Homework # 11

1. Let \( T, R^T, R_0^T \) be as in class. Show that
   (a) \( R_0^T \) is a Field.
   (b) \( \sigma(R_0^T) = R^T \)

   \textbf{Hint.} Try to write the cylinder sets in terms of the canonical coordinate process.

2. Construct a probability space on which is defined an i.i.d. sequence of \( N(0, 1) \) random variables.

3. Let \( \nu \) be a probability measure on \( (\mathbb{R}, \mathcal{B}(\mathbb{R})) \). Also let \( p: \mathbb{R} \times \mathcal{B}(\mathbb{R}) \rightarrow [0, 1] \) be a function such that for every \( x \in \mathbb{R} \), \( p(x, \cdot) \) is a probability measure on \( (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) and for every \( B \in \mathcal{B}(\mathbb{R}) \), \( p(\cdot, B) \) is a measurable function from \( (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) to \( (\mathbb{R}, \mathcal{B}(\mathbb{R})) \). Show that there is a sequence of random variables \( \{X_n\} \) defined on some probability space \((\Omega, \mathcal{F}, P)\) such that for all \( k \geq 1 \) and \( A_i \in \mathcal{B}(\mathbb{R}) \), \( i = 1, \ldots, k \)

\[
P(X_1 \in A_1, \cdots X_k \in A_k) = \int_{A_1} \left( \int_{A_2} \left( \cdots \left( \int_{A_{k-1}} p(x_k, A_k)p(x_{k-1}, dx_k) \right) \cdots \right) p(x_1, dx_2) \right) \nu(dx_1).
\]

Such a sequence of random variables is called a Markov chain with initial distribution \( \nu \) and transition probability function \( p(x, dy) \).

   \textbf{Hint.} Recall the remark made in class regarding the simplified form of Kolmogorov Consistency Theorem for ordered sets.

4. Let \( \nu \) and \( p \) be as in 3. Suppose that \( \nu \) satisfies the condition

\[
\int p(x, A) \nu(dx) = \nu(A), \quad \forall A \in \mathcal{B}(\mathbb{R}).
\]

(We say that \( \nu \) is an invariant measure for the Markov chain.) Show that there exists a sequence \( \{Z_n\}_{n \in \mathbb{Z}} \), where \( \mathbb{Z} = \{-1, 0, 1, \cdots\} \), defined on some probability space satisfying:

   (a) \( P(Z_n \in A) = \nu(A) \) for all \( A \in \mathcal{B}(\mathbb{R}) \).
   (b) \[
P(Z_n \in A_1, \cdots Z_{n+k} \in A_k) = \int_{A_1} \left( \int_{A_2} \left( \cdots \left( \int_{A_{k-1}} p(x_k, A_k)p(x_{k-1}, dx_k) \right) \cdots \right) p(x_1, dx_2) \right) \nu(dx_1),
\]

for all \( n \in \mathbb{Z} \) and \( A_i \in \mathcal{B}(\mathbb{R}) \), \( i = 1, 2, \cdots k \).

   \textbf{Hint.} Same as problem 3.
5. Let $T = [0, \infty)$. Show that there exists a stochastic process $\{W_t\}_{t \in T}$ on some probability space satisfying:

(a) $W_t - W_s \sim N(0, (t - s))$ for all $0 \leq s < t < \infty$.

(b) $W(0) \equiv 0$.

(c) For all $k > 1$ and $0 \leq t_1 < t_2 < \cdots < t_k$, $(W_{t_1}, W_{t_2} - W_{t_1}, \cdots, W_{t_k} - W_{t_{k-1}})$ are independent random variables.

This is the first step in the construction of a Wiener process.

**Hint.** Try to write down what should be the measure induced by $(W_{t_1}, W_{t_2}, \cdots, W_{t_k})$.

6. Let $T = [0, \infty)$. A function $R : T \times T \to \mathbb{R}$ is called a positive definite kernel (also called a covariance kernel) if for all $k \geq 1$, $t_1, \cdots, t_k \in T$ and $a_1, \cdots, a_k \in \mathbb{R} \setminus \{0\}$,

$$
\sum_{i=1}^{k} \sum_{j=1}^{k} a_i a_j R(t_i, t_j) > 0.
$$

A stochastic process $\{X_t\}_{t \in T}$ is called a Gaussian process if all of its finite dimensional distributions are Gaussian, i.e. for all $k \geq 1$ and $t_1, \cdots, t_k \in T$,

$$(X_{t_1}, \cdots, X_{t_k})$$

is a multivariate Normal random variable.

Let $R$ be a positive definite kernel. Show that there is a Gaussian process $X_t$ defined on some probability space such that $E(X_t) = 0$ for all $t \in T$ and $\text{cov}(X_t, X_s) = R(t, s)$ for all $s, t \in T$.

**Hint.** Exactly the same hint as 5.