1. Rewriting the problem

\[
\begin{align*}
\text{max} & \quad \sum_{i=1}^{k} \max \{ c_i^T x, d_i^T x \} \\
\text{st.} & \quad Ax \leq b
\end{align*}
\]  

(0.1)

as

\[
\begin{align*}
\text{max} & \quad \sum_{i=1}^{k} \min \{ -c_i^T x, -d_i^T x \} \\
\text{st.} & \quad Ax \leq b
\end{align*}
\]  

(0.2)

does not work. For

\[
\begin{align*}
\text{max} & \quad \max \{ x_1, 2x_1 \} \\
\text{st.} & \quad 1 \leq x_1 \leq 2
\end{align*}
\]  

(0.3)

the original model’s optimal solution value is 4, and of the formulation it is \(-2\).

2. First note that to model

\[
\begin{align*}
\text{min} & \quad \sum_{i=1}^{k} \max \{ c_i^T x, d_i^T x \} \\
\text{st.} & \quad Ax \leq b
\end{align*}
\]  

(0.4)

an LP is enough: write

\[
\begin{align*}
\text{min} & \quad \sum_{i=1}^{k} z_i \\
\text{st.} & \quad Ax \leq b \\
& \quad (*) z_i \geq c_i^T x \\
& \quad (**) z_i \geq d_i^T x
\end{align*}
\]  

(0.5)

The constraints (*) and (**) make sure that

\[
z_i \geq \max \{ c_i^T x, d_i^T x \}
\]  

(0.6)

and since we minimize, it will be equal in any optimal solution.

If we maximize, in addition to (*) and (**) we have to make sure that

\[
z_i \leq \max \{ c_i^T x, d_i^T x \}
\]  

(0.7)

But (0.7) is equivalent to

\[
z_i \leq c_i^T x \lor z_i \leq d_i^T x,
\]  

(0.8)

and (0.8) can be enforced by the constraints

\[
z_i - c_i^T x \leq 2M y_i, \quad z_i - d_i^T x \leq 2M(1 - y_i), \quad y_i \in \{0, 1\}.
\]  

(0.9)

So the complete model is

\[
\begin{align*}
\text{max} & \quad \sum_{i=1}^{k} z_i \\
\text{st.} & \quad Ax \leq b \\
& \quad z_i \geq c_i^T x \\
& \quad z_i \geq d_i^T x \\
& \quad z_i - c_i^T x \leq 2M y_i \\
& \quad z_i - d_i^T x \leq 2M(1 - y_i) \quad (i = 1, \ldots, k).
\end{align*}
\]  

(0.10)
3. To apply this to compute the maximum adr, note that if \( d_i \) denotes the downside risk for a portfolio, then

\[
d_i = \max \{ 0, -r_i \},
\]

where \( r_i \) is the return in scenario \( i \). We can choose \( M \) as an upper bound on \( r_i \), say the largest \( |r_{ij}| \), where the \( r_{ij} \) are the entries of the return matrix. So the complete model is

\[
\begin{align*}
\max & \quad ADR \\
\text{s.t.} & \quad \sum_{j=1}^{n} r_{ij} x_j = r_i \\
& \quad \sum_{i=1}^{m} p_i r_i = m \\
& \quad \sum_{j=1}^{n} x_j = 1 \\
& \quad \sum_{i=1}^{m} p_i d_i = ADR \\
& \quad x_j \geq 0 \ \forall j \\
& \quad d_i \geq 0 \ \forall i \\
& \quad d_i \geq -r_i \ \forall i \\
& \quad d_i \leq 2M y_i \\
& \quad d_i + r_i \leq 2M (1 - y_i) \\
& \quad r_P \text{ free, } r_i \text{ free } \forall i, \ y_i \in \{ 0, 1 \}.
\end{align*}
\]

(a) The result with risk2.dat is

\[
\begin{align*}
ADR &= 3.224 \\
x &= (1, 0, 0, 0, 0, 0) \\
r &= (4.86, -2.74, -8.92, 1.6, 2.81) \\
d &= (0, 2.74, 8.92, 0, 0)
\end{align*}
\]

In comparison, when looking for the smallest possible ADR, we get

\[
\begin{align*}
ADR &= 0.00 \\
x &= (0.235378, 0.372825, 0.0556278, 0, 0, 0.33617) \\
r &= (0, 0, 0, 1.20662, 1.2944) \\
d &= (0, 0, 0, 0, 0)
\end{align*}
\]

(b) To find the worst risk, i.e. the maximum ADR among the portfolios with the maximum return, we add the constraint \( \sum_{i=1}^{m} p_i r_i \geq 1.238 \), and resolve. The result is

\[
\begin{align*}
ADR &= 1.0065 \\
x &= (0, 0, 0, 0, 0, 1) \\
d &= (1.44, 0, 0, 0, 3.45) \\
r &= (-1.44, 5.2, 2.32, 3.39, -3.45)
\end{align*}
\]
In comparison, when looking for the smallest possible ADR among the portfolios with the maximum return, we get

$$ADR = 0.00$$

$$x = (0, 0, 0.488669, 0, 0, 0.511331)$$

$$r = (1.03266, 5.0534, 0.497266, -0.167507, 0)$$

$$d = (0, 0, 0.167507, 0)$$

(0.15)

4. In both cases, the portfolio with the smaller ADR is better diversified, i.e. the investment is more “spread” among the securities.