Homework # 4

1. Let $\mathcal{F}$ be a $\sigma$-field on $\mathbb{R}$.

   (i) Show that $\mathcal{B}(\mathbb{R}) \subset \mathcal{F}$ if and only if every continuous function from $\mathbb{R}$ to $\mathbb{R}$ is $\mathcal{F}\setminus\mathcal{B}(\mathbb{R})$ measurable (i.e measurable from $(\mathbb{R}, \mathcal{F})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$).

   (ii) Define
   $$\mathcal{F}_0 = \sigma\{f^{-1}(B) | B \in \mathcal{B}(\mathbb{R}) \text{ and } f \in C(\mathbb{R} : \mathbb{R})\},$$
   where $C(\mathbb{R} : \mathbb{R})$ is the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Note that $\mathcal{F}_0$ is the $\sigma$-field generated by all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Show that $\mathcal{F}_0 = \mathcal{B}(\mathbb{R})$.
   [This shows that $\mathcal{B}(\mathbb{R})$ is the smallest $\sigma$-field with respect to which all continuous functions are measurable.]

2. Let $f_n, f$ be measurable maps from $(\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let $\mu$ be a measure on $(\Omega, \mathcal{F})$. Suppose that there exists $A \in \mathcal{F}$ such that $\mu(A) < \infty$ and $f_n(\omega) \to f(\omega)$ for all $\omega \in A$. Show that there exists $B \in \mathcal{F}$ such that $B \subset A$, $\mu(B) < \epsilon$ and
   $$\sup_{\omega \in A \setminus B} |f_n(\omega) - f(\omega)| \to 0$$
as $n \to \infty$. This is known as Egoroff’s Theorem.
   [Hint: Define $B_n^{(k)} = \{\omega \in A | |f(\omega) - f_m(\omega)| < \frac{1}{k} \text{ for some } m \geq n\}$. Show that for all $k$, $B_n^{(k)} \downarrow \emptyset$ as $n \to \infty$. Now use continuity from above to get, for each fixed $k$, a $n_k$ such that $\mu(B_n^{(k)}) < \frac{\epsilon}{2^k}$. Define $B = \bigcup_{k=1}^{\infty} B_n^{(k)}$. Check that $B$ has the desired properties.]

3. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be upper semi-continuous (u.s.c) at $x \in \mathbb{R}$ if for each $\epsilon > 0$ there is a $\delta > 0$ such that $f(y) < f(x) + \epsilon$ whenever $|x - y| < \delta$.
   [This says that if $x_n \to x$ then $\limsup_{n \to \infty} f(x_n) \leq f(x)$.] Suppose that $f$ is u.s.c at every $x \in \mathbb{R}$. Show that $f$ is $\mathcal{B}(\mathbb{R})\setminus\mathcal{B}(\mathbb{R})$ measurable.
   [Hint. Show that for every $M \in \mathbb{R}$, the set $f^{-1}[M, \infty)$ is closed.]

4. Do Exercise 2.2 (page 28) from the text book. (Text book is on reserve in the M/P library.)