1. The data for the ULS is:

- \( f_1 = 10, c_1 = 5, h_1 = 1, d_1 = 5 \).
- \( f_2 = 10, c_2 = 10, h_2 = 2, d_2 = 1 \).
- \( f_3 = 0, c_3 = 8, d_3 = 3 \).

Solution by shortest path method:

Denote the costs of the arc \((i, j)\) by \( m_{ij} \). Note that going on arc \((i, j)\) means producing amount \( d_{i+1} + \cdots + d_j \) in period \( i + 1 \), and satisfying all demands in periods \( i + 1, \ldots, j \) from it.

- \( m_{01} = f_1 + c_1 d_1 = 10 + 5 \times 5 = 35 \).
- \( m_{02} = f_1 + c_1 (d_1 + d_2) + h_1 d_2 = 10 + 5 \times (5 + 1) + 1 \times 1 = 41 \).
- \( m_{03} = f_1 + c_1 (d_1 + d_2 + d_3) + h_1 (d_2 + d_3) + h_2 d_3 = 10 + 5 \times (5 + 1 + 3) + 1 \times (1 + 3) + 2 \times 3 = 65 \).
- \( m_{12} = f_2 + c_2 d_2 = 10 + 10 \times 1 = 20 \).
- \( m_{13} = f_2 + c_2 (d_2 + d_3) + h_2 d_3 = 10 + 10 \times (1 + 3) + 2 \times 3 = 56 \).
- \( m_{23} = f_3 + c_3 d_3 = 0 + 8 \times 3 = 24 \).

Denote by \( p_i \) the shortest path from node 0 to node \( i \). Then

- \( p_0 = 0 \).
- \( p_1 = p_0 + m_{01} = 25 \).
- \( p_2 = \min \{ p_0 + m_{02}, p_1 + m_{12} \} = \min \{ 41, 25 + 20 \} = 41 \).
- \( p_3 = \min \{ p_0 + m_{03}, p_1 + m_{13}, p_2 + m_{23} \} = \min \{ 65, 25 + 50, 41 + 24 \} = 65 \).

There are two shortest paths from 0 to 3: 0 → 3, and 0 → 2 → 3.

(*) The path 0 → 3 corresponds to \( x_1 = d_1 + d_2 + d_3 = 9, x_2 = 0, x_3 = 0 \), and

(**) the path 0 → 2 → 3 corresponds to \( x_1 = d_1 + d_2 = 6, x_2 = 0, x_3 = d_3 = 3 \).

On figure 1

we have \( c_1 + h_1 + h_2 - c_3 = 5 + 1 + 2 - 8 = 0 \), so both solutions obtained from

- increasing \( x_1, s_1, s_2 \) and decreasing \( x_3 \), and

- decreasing \( x_1, s_1, s_2 \) and increasing \( x_3 \)

are optimal. In the first way, we will get the solution (*), and in the second way we will get (**).

2. With the data

\( n = 4, c = (1, 1, 1, 2), h = (1, 1, 1, 1), f = (20, 10, 45, 15), d = (8, 5, 13, 4) \).

the costs of the arcs in the graph will be

- \( m_{01} = f_1 + c_1 d_1 = 20 + 8 = 28 \).
Figure 1: Lotsizing instance
\[ m_{02} = f_1 + c_1(d_1 + d_2) + h_1d_2 = 20 + 13 + 15 = 38. \]
\[ m_{03} = f_1 + c_1(d_1 + d_2 + d_3) + h_1(d_2 + d_3) + h_2d_3 = 20 + 26 + 18 + 13 = 77. \]
\[ m_{04} = f_1 + c_1(d_1 + d_2 + d_3 + d_4) + h_1(d_2 + d_3 + d_4) + h_2(d_3 + d_4) + h_3d_4 = 93. \]
\[ m_{12} = f_2 + c_2d_2 = 10 + 5 = 15. \]
\[ m_{13} = f_2 + c_2(d_2 + d_3) + h_2d_3 = 10 + 18 + 13 = 41. \]
\[ m_{14} = f_2 + c_2(d_2 + d_3 + d_4) + h_2(d_3 + d_4) + h_3d_4 = 10 + 22 + 17 + 4 = 53. \]
\[ m_{23} = f_3 + c_3d_3 = 45 + 13 = 58. \]
\[ m_{24} = f_3 + c_3(d_3 + d_4) + h_3d_4 = 45 + 17 + 4 = 66. \]
\[ m_{34} = f_4 + c_4d_4 = 15 + 8 = 23. \]

The shortest path from 0 to 4 is \( 0 \rightarrow 1 \rightarrow 4 \), corresponding to the production plan:

\[ x_1 = 8, \quad x_2 = d_2 + d_3 + d_4 = 22, \quad s_2 = d_3 + d_4 = 17, \quad s_3 = d_4 = 4. \]