On the Closedness of the Linear Image of a Closed Convex Cone

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The Problem

We are given a closed, convex cone, and a linear mapping. Under what conditions is the image of the cone closed?

- A very simple question in convex analysis → interesting on its own right.
- Fundamental in studying duality theory.
The setup

Let

- $K$ be a closed, convex cone, $(x \in K, \lambda \geq 0 \Rightarrow \lambda x \in K)$.
- $K^* = \{y \mid \langle y, s \rangle \geq 0 \ \forall s \in K\}$ the dual of $K$.
- $M$ a linear map, $M^*$ its adjoint (transpose).
The question

- Under what conditions is $M^*K^*$ is closed?

Classical results

- If $K$ is polyhedral,
- Or $\mathcal{R}(M) \cap \text{ri } K \neq \emptyset$ ("Slater-condition"),
- Then $M^*K^*$ is closed.
More recent results

- Waksman and Epelman (76): a simple condition, that reduces to the classical ones in most important cases.
- Auslender (96): a more complicated, necessary and sufficient condition for arbitrary closed convex sets.
- Bauschke and Borwein (99): a necessary and sufficient condition for the continuous image of a closed convex cone, in terms of the CHIP property.
- Ramana (98): An extended dual for semidefinite programs, without any CQ: related to work of Borwein and Wolkowicz in 84 on facial reduction.
Outline of main results

We provide simple, equivalent conditions that are

- necessary for all cones,
- necessary and sufficient for a large class of cones, that we call nice cones. (Technical condition, more about it later).
- Fact: Most cones occurring in optimization (polyhedral, semidefinite, quadratic, lp-norm cones etc.) are nice.
Some important basics

- $C$ convex set. $\text{dir} \ (x, C) := \{ y \mid x + \alpha y \in C \text{ for some } \alpha > 0 \}$: the feasible directions at $x$ in $C$.
- Fact: $\text{dir} \ (x, C')$ is a convex cone, but it may not be closed!
Figure 1: Feasible directions
Main Result, Part 1

Let $K$ be a closed cone, $M$ a linear map, $x \in \text{ri} (\mathcal{R}(M) \cap K)$ (nonneg. orthant: max # of nonzeros; semidef. cone: max. rank).

Then

$$M^* K^* \text{ is closed } \Rightarrow \mathcal{R}(M) \cap \text{cl dir} \ (x, K) = \mathcal{R}(M) \cap \text{dir} \ (x, K) \ (\text{Condition 1})$$

If $K$ is nice, then $\Leftrightarrow$ is true.

Obviously,

$K$ is polyhedral or $x \in \text{ri } K \Rightarrow \text{dir} \ (x, K)$ is closed $\Rightarrow$ Condition 1.
Example 1 $K = K^* = S^2_+ = 2 \times 2$ psd matrices.

\[ M \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = z_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + z_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]

\[ x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in \text{ri} (\mathcal{R}(M) \cap K) \]

\[ y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathcal{R}(M) \cap (\text{cl dir} (x, K) \setminus \text{dir} (x, K)) \]

The $x$ and $y$ are certificates of the nonclosedness of $M^*K^*$. 
Indeed, we can check the nonclosedness of $M^*K^*$ directly:

- $M^* \begin{bmatrix} a & c \\ c & b \end{bmatrix} = \begin{bmatrix} a \\ 2c \end{bmatrix}$.

- Then $\begin{bmatrix} 0 \\ 2 \end{bmatrix} \in \text{cl} \left( M^*S_+^2 \right)$, since $M^* \begin{bmatrix} \epsilon & 1 \\ 1 & 1/\epsilon \end{bmatrix} = \begin{bmatrix} \epsilon \\ 2 \end{bmatrix}$.

- But $\begin{bmatrix} 0 \\ 2 \end{bmatrix} \not\in M^*S_+^2$, since $\begin{bmatrix} 0 & 1 \\ 1 & b \end{bmatrix} \not\in S_+^2$ for any $b$. 
Next: some equivalent variants of Condition 1. Let

\[ x \in \text{ri} (\mathcal{R}(M) \cap K) \]

\[ F = \text{the minimal face of } K \text{ that contains } x \]

\[ F^\perp = \{ y \mid y^T x = 0 \ \forall x \in F \} \quad \text{(a subspace)} \]

\[ F^\Delta = K^* \cap F^\perp \quad \text{(a face of } K^*) \]

\[ F^\Delta \text{ is called the complementary (conjugate) face of } F. \]
**Example** If $K = K^* = S^n_+$, a typical $F$, $F^\perp$, and $F^\triangle$ look like

\[
F = \left\{ \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix} \mid U \succeq 0 \right\}
\]

\[
F^\perp = \left\{ \begin{bmatrix} 0 & V \\ V^T & W \end{bmatrix} \mid V, W \text{ free} \right\}
\]

\[
F^\triangle = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & W \end{bmatrix} \mid W \succeq 0 \right\}
\]
Main Result, Part 2

Let $K$, $M$ and $F$ be as before. Then

- $M^* K^*$ is closed $\Rightarrow M^* F^\Delta = M^* F^\perp$ (Condition 2)

If $K$ is nice, then $\Leftrightarrow$ is true.
Condition 2 rephrased for $K =$ the nonnegative orthant. Suppose that in

\[ M_0 y \geq 0 \]
\[ M_+ y \geq 0 \]

the first group of inequalities always hold at equality, and it is maximal w.r.t. this property (i.e. $\exists \bar{y} : M_0 \bar{y} = 0, M_+ \bar{y} > 0$).

Then Condition 2 $\iff$

\[ \{ y^T M_0 \mid y \geq 0 \} = \{ y^T M_0 \mid y \text{ free} \} \]
Main Result, Part 3

\[ F^\triangle := K \cap F^\perp: \text{ the complementary face of } F. \text{ Then} \]

\[ M^*K^* \text{ is closed } \Rightarrow \]

(1) \( \exists u \in \text{ri} \ F^\triangle \cap \mathcal{N}(M^*), \text{ and} \)

(2) \( M^*(\tan(u, K^*)) = M^*(\text{lin } F^\triangle). \)

If \( K \) is nice, then \( \Leftrightarrow \) is true.

(1) \( \Leftrightarrow x \text{ and } u \text{ are a strictly complementary pair, that is,} \)
\( x \in \mathcal{R}(M) \cap \text{ri}F \text{ and } u \in \mathcal{N}(M^*) \cap \text{ri}F^\triangle. \)
• (1) for $K$ polyhedral: true by Goldman-Tucker.

• (2) for $K$ polyhedral: the tangent space and the linear span are the same.
Example 3

\[
M z = z_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + z_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}
\]

Certificates of closedness:

- \((SC)\) points in \(\mathcal{R}(M) \cap K\) and \(\mathcal{N}(M^*) \cap K^*\):

\[
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

- \(\text{lin } F^\Delta\) and \(\tan (u, K^*)\):

\[
\begin{bmatrix} 0 & 0 & 0 \\ 0 & \times & \times \\ 0 & \times & \times \end{bmatrix} \quad \begin{bmatrix} 0 & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix}
\]

Hence \(M^* K^*\) is closed.
To verify that $M^*K^*$ is closed, we need to check:

- The pair $(x, u)$ is strictly complementary, and

- Two subspaces are equal. This is easy, as opposed to checking the equality of two arbitrary sets.

Hence, if $K = K^* = S^n_+$, we can verify the closedness of $M^*K^*$ in polynomial time, in the real number model of computing.
The examples so far were easy . . . But:

**Example 4** Using Condition 3, it is easy to verify the closedness of $M^* S_4^+$, where

$$M^* : S_4^+ \ni Y \rightarrow \begin{bmatrix} y_{11} \\ 2y_{12} - y_{22} + y_{33} + 2y_{24} \\ 2y_{13} + y_{22} - y_{33} \\ 2y_{14} + 2y_{23} \end{bmatrix}$$

The verification seems quite hard **without** Condition 3.
So, what are nice cones?

**Definition** $K$ is *nice*, if for all faces $F$ of $K$, $F^* = K^* + F^\perp$.

For $K = K^* = \text{nonnegative orthant}$:

\[
F = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \mid x \geq 0 \right\}
\]

\[
F^* = \left\{ \begin{bmatrix} z \\ y \end{bmatrix} \mid z \geq 0, \ y \text{ free} \right\}
\]

\[
F^\perp = \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} \mid y \text{ free} \right\}
\]

They first appear in a paper by Borwein and Wolkowicz in 1980. Niceness seems like a reasonable “relaxation” of polyhedrality.
Theorem

1. \( K \) is nice \( \Rightarrow \) \( K \) is facially exposed.

2. \( K \) is facially exposed, and for all faces \( F \) of \( K \), \( F^* \) is facially exposed \( \Rightarrow \) \( K \) is nice.

Figure 2: A facially not exposed convex set
Conclusion, and further work

- Very simple, necessary condition for the closedness of the image of a closed convex cone;
- Exact for most relevant cones occurring in optimization.
- Certificates for
  - Nonclosedness of the image.
  - Closedness of the image.
- Ongoing work:
  - What are nice cones?
  - What about cones, which are not nice?
  - Applications …