AMO 3.5

(a) $f(n)$ is $\Omega(g(n))$, but NOT $O(g(n))$

$f(n)$ is not $O(g(n))$: Suppose that $f(n)$ is $O(g(n))$. So there exist $n_0$ and $c$ such that $f(n) \leq cg(n)$, $\forall n \geq n_0$. So in particular, for all even $n \geq n_0$ we have $(*) n^2 \leq cn$. However, if $n > \max\{n_0, c\}$ then $(*)$ does \textit{not} hold, a contradiction.

$f(n)$ is $\Omega(g(n))$: True, because $f(n) \geq g(n)$ for all even $n$ such that $n \geq 1$.

(b) $f(n)$ is $O(g(n))$ and $\Omega(g(n))$, hence $\Theta(g(n))$.

$f(n)$ is $O(g(n))$: We claim that $f(n) \leq g(n)$, $\forall n \geq 1$. This is trivially true, when $n$ is odd. If $n$ is even, then $n$ is not prime. So $(**)$ is true again.

$f(n)$ is $\Omega(g(n))$: This follows, since $f(n) \geq g(n)$, $\forall n \geq 3$ when $n$ is an even number.

Hence by definition of $\Theta(g(n))$, $f(n)$ is $\Theta(g(n))$.

(c) $f(n)$ is $O(g(n))$ and $\Omega(g(n))$, hence $\Theta(g(n))$.

$f(n)$ is $O(g(n))$: Observe that $f(n) \leq 4$ and $g(n) \geq 1$, $\forall n \geq 10$. So $f(n) \leq 4g(n)$, $\forall n \geq 10$.

$f(n)$ is $\Omega(g(n))$: Notice that $f(n) \geq 3$ and $g(n) \leq 2$, $\forall n \geq 2$. So $f(n) \geq g(n)/2$, $\forall n \geq 2$.

Hence by definition of $\Theta(g(n))$, $f(n)$ is $\Theta(g(n))$. 

1