Math 31 Sec 7

EXAM I

26 September 2003

Name: ____________________________

I pledge that I have neither given nor received any unauthorized assistance on this exam.

_______________________________
(signature)

DIRECTIONS

1) Print your name and sign the honor pledge above. If the pledge is not signed, your exam will not be graded.

2) Check now that your test contains all 6 pages and 9 problems.

3) You may use a calculator (except symbolic manipulators such as a TI-89, TI-92, or similar), but your answers must be given in their exact form. (i.e. √3 and not 1.73, π and not 3.14)

4) All work must be shown on this exam. No credit will be given for a correct answer without supporting work that leads to the answer. When it is indicated that calculators are not to be used, clear non-calculator work must be shown.

5) Place a box around all of your final answers. Include units when necessary.

6) Notation and clarity count. Your job is to communicate mathematically; make what you are thinking clear.

7) Work quickly but thoroughly through the test. If you get stuck on a problem, move on to the next and return to it later after you’ve completed the problems you know how to do. Good Luck.
(5) 1. Let \( g(x) = \begin{cases} \frac{1- \cos x}{x} & x \neq 0 \\ c & x = 0 \end{cases} \) for what value of \( c \) is \( g(x) \) continuous at \( 0 \)? Explain.

\[ g(x) \text{ is continuous at } 0 \text{ if } \lim_{x \to 0} g(x) = g(0). \]

\[ \lim_{x \to 0} g(x) = \lim_{x \to 0} \frac{1- \cos x}{x} = 0. \]

So \( g(x) \) is continuous at \( 0 \) if \( c = 0 \).

(10) 2. Write the equations for the horizontal asymptotes of the graph of \( f(x) = \frac{x-3}{\sqrt{3x^2+1}} \).

Horizontal asymptotes occur at \( y = L \) if

\[ \lim_{x \to \infty} f(x) = L \text{ or } \lim_{x \to -\infty} f(x) = L. \]

\[ \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x-3}{\sqrt{3x^2+1}} = \lim_{x \to \infty} \frac{(x-3)/x}{\sqrt{3x^2+1}/x} = \lim_{x \to \infty} \frac{1- \frac{3}{x}}{\sqrt{3 + \frac{1}{x^2}}} = 1. \]

Since as \( x \to \infty \), \( x > 0 \) and \( \sqrt{x^2} = |x| = x \).

So: \[ \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1- \frac{3}{x}}{\sqrt{3 + \frac{1}{x^2}}} = \frac{1}{\sqrt{3}} \]

as \( x \to \infty \), \( x > 0 \), and \( \sqrt{x^2} = |x| = x \), so we have

\[ \lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{(x-3)/x}{\sqrt{3x^2+1}/x} = \lim_{x \to -\infty} \frac{1- \frac{3}{x}}{\sqrt{3 + \frac{1}{x^2}}} = \lim_{x \to -\infty} \frac{1- \frac{3}{x}}{\sqrt{3 + \frac{1}{x^2}}} = \frac{1}{-\sqrt{3}}. \]

Horizontal asymptotes occur at \( y = \frac{1}{\sqrt{3}} \) and \( y = -\frac{1}{\sqrt{3}} \).
3. Calculate the following limits (if they exist) or state why it does not exist.

(a) \( \lim_{t \to -2} \frac{t^2 - 3t - 10}{t^2 - 4} = \lim_{t \to -2} \frac{(t+2)(t-5)}{(t+2)(t-2)} = \lim_{t \to -2} \frac{t-5}{t-2} = -\frac{7}{4} = \frac{7}{4} \)

(b) \( \lim_{x \to 1} \frac{|x-1|}{x-1} \) Consider the one-sided limits:
\[ \lim_{x \to 1^-} \frac{|x-1|}{x-1} = \lim_{x \to 1^-} \frac{-1}{x-1} = \lim_{x \to 1^-} -1 = -1 \]
\[ \lim_{x \to 1^+} \frac{|x-1|}{x-1} = \lim_{x \to 1^+} \frac{x-1}{x-1} = \lim_{x \to 1^+} 1 = 1 \]
Since the one-sided limits are not equal, the limit does not exist.

(c) \( \lim_{\theta \to 0} \frac{\pi \cos(3\theta)}{\sin(4\theta)} = \lim_{\theta \to 0} \pi \cdot \cos(3\theta) \cdot \frac{4\theta}{\sin(4\theta)} = \lim_{\theta \to 0} \frac{\pi \cdot \cos(3\theta)}{4} \cdot \lim_{\theta \to 0} \frac{4\theta}{\sin(4\theta)} = \frac{\pi}{4} \cdot 1 = \frac{\pi}{4} \)

(d) \( \lim_{x \to 3^+} \frac{x^2 - 7x + 10}{x^2 - 9} = \lim_{x \to 3^+} \frac{(x-2)(x-5)}{(x-3)(x+3)} = -\infty \)
4. Sketch the graph of a function $f(x)$ satisfying all of the following:

A. $\lim_{x \to \infty} f(x) = 2$

B. $\lim_{x \to -\infty} f(x) = -4$

C. $f(0) = 1$ and $f'(0) = 0$

D. $\lim_{x \to 1} f(x) = -1$, but $f$ is not continuous at 1

Solutions will vary, this is one example.

5. Write the equation of the tangent line to the graph of $f(x) = \frac{x^2}{x+4}$ at the point $(2, \frac{2}{3})$.

$m_{\text{tan}} = f'(x) = \frac{(x+4) \cdot 2x - x^2 \cdot 1}{(x+4)^2}$

$f'(2) = \frac{6 \cdot 4 - 4}{6^2} = \frac{24}{36} = \frac{5}{9}$.

The line with slope $\frac{5}{9}$ through the point $(2, \frac{2}{3})$ has the equation:

$y - \frac{2}{3} = \frac{5}{9}(x - 2)$
6. (a) State the limit definition of \( f'(x) \).

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

(b) Use the limit definition to compute \( f'(x) \) for \( f(x) = \sqrt{8x+3} \).

You must use the limit definition for this; using any other method will result in zero credit.

\[
f'(x) = \lim_{h \to 0} \frac{\sqrt{8(x+h)+3} - \sqrt{8x+3}}{h} \cdot \frac{\sqrt{8(x+h)+3} + \sqrt{8x+3}}{\sqrt{8(x+h)+3} + \sqrt{8x+3}}
\]

\[
= \lim_{h \to 0} \frac{8(x+h)+3 - (8x+3)}{h(\sqrt{8(x+h)+3} + \sqrt{8x+3})}
\]

\[
= \lim_{h \to 0} \frac{8x + 8h + 3 - 8x - 3}{h(\sqrt{8(x+h)+3} + \sqrt{8x+3})}
\]

\[
= \lim_{h \to 0} \frac{8h}{h(\sqrt{8(x+h)+3} + \sqrt{8x+3})}
\]

\[
= \lim_{h \to 0} \frac{8}{\sqrt{8(x+h)+3} + \sqrt{8x+3}} = \frac{8}{2 \sqrt{8x+3}} = \frac{4}{\sqrt{8x+3}}
\]

(5) 7. Use the Intermediate Value Theorem to show that \( f(x) = -3 \sin^2 x - 5x + 6 \) has a root in the interval \([0, \pi]\). Do not actually find this root!

\( f(0) = -3 \cdot 0 - 5 \cdot 0 + 6 = 6 \)

\( f(\pi) = -3 \cdot 0 - 5 \cdot \pi + 6 = -3 \cdot 5 \pi + 6 \approx -9.708 \)

\( f \) is continuous, \( f(0) > 0 \), \( f(\pi) < 0 \). So by the Intermediate Value Theorem, there is an \( x \), \( 0 \leq x \leq \pi \), such that \( f(x) = 0 \).
8. Find \( D_x y \) for the following \( y \).

(a) \( y = (3x^2 + 2x^{-3}) \cdot (\frac{4}{x} - 3x^4) \)

\[
D_x y = (3 \cdot 2x + 2 \cdot 3x^4) \left( \frac{4}{x} - 3x^4 \right) + (3x^2 + 2x^{-3}) \left( -\frac{4}{x^2} - 3 \cdot 4x^3 \right)
\]

\[
= \left( 6x - 6x^4 \right) \left( \frac{4}{x} - 3x^4 \right) + (3x^2 + 2x^{-3}) \left( -\frac{4}{x^2} - 12x^3 \right)
\]

(b) \( y = 3x^3 - 4x^3 \)

\[
D_x y = 3\pi^2 - 4 \cdot 3x^2
\]

\[
= 3\pi^2 - 12x^2
\]

9. Suppose an arrow is shot upward with velocity of 58 m/s, and its height in meters after \( t \) seconds given by \( h(t) = 58t - 0.83t^2 \).

(a) Find the average velocity of the arrow between 1 and 3 seconds.

Average velocity is \( \frac{h(3) - h(1)}{3 - 1 \text{ sec}} = \frac{58(3) - 0.83(3) - 58 + 0.83}{2} \)

\[
= \frac{109.36}{2} \text{ m/s} = 54.68 \text{ m/s}
\]

(b) Find the instantaneous velocity when the arrow lands (\( h(t) = 0 \)).

Instantaneous velocity is \( h'(t) = 58 - 1.66t \text{ m/s} \)

\( h(t) = 0 \) for \( 58t - 0.83t^2 = 0 \)

\[
t(58 - 0.83t) = 0 \]

\[ t = 0 \text{ or } t = \frac{58}{0.83} \]

\[
h'(\frac{58}{0.83}) = 58 - 1.66 \cdot \frac{58}{0.83} \text{ m/s}
\]

\[
= -58 \text{ m/s}
\]