac{e^{-b(x)p}}{1 + e^{-b(x)p}} \quad \text{for } p \in [0, \infty). \]

For each one of these four cases, one can show that the price elasticity of demand increases with \( b(x) \). Then, for each case we can further show that Assumption 1 holds if and only if \( b(x) \) is non-increasing in \( x \), i.e., if and only if higher signals indicate lower price elasticity.

### 4.4. Target vs. Non-target segment formulation

Suppose that the customer population consists of two segments and \( F_i \) is the reservation price distribution for segment \( i \) for \( i = 1, 2 \). We assume that, for this example, customer signal has a very specific meaning, namely, signal \( x \) for a particular customer is equal to the probability of that customer being of segment 1. Then, the purchase probability function is given by

\[ q(x, p) = x(1 - F_1(p)) + (1 - x)(1 - F_2(p)). \]

One can then show that Assumption 1 holds if and only if \( F_1 \geq_f F_2 \), i.e., customers of segment 2 are more price sensitive than customers of segment 1.

This example is a slightly generalized version of the model considered by Aydin and Ziya (2009). In their model, the firm has a prior distribution for segment identities of the customers and when a customer arrives and reveals her signal, it carries out simple Bayesian updating to compute the segment 1 probability for the customer. In the example we provide here, we simply skip the updating step and take the updated probabilities as the signals. Note that Aydin and Ziya (2009) find that optimal policies will have the properties described in Section 3 if \( F_1 \geq_f F_2 \) (as we also
determined here) and signals that come from segment 1 (the segment which has higher reservation price in failure rate) are larger than the signals that come from segment 2 in likelihood ratio ordering, which ensure that higher signals imply a higher probability of belonging to segment 1.

5. Comparison of Signals

If the firm has extensive information about its customers and can use different pieces of this information to infer something about the customers’ reservation prices (e.g. any demographic information such as age, purchase history, or income level), the question arises as to what particular information to use when setting prices. Are there any characteristics that would make one piece of information superior than the others? To be more precise, if there is more than one customer signal at the firm’s disposal, which signal should the firm use so that the expected profit is the highest? This is the central question we investigate in this section.

We investigate this question using the customer signal formulation described in Section 4.4. (We provide an explanation for this choice at the end of this section.) Accordingly, we assume that there are two customer segments in the population: segment 1 and segment 2, with respective reservation price distributions, \( F_1(\cdot) \) and \( F_2(\cdot) \). We assume that \( F_1 \geq_{f_r} F_2 \) so that customers of segment 2 are more price sensitive than those of segment 1. Segment 1 can be seen as the target segment for the product the firm is selling. As we described in Section 4.4, the firm cannot observe the segment identities of the arriving customers, but using the information available about each customer, computes the probability that the customer belongs to segment 1. We call this probability the customer signal. Now, suppose that the firm has two different ways of computing these probabilities (signal \( Y \) and signal \( Z \)) using different pieces of the information available about each customer. For example, the firm can use the purchase history from the last three months, or from the last twelve months. Or, the firm can choose to use customers’ zipcodes if they are known to be correlated with customers’ segment identities. Each piece of information would possibly give a different estimate on the segment identity probability for each customer. One might guess that using the purchase history from the last twelve months would be better than using the history from the last three
months since more information would help us better estimate the segment identities of individual customers and make better pricing decisions. But, what exactly would ensure that one signal is indeed better than the other?

Before we move on to our analysis, the reader might consider the following simple example to get a better sense as to what we mean by different signals:

**Example:** Suppose that the product in question is the newly-released DVD of an “action” movie. The firm has an estimate for the reservation price distributions of two segments in the population: “action-movie lovers” (target segment) and the rest of the population (non-target segment). The firm does not know whether a given customer is an action-movie lover, but can use the customer’s purchase history to determine its probability. Let us assume that the firm does this in a very straightforward manner by simply dividing the total number of action movie DVD purchases by the total number of DVD purchases over a given horizon. Suppose that a particular customer has purchased 5, 19, and 10 DVDs over the last three months from this firm (starting with the most recent month) and the number of action-movie purchases were 2, 9, and 6, respectively. If the firm uses the data from the last month only, the signal of the customer would be 2/5=0.4. If it uses three-month data, it would be (2+9+6)/(5+19+10) = 0.5.

Now, let $B_Y(\cdot)$ and $B_Z(\cdot)$ denote the cumulative distribution functions and $b_Y(\cdot)$ and $b_Z(\cdot)$ denote the probability density functions for signal $Y$ and signal $Z$, respectively, and define $\alpha_1$ to be the proportion of segment 1 customers in the population. We assume that

$$\alpha_1 = \int_0^1 xb_Y(x)dx = \int_0^1 xb_Z(x)dx$$

which means that both signals give the same true estimate for the fraction of segment 1 customers in the population. This also ensures that both signals’ estimates for the unconditional purchase probability $q(p)$ are the same as well:

$$q(p) = \int_0^1 q(x,p)b_Y(x)dx = \int_0^1 q(x,p)b_Z(x)dx$$

where

$$q(x,p) = x(1-F_1(p)) + (1-x)(1-F_2(p)).$$
Before we state our result on the comparison of the expected revenues under the two signals, we first define convex ordering for random variables (see Shaked and Shanthikumar 2007):

**Definition 2. Convex Ordering:** Suppose that \( W \) and \( X \) both have finite means with respective cumulative distribution functions \( F_W(\cdot) \) and \( F_X(\cdot) \). If \( E[g(W)] \geq E[g(X)] \) for all real convex functions \( g(\cdot) \) such that the expectations exist, then, we say that \( F_W \) is greater than \( F_X \) in convex ordering (denoted by \( F_W \preceq_{cx} F_X \)).

Now, for \( i = Y, Z \), define \( V_i(n,t) \) to be the optimal expected revenue starting at time \( t \) with an inventory level of \( n \) under signal \( i \) and under the full personalization policy, and define \( V_i^P(n,t) \) to be the same expectation under the partial personalization policy. Then, we can show the following:

**Theorem 3.** Suppose that \( B_Y \preceq_{cx} B_Z \). Then, for any firm employing dynamic pricing with full personalization or dynamic pricing with partial personalization, the expected revenue under signal \( Y \) is larger than that under signal \( Z \), i.e., \( V_Y(n,t) \geq V_Z(n,t) \) and \( V_Y^P(n,t) \geq V_Z^P(n,t) \).

Convex ordering is essentially an ordering of variability. Signals that are larger in convex ordering are said to be more variable. In particular, an ordering in convex ordering implies an ordering of variances in the same direction. Thus, the insight that comes out of Theorem 3 is that signals that are more spread-out are more preferable since the information they provide helps the firm better differentiate among their customers.

To better understand the intuition behind Theorem 3, we consider two extreme cases. First, consider the case with zero variance where all customers give the same signal \( x = \alpha_1 \). In this case, even though customers either belong to segment 1 or segment 2, for the firm, they are all the same. There is no information that the firm can use to differentiate among the customers and therefore the optimal expected revenue is small. If there is some variance in the customer signal no matter how small, that will provide the firm with some basis to differentiate among the customers and that will improve the expected revenue. Now, consider the maximum variance case, where all customers of segment 1 give signal 1 while all customers of segment 2 give signal 0. This is the perfect scenario for the firm since customers’ segment identities are readily available and all customers of segment 1
are charged the segment 1 optimal price while all customers of segment 2 are charged the segment 2 optimal price. Thus, under this perfect information case, the optimal revenue is the largest. It is easy to compare these two cases and see how variance can come into play since they are at the two opposite ends of the class of signal distributions. In general, it is difficult to make the same comparison for any other two distributions, but Theorem 3 provides us with one way of doing that. If there is a convex ordering between the two distributions, we know that the larger one is more preferable.

One way of establishing whether or not there is a convex ordering between two distributions is to directly use Definition 2. However, this might be difficult since it requires checking to see whether the expectations of any convex function of these random variables are ordered. Alternatively, there are known sufficient conditions that are much easier to check. For example, if \( B_Y - B_Z \) starts positive, becomes negative at some point and stays negative from then on, then \( B_Y \geq_{cx} B_Z \). For a proof and for more on sufficient conditions for convex ordering, see Shaked and Shanthikumar (2007).

Argon and Ziya (2009) use the signal formulation we use in this section to investigate the same signal comparison question in a completely different context and find a similar result. To be more precise, Argon and Ziya (2009) consider a queueing model where customers’ class identities are not available, but customers provide signals that reveal their class identity probabilities. The service provider uses these signals to assign priority levels for the customers. The authors find that using signals that are larger in convex ordering are more preferable since they lead to lower long-run average waiting costs for the system.

The main reason behind the choice of this particular signal formulation in this section is that it permits a fair comparison among different signals. Unlike the general signal formulation we described in Section 2 and the other example formulations described in Section 4, this formulation allows us to fix the customer population (consisting of two segments that make up of fixed fractions of the customer population and corresponding reservation price distributions) and choose different signals. For example, if one were to use the general formulation of Section 2 and pick two signals \( Y \)
and $Z$ that lead to two purchase probability functions $q_Y(\cdot)$ and $q_Z(\cdot)$, then it is not clear how one can choose these functions. The choice cannot be arbitrary since that would not guarantee that the customer population is the same under both signals. It is not possible to isolate the effect of using a different signal from the effect of selling to a different customer population. In the formulation we use in this section, we ensure that the population remains the same by requiring that the expected values of the two signals are the same. Such a simple condition would not be sufficient for the general formulation.

6. Conclusions

One of the first decisions for a firm which is planning to implement price personalization for a certain product involves determining the piece of information that will be used when setting prices. Obviously, there must exist some correlation between the reservation prices and the information (which we call signal in this paper) used by the firm, but beyond that not much is known regarding what type of information would be more preferable. This paper provides some answers to this question.

In particular, there are two factors that should be taken into consideration when choosing the customer signal. The first concern is the profits. The signal should lead to as much profits as possible. Second, it should ideally lead to optimal policies with simple structures so that they will be easier to implement. In this paper, we first provide some results for the latter. We consider a very general customer signal formulation and identify a condition for the customer signal, which leads to optimal prices that are monotonically increasing in the customer signal if the firm chooses to implement full personalization, and a simple threshold-type optimal price offering policy if the firm chooses to implement partial personalization. Knowing the existence of these structural properties would be useful in practice. For example, in the partial personalization case, if the signal satisfies the condition we determined in this paper, the firm would simply need to determine threshold values (a single threshold value if there are only two prices) on the signal space that will separate one price region from another.
As for the comparison of the signals in terms of the expected revenues, we consider a formulation that is a slight generalization of the model of Aydin and Ziya (2009). The formulation assumes that there are two customer segments: target and non-target, and customer signal simply reveals the probability of the customer belonging to the target segment. For this formulation, we find that signals that are more variable lead to higher expected profits since they essentially provide more opportunities for the firm to differentiate among the customers. More precisely, if there is a convex ordering between any two given signals, the firm should choose the larger. Even though this two-segment formulation is one natural way of segmenting the population and use price personalization (it is also used frequently in the literature), it is of interest to investigate the same question under different and more general customer signal formulations. This is an important, but a challenging avenue for future work.

Appendix.

Proof of Theorem 1:

First, we can write

$$V(n,t) = V(n,t-1) + E_S{\sup_p}\{\lambda q(S,p)(p - \Delta(n,t-1))\}$$

where

$$\Delta(n,t-1) = V(n,t-1) - V(n-1,t-1).$$

Suppose for contradiction that $p^*(x_2,n,t) > p^*(x_1,n,t)$. Then, by the optimality of $p^*(x_1,n,t)$ and $p^*(x_2,n,t)$, and the fact that when there are multiple optima $p^*(x,n,t)$ is defined to be the smallest for any $x$, we have

$$(p^*(x_2,n,t) - \Delta(n,t-1))q(x_2,p^*(x_2,n,t)) > (p^*(x_1,n,t) - \Delta(n,t-1))q(x_2,p^*(x_1,n,t))$$

and

$$(p^*(x_1,n,t) - \Delta(n,t-1))q(x_1,p^*(x_1,n,t)) \geq (p^*(x_2,n,t) - \Delta(n,t-1))q(x_1,p^*(x_2,n,t)),$$

from which it immediately follows that

$$\frac{q(x_2,p^*(x_2,n,t))}{q(x_2,p^*(x_1,n,t))} > \frac{p^*(x_1,n,t) - \Delta(n,t-1)}{p^*(x_2,n,t) - \Delta(n,t-1)} \geq \frac{q(x_1,p^*(x_2,n,t))}{q(x_1,p^*(x_1,n,t))}.$$
Thus,
\[ \frac{q(x_2,p^*(x_2,n,t))}{q(x_2,p^*(x_1,n,t))} > \frac{q(x_1,p^*(x_2,n,t))}{q(x_1,p^*(x_1,n,t))} \]  \\
But then since we assumed that \( p^*(x_2,n,t) > p^*(x_1,n,t) \), (1) is an immediate contradiction to Assumption 1. Thus, we must have \( p^*(x_1,n,t) \geq p^*(x_2,n,t) \). \( \square \)

**Proof of Theorem 2:**

First, we can rewrite \( V^P(n,t) \) as follows:
\[ V^P(n,t) = V^P(n,t-1) + \max_{p_i \in P_i} \text{ for } i = 1, \ldots, K \ E_S \left[ \max_{i = 1, \ldots, K} \{ \lambda(q(S,p_i)(p_i - \Delta^P(n,t-1))) \} \right]. \]
where
\[ \Delta^P(n,t-1) = V^P(n,t-1) - V^P(n-1,t-1). \]

Any price offering policy can be described by a function \( p(\cdot) \) from the signal space \( S \) to the price set. Fix \( n \) and \( t \) and let \( \Theta^* \) denote the set of optimal price offering functions for a given set of optimal prices \( p'_1, p'_2, \ldots, p'_K \).

Suppose for contradiction that there exists no optimal price offering policy conforming to the description in the statement of the theorem. Then, for any \( p^*(\cdot) \in \Theta^* \) we must have for some \( y < z, p^*(y) > p^*(z) \) and as a consequence
\[ (p^*(z) - \Delta^P(n,t-1))q(z,p^*(z)) \geq (p^*(y) - \Delta^P(n,t-1))q(z,p^*(y)), \]
\[ (p^*(y) - \Delta^P(n,t-1))q(y,p^*(y)) \geq (p^*(z) - \Delta^P(n,t-1))q(y,p^*(z)), \]
where at least one of the inequalities (2) and (3) is strict. It then follows that
\[ \frac{q(y,p^*(y))}{q(z,p^*(z))} > \frac{p^*(z) - \Delta^P(n,t-1)}{p^*(y) - \Delta^P(n,t-1)} = \frac{q(z,p^*(y))}{q(z,p^*(z))} \]
where at least one of the two inequalities is strict. Thus, we have
\[ \frac{q(y,p^*(y))}{q(y,p^*(z))} > \frac{q(z,p^*(y))}{q(z,p^*(z))}, \]
which is a contradiction to Assumption 1 together with the fact that \( p^*(y) > p^*(z) \). Thus, the result follows. \( \square \)

**Proof of Theorem 3:**

Before we give the proofs, we first show an identity that will simplify the presentation. Consider a single-period full personalization problem. If the signal for an arriving customer is \( x \), the firm will simply charge the optimal price \( p^*(x) \) that maximizes
\[ pq(x,p) = p((x(1 - F_1(p)) + (1 - x)(1 - F_2(p))). \]
In order to write the optimal expected revenue $\Pi^*$, first let $b^j(\cdot)$ denote the probability distribution function for the signal of a random segment $j$ customer. Then, it is straightforward to show that

$$b^1(x) = \frac{xb(x)}{\alpha_1} \quad \text{and} \quad b^2(x) = \frac{(1-x)b(x)}{1-\alpha_1}.\$$

Then,

$$\Pi^* = \alpha_1 \int_0^1 p^*(x)(1 - F_1(p^*(x)))b^1(x)dx + (1 - \alpha_1) \int_0^1 p^*(x)(1 - F_2(p^*(x)))b^2(x)dx$$

$$= \alpha_1 \int_0^1 p^*(x)(1 - F_1(p^*(x))) \frac{xb(x)}{\alpha_1} dx + (1 - \alpha_1) \int_0^1 p^*(x)(1 - F_2(p^*(x))) \frac{(1-x)b(x)}{1-\alpha_1} dx$$

$$= \int_0^1 \sup_p \{pq(x,p)\} b(x)dx.$$ 

Thus, the optimal revenue $\Pi^*$ can directly be written as an expectation over the customer signal distribution. It can also easily be shown that the same property holds for multi-period pricing problems. For brevity, in the following, we write the expected revenue directly using this equivalent form without going through a similar justification for each case.

**Proof for full personalization:**

First let $t = 1$. Then, for $i = Y, Z$, we have

$$V_i(n,1) = \lambda \int_0^1 \sup_p \{pq(x,p)\} b_i(x)dx$$

where

$$q(x,p) = x(1 - F_1(p)) + (1 - x)(1 - F_2(p)).$$

For any fixed $p$, $q(x,p)$ is linear in $x$ and therefore $\sup_p \{pq(x,p)\}$ is convex in $x$ (see, e.g., Porteus 2002). It then follows from the definition of convex ordering that $V_Y(n,1) \geq V_Z(n,1)$ for any $n \geq 0$.

Now, suppose that for some $t \geq 1$, $V_Y(n, t-1) \geq V_Z(n, t-1)$ for any $n \geq 0$. We will show that $V_Y(n, t) \geq V_Z(n, t)$.

For any $i = Y, Z$, we have

$$V_i(n, t) = \int_0^1 \left[ \sup_p \{\theta_i(x, p, n, t)\} \right] b_i(x)dx$$

where

$$\theta_i(x, p, n, t) = \lambda q(x, p) (p + V_Y(n-1, t-1)) + (1 - \lambda q(x, p)) V_i(n, t-1).$$
First, since \( V_Y(n, t - 1) \geq V_Z(n, t - 1) \) for all \( n \geq 0 \), we have

\[
\theta_Y(x, p, n, t) \geq \theta_Z(x, p, n, t).
\]

This immediately implies that

\[
\int_0^1 \sup_p \{\theta_Y(x, p, n, t)\} b_Y(x) dx \geq \int_0^1 \sup_p \{\theta_Z(x, p, n, t)\} b_Y(x) dx. \tag{4}
\]

Since \( \theta_i(x, p, n, t) \) is linear in \( x \) (and thus convex) for \( i = Y, Z \) and any fixed \( p, n, \) and \( t \), and convexity is preserved under the supremum operator, \( \sup_p \{\theta_Z(x, p, n, t)\} \) is also convex in \( x \). Then, since \( B_Y \geq_{cx} B_Z \), it follows from the definition of convex ordering that

\[
\int_0^1 \sup_p \{\theta_Z(x, p, n, t)\} b_Y(x) dx \geq \int_0^1 \sup_p \{\theta_Z(x, p, n, t)\} b_Z(x) dx. \tag{5}
\]

It then follows from (4) and (5) that

\[
V_Y(n, t) = \int_0^1 \sup_p \{\theta_Y(x, p, n, t)\} b_Y(x) dx \geq \int_0^1 \sup_p \{\theta_Z(x, p, n, t)\} b_Z(x) dx = V_Z(n, t).
\]

This completes the proof for the full personalization model.

**Proof for partial personalization:**

First let \( t = 1 \). Then, for \( i = Y, Z \), we have

\[
V_i^p(n, 1) = \max_{p_k \in P_{kt}} \lambda \int_0^1 \max_{k=1, \ldots, K} \{p_k q(x, p_k)\} b_i(x) dx.
\]

For any fixed \( p, q(x, p) \) is linear in \( x \) and therefore \( \max_{p} \{pq(x, p)\} \) is convex in \( x \) for each fixed \( p = (p_1, p_2, \ldots, p_K) \) (e.g., see Porteus 2002). It then follows from the definition of convex ordering that \( V_Y^p(n, 1) \geq V_Z^p(n, 1) \) for any \( n \geq 0 \).

Now, suppose that \( V_Y^p(n, t - 1) \geq V_Z^p(n, t - 1) \) for any \( n \geq 1 \) and for some \( t \geq 2 \). To complete the induction argument, we will show that \( V_Y^p(n, t) \geq V_Z^p(n, t) \). For \( i = Y, Z \), let

\[
\beta_i(x, p, n, t) = pq(x, p) + q(x, p)V_Y^p(n - 1, t - 1) + (1 - q(x, p))V_Z^p(n, t - 1).
\]

Then, we have

\[
V_Y^p(n, t) = \max_{p_k \in P_{kt}} \lambda \int_0^1 \max_{k=1, \ldots, K} \beta_Y(x, p_k, n, t)b_Y(x) dx + (1 - \lambda)V_Y^p(n, t - 1)
\]

\[
\geq \max_{p_k \in P_{kt}} \lambda \int_0^1 \max_{k=1, \ldots, K} \beta_Y(x, p_k, n, t)b_Z(x) dx + (1 - \lambda)V_Y^p(n, t - 1)
\]

\[
\geq \max_{p_k \in P_{kt}} \lambda \int_0^1 \max_{k=1, \ldots, K} \beta_Z(x, p_k, n, t)b_Z(x) dx + (1 - \lambda)V_Z^p(n, t - 1)
\]

for each fixed \( \lambda \).
where the first inequality follows from the definition of convex ordering and the fact that $\beta_i(x, p, n, t)$ is linear in $x$ for any fixed $p, n,$ and $t,$ so that $\max_{k=1,\ldots,K} \beta_i(x, p_k, n, t)$ is convex in $x.$ The second inequality follows from the induction assumption $V_{P}^{i}(n, t - 1) \geq V_{Z}^{i}(n, t - 1).$ Thus, the result follows. □

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References


