Observing the Crisis: Characterising the spectrum of financial markets with high frequency data, 2004-2008.*

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Abstract

Financial market data in crises are usually modelled as possessing common characteristics with non-crisis data with some additional peculiarities. Recent advances in the analytical tools available for high frequency data make it possible to characterise which components of the data generating process change in crisis, and which do not. A set of new statistics is introduced which particularly indicates changes in tail behaviour across different sample periods. Using data from US Treasury market this paper provides a thorough examination of the behaviour of financial markets during the crisis of 2007-2008. We clearly identify increased identification of price discontinuities during the crisis period with the new statistics indicating increased tail activity across maturities, and particularly flight to quality and cash in short maturity bonds.

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1 Introduction

The observed changes in the behavior of asset returns between a non-crisis and crisis period are generally modelled as comprising some underlying driving process, which develops additional peculiarities during crisis periods. These peculiarities have been attributed to such things as idiosyncratic shocks, the transmission of shocks from other markets, and changing market conditions. For example, almost all existing models of contagion during financial crises be characterized in this way, see Dungey et al (2005).

This paper examines the behavior of markets using high frequency data. It extends the recent framework of Aït-Sahalia and Jacod (2010) to formally test the proposition that the underlying data generating process for an asset remains the same across non-crisis and crisis periods. In doing so the paper makes a number of contributions. First, it provides the first usage of high frequency spectrum statistics to compare financial markets across sample periods, and indirectly provides the first characterization of the US Treasury market using these tools. Second, it provokes a thorough evaluation of the proposed high frequency spectrum statistics, particularly with respect to variations in the chosen level of power for calculation, culminating in a series of experiments to extend the 2 dimensional representations of Aït-Sahalia and Jacod (2010) to 3 dimensional representations. Third, it introduces a new statistic which specifically focuses on changing tail behavior between two sample periods, drawing on the findings that the most identifiable change in a crisis is contained in the tail observations, with potentially asymmetric tail changes being specifically acknowledged.

We examine data for US Treasuries during the global financial crisis period from July 2007 until December 2008 and a pre-crisis period from July 2004 to July 2007. The data originate from electronic trade on the BG Cantor secondary market trading for 2, 5, 10 and 30 year maturities. We find clear evidence that these

\footnote{While they specifically examine threshold models of Bae, Karolyi and Stulz (2003), outlier models of Favero and Giavazzi (2002) and correlation models of Forbes and Rigobon (2002), the approach can be readily extended to more recent work such as Dungey, Milunovich and Thorp (2010) and Ait-Sahalia, Cacho-Diaz and Laeven (2010).}

\footnote{Aït-Sahalia and Jacod (2009a) summarises the research produced by these authors in Aït-Sahalia and Jacod (2008a,b,2009b,c) and is related particularly to work by Tauchen and Todorov (2009a,b).}
financial assets retain Brownian motion and the presence of jumps across non-crisis and crisis periods, and importantly increased identification of jump activity during the crisis period.\footnote{In work in progress similar results are observed for equity futures trade for the S&P500 and Nasdaq100 indices on the Globex electronic platform for after-hours trading.} Importantly, this result establishes that while price discontinuities are not necessarily more prevalent during crises, they are more easily identified, which may mean they are more readily transmitted across markets.

The Aït-Sahalia and Jacod framework uses a two dimensional graphical representation of what is actually a multidimensional statistic. In particular results compiled with different power are commonly grouped. To overcome this we extend the representation to three dimensions and simulate a number of standard models to show how the statistics behave in both two and three dimensions, revealing further refinements about the data generating process. The three dimensional representation can separate two data sets which have seemingly identical properties in two dimensions.

Given the results that changing high frequency characteristics between non-crisis and crisis periods are contained in the tails we propose new measures which extend the Aït-Sahalia and Jacod (2010) tests to include measures of tail behavior only. Comparing these statistics between the non-crisis and crisis periods provides an easily implementable test of the extent to which market behavior has changed between two periods. When applied to the US Treasury market data the new test provides strong evidence of the flight to cash effect - the mass of returns in the left tail for longer dated maturities increases, but for short dated maturities is dramatically reduced.

The paper proceeds as follows. Section 2 outlines the framework suggested for characterizing high frequency data drawing on the papers by Aït-Sahalia and Jacod (2010, 2009a,b) and Todorov and Tauchen (2010a). Section 3 gives some background to the features of the US Treasury data over the sample, with particular attention to the changes evident during the global financial crisis. Using the two dimensional representation of the statistics suggested by Aït-Sahalia and Jacod, the identified changes in the data generating processes for these data between the non-crisis and crisis periods are given in Section 4. The representation of the statistics is extended to three dimensions, with simulation results for both
two and three dimensional representations for common theoretical representations with and without jump activity in Section 5 as well as to our datasets in Section 6. In Section 7 we introduce a new statistic measuring the extent to which the market tails have changed between the two periods. Section 8 concludes.

2 Measures of high frequency market characteristics

Following Aït-Sahalia and Jacod (2010) assume that the process describing prices for an individual asset, denoted $X_t$, is described by a semimartingale process of the form

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW +$$

$$\int_0^t \int_{|x| \leq \varepsilon} x(\mu - v)(ds_x, dx) + \int_0^t \int_{|x| > \varepsilon} x\mu(ds_x, dx)$$

(1)

consisting of a sum of 5 potential components which are, in sequential order, a potentially non-zero mean, drift, a continuous component with Brownian motion and finally two jump terms, with $\mu$ the jump measure of $X_t$ and $v$ the Levy measure of its predictable component. The two jump terms represent firstly potentially many small jumps, no larger than some chosen threshold, $\varepsilon$, and a (finite) number of larger jumps which exceed the threshold size. The first step in characterizing the data is to decide which of the components given in equation (1) are useful in describing the particular data series of interest. A given series may be comprised of a pure jump process, not requiring the continuous component, or a purely continuous process with no jumps, or some mixture between these. The activity signature function of Todorov and Tauchen (2010a) examines the same decomposition.

In practice we work with the discretized form of equation (1) and use equally spaced transaction data. $\Delta X^n_i$ refers to the change in the price in $X$ between observation $i$ and $i - 1$ where the data are discretized by sampling at regular intervals given by the interval $\Delta_n$. Asymptotic results apply around $\Delta_n \to 0$, and
there are \( T/\Delta_n \) intervals in the given period of interest which contains a fixed \( T \) (usually a single day).

There are 3 main characteristics used to construct descriptive statistics for the discretely sampled data series. These are (i) realized power without truncation (ii) realized power with truncation and (iii) simple counts of the number of increments in prices. Realized power variation without truncation is the most immediately recognizable as

\[
B(p, \infty, \Delta_n) = \sum_{i=1}^{T/\Delta_n} |\Delta_i^n X|^p
\]

where \( p \geq 0 \) describes the degree of the power such that realized variance is given when \( p = 2 \). At other times it is useful to introduce a truncation point for the size of the price movement given as realized power variation with truncation

\[
B(p, u_n, \Delta_n) = \sum_{i=1}^{T/\Delta_n} |\Delta_i^n X|^p 1\{|\Delta_i^n X| \leq u_n\}
\]

where \( u_n \) is the truncation level and in this case captures movements no larger than \( u_n \). To keep increments which mainly consist of Brownian motion one can choose \( u_n = \alpha \Delta_n^\varpi \) where \( \varpi \in (0, 1/2) \) and \( \alpha > 0 \). The realized power measures with and without truncation are then used to generate a measure of the non continuous increments in the data generating process as

\[
U(p, u_n, \Delta_n) = B(p, \infty, \Delta_n) - B(p, u_n, \Delta_n)
\]

with \( u_n = \alpha \Delta_n^{1/2} \).

Finally, a simple count of the number of price increments greater (or less) than the truncation level can be expressed as

\[
U(0, u_n, \Delta_n) = \sum_{i=1}^{T/\Delta_n} 1\{|\Delta_i^n X| > u_n\}.
\]

The convenience of the measures given in equations (2) to (5) is that the behavior of a given series can be characterized by examining this set of statistics with changing power, \( p \), truncation level, \( u_n \) and sampling frequency, where \( k\Delta_i^n, k \geq 2 \) denotes that the series \( \Delta_i^n \) is sampled \( k \) times more frequently than one generated
at frequency \( k\Delta^p \). Aït-Sahalia and Jacod (2010) provide a formal set of jumps for: the presence of jumps, whether jumps represent finite activity and finally, whether the data is best represented as a pure jump process, that is there is no evidence of Brownian motion. These tests are summarized below.

1. \( H_0 : \text{Presence of Jumps} \)

\[
S_J (p, k, \Delta_n) = \frac{B(p, \infty, k\Delta_n)_T}{B(p, \infty, \Delta_n)_T}
\]

for \( p > 2, k \geq 2 \) and \( T \) the total sample. This statistic converges in the limit to 1 for a process with jumps, and to \( k^{p/2-1} \) for the no jumps case.

2. \( H_0 : \text{Finite Activity} \)

\[
S_{FA} (p, u_n, k, \Delta_n) = \frac{B(p, u_n, k\Delta_n)}{B(p, u_n, \Delta_n)}
\]

where \( p > 2, k \geq 2 \) and the addition of a truncation point, \( u_n > 0 \), helps to distinguish larger jumps. Under the null hypothesis of finite activity \( S_{FA} \) goes asymptotically to the value \( k^{p/2-1} \) and under the alternative of infinite jumps to the value 1.

3. \( H_0 : \text{Brownian Motion} \)

\[
S_W (p, u_n, k, \Delta_n) = \frac{B(p, u_n, \Delta_n)}{B(p, u_n, k\Delta_n)}
\]

where \( p < 2 \) and \( k \geq 2 \) and an integer. Note that this statistic is the simple inverse of \( S_{FA} \) but is being assessed over a different range of \( k \), which is where the direct correspondence with the graphical activity signature function of Todorov and Tauchen (2010a) arises. In the limit the statistic \( S_W \) converges to \( k^{1-p/2} \) under the null hypothesis and alternatively to 1.

The measures given above are for the case of no microstructure noise. In the case where there is potentially additive market microstructure noise, \( S_J \) and \( S_{FA} \) converge to \( 1/k \) where noise dominates and \( 1/k^{1/2} \) when rounding errors dominate. As \( S_W \) is the inverse of \( S_{FA} \) in the presence of additive noise \( S_W \) logically converges to \( k \) when noise dominates and \( k^{1/2} \) when rounding error dominates.

\footnote{Aït-Sahalia and Jacod (2009a) also include tests for infinite activity and no Brownian motion, but as they are not germane to the application here they are ommitted from our discussion.}
Aït-Sahalia and Jacod (2009a) propose an estimate of the Blumenthal-Getoor index, $\beta_n$, which differentiates over the range of processes which look like processes with jumps arriving according to a Poisson process to very active jump processes which closely mimic Brownian motion. Their first estimator for $\beta_n$ is based on a ratio of the number of increments greater than some cutoff point over alternative truncation points

$$\hat{\beta}_n(\omega, \alpha, \alpha') = \log \left( \frac{U(0, u_n, \Delta_n)}{U(0, \gamma u_n, \Delta_n)} \right) \cdot \frac{1}{\log(\gamma)}$$

where $\gamma = \alpha' / \alpha$ representing an increase in the truncation point, while their second estimator is based on varying the sampling frequency $k$,

$$\hat{\beta}'_n(\omega, \alpha, k) = \log \left( \frac{U(0, u_n, \Delta_n)}{U(0, u_n, k\Delta_n)} \right) \cdot \frac{1}{\omega \log(k)}.$$  

A difficulty with the second of these ratios is that it is technically possible to obtain meaningless negative values for $\hat{\beta}'_n$ when the ratio in the first bracket of equation (7) is smaller than 1. Hence, the current paper implements the first measure of $\hat{\beta}_n$ calculated with changing cutoff points. In the application of the next section we consider particularly how $\hat{\beta}_n$ has changed over the period prior to the global financial crisis to the crisis data.

A closely related concept to the indicators described thus far is the activity signature index proposed in Todorov and Tauchen (2010a,b). Unlike the calculation of $\hat{\beta}_n$ in equations (6) and (7), which restrict attention to the case of $p = 0$, the activity signature index incorporates information across a range of $p > 0$. Using the notation of the current paper, the activity signature index has the form

$$\hat{b}x, t(p) = \frac{\ln(k)p}{\ln(k) + \ln[B(p, \infty, k\Delta_n)_T / B(p, \infty, \Delta_n)_T]}, p > 0$$

where $k$ is an integer ($k = 2$). A critical difference between the activity signature index and $\hat{\beta}_n$, is that $\hat{\beta}_n$ relies on truncation. The activity signature index has no truncation while it is clear from the definition of $U(0, u_n, \Delta_n)$ in equation (5) that with no truncation value for $u_n$ then $U(0, u_n, \Delta_n) = 0$ and $\hat{\beta}_n$ is not defined. The activity signature function $bx, t(p)$ can be calculated across a range of $p$, and the outcomes have properties which allow identification of the form of the underlying data generating process. Specifically;
indicates a pure-continuous model

indicates a pure jump model

indicates a continuous with jumps model,

where \( u.c.p. \) denotes convergence in probability, locally uniform in time and \( 0 < bx < 2 \) characterises the decomposition of the Lévy density; see Todorov and Tauchen (2010a) for details. This creates a kinked form to the activity signature index for those data which do not conform to a pure-continuous model, and the location of the kink distinguishes between the continuous with jumps model (which kinks at a value of 2) and the pure jump model which kinks as some other index value.

Once the data have been characterized an estimate of the relative contribution of the continuous component can be provided as a variance decomposition of the quadratic variation, that is dividing the power variation in the special case of \( p = 2 \). The contribution of each of the components is given by:

\[
\text{Contribution of continuous component} = \frac{B(2,u_n,\Delta_n)}{B(2,\infty,\Delta_n)}
\]

\[
\text{Contribution of jumps} = \frac{B(2,\infty,\Delta_n) - B(2,u_n,\Delta_n)}{B(2,\infty,\Delta_n)}
\]

Of which:

\[
\text{Contribution of large jumps} = \frac{U(2,\varepsilon,\Delta_n)}{B(2,\infty,\Delta_n)}
\]

\[
\text{Contribution of small jumps} = \frac{B(2,\infty,\Delta_n) - B(2,u_n,\Delta_n) - U(2,\varepsilon,\Delta_n)}{B(2,\infty,\Delta_n)}
\]

where the division between small and large jumps depends on some exogenously chosen threshold value, \( \varepsilon \).

### 3 US Treasury Data

The secondary market for US Treasuries is one of the largest individual asset markets in the world with turnover of over $US120 trillion in 2008. The majority of trade since the turn of the century has migrated to two dominant electronic communication networks (ECNs), one now known as BGCantor (formerly eSpeed) and
the other BrokerTec. The existing empirical evidence suggests that total turnover is reasonably evenly divided between them - Mizrach and Neely (2006) find approximately 40% of ECN turnover on BG Cantor, but more recent comparisons in Jiang et al (2010) and Dungey, McKenzie and Smith (2009) make it closer to 50% each. The US Treasury market played a key role in the flight to liquidity and quality which occurred in the crisis of 2007-2008 as it did during for the period of the Hong Kong dollar speculative double play and the near collapse of the US based hedge fund Long-Term Capital Markets in August and September 1998 in Dungey, Goodhart and Lubhrutdinova (2009).

In order to examine the differences between non-crisis and crisis period trading, we divide our total sample period of July 1, 2004 to December 31, 2008 into a calm period beginning with the upswing of interest rates from the previous US monetary policy cycle, and a crisis period beginning July 17, 2007. The breakpoint between the pre-crisis and crisis data is consistent with the first indications of troubles from hedge funds at Bear-Stearns and precedes the changes in policy at the European Central Bank on August 9, 2009. Both of these dates have been used to mark the start of the crisis elsewhere. The data cover the main US trading hours of 8:00am to 5:30pm EST for the secondary trading in 2, 5, 10 and 30 year bonds.

Figure 1 shows the daily volume of US Treasury trade in 4 different maturities. It is clear that volume increased dramatically in mid-2007 associated with the freezes in interbank markets and fears of collapsing hedge fund exposures to mortgage backed securities markets in July-August 1998. Daily volume then settles, although at a heightened level to that prior to July 2007, before spiking dramatically in the period of September - December 2008, the period of the Lehman Bros. bankruptcy, the rescue of AIG, collapse of Fannie Mae and Freddie Mac and numerous other instances of institutional stress. This is particularly evident for the shorter maturities, reflecting the flight to cash aspect of the markets.

Figure 2 presents the realized variance from 5 minute returns in each maturity across the sample period, the average daily realized variances for the two subperiods are given in Table 1. As with volume, there is a clear increase in variance (and covariance which is not reported) between assets during the crisis period compared

5There are by now, many chronologies of the 2007-2008 crisis, including Rose and Speigel (2009) and Bordo (2008).
with the relatively tranquil non-crisis period. Large increases occur in realized variance in the July-August 2007 period with greater rises in the September-December 2008 sample.

4 Results

Unlike the sample applications reported in Aït-Sahalia and Jacod (2010) sampling US Treasury transaction data in sub-minute intervals is not appropriate, as there would simply be too many intervals with no transaction. Evidence in Dungey, McKenzie and Smith (2009) and Dungey and Hvozdyk (2010) suggests that 5 minutes is an appropriate sampling frequency. We therefore proceed with the interval \( \Delta_T^\alpha = 5 \) minutes.\(^6\)

The empirical distribution of the test statistic for the presence of jumps \( S_J \) for the pre-crisis sample (solid line) and the crisis sample (dashed line) is shown in Figure 3a). Each line represents the values obtained for \( S_J \) as \( p \) varies across \( 2.1 \leq p \leq 6.0 \) in 0.1 increments and \( k = 2, 3 \) pooled for all maturities. Pooling across assets and \( p \) follows the suggestion in Aït-Sahalia and Jacod (2010). It is clear that the median is just larger than 1 in each case, supporting evidence for jump activity. Comparing the two subsample distributions, it is apparent that the mass of the distribution has shifted to become right skewed during the crisis rather than left skewed in the pre-crisis period. The implication of this shift is that in the pre-crisis period jumps were less distinguishable from noise (which are values of \( S_J < 1 \)), whereas in the crisis period, almost all of the distribution supports the null of jump activity. The long right tail is more difficult to interpret, as right tails are more appropriately associated with Brownian motion - an aspect to which we return in Section 5.

Given the evidence supports the presence of jumps, the next step is to test whether the jump activity is finite. To do so requires specification of \( u_n \), which is chosen here to be the 1% tail of the absolute values of the 5 minute interval log

\(^6\)The choice of optimal sampling regime for different assets is a topic of ongoing research, but efforts include Russell and Bandi (2006), Aït-Sahalia, Mykland and Zhang (2005). The consequences of incorrect sampling are either estimates corrupted by noise from over frequent sampling (and biased covariance estimators see Sheppard, 2006) or the exclusion of information from insufficiently frequent sampling.
returns in the pre-crisis period. The distribution of the calculated test statistic for whether jump activity is finite, \( S_{FA} \), is given in Figure 3b), where the statistics are calculated across all maturities for \( 2.1 \leq p \leq 6.0 \) in 0.1 increments and \( k = 2, 3 \). The mass of the distribution covers the range from 0.5 to 1.5 but both the pre-crisis and crisis distributions are centred on 1, consistent with infinite activity jumps. The distribution has a lower variance in the crisis period, centring its mass more tightly on 1.

Having confirmed that the jump activity is infinite, Figure 3c) plots the distribution of the \( S_W \) statistics calculated for each subsample for \( 1 \leq p \leq 1.75 \) incrementing by 0.05, and for \( k = 2, 3 \). The striking aspect of the figure is that the distribution is barely changed across the pre-crisis and crisis samples, and clearly supports the presence of Brownian motion in the data generating process.

Figure 4 plots the distribution of activity signature indices calculated for each maturity for \( 0 \leq p \leq 4 \) in 0.1 increments.\(^7\) The dashed line represents the pre-crisis activity signature index, shown with 25% quartile bands. The solid line is the crisis period result. This figure shows that the data are not consistent with a pure continuous model, when the value of this index should be unchanging at 2 for all \( p \). Instead, the mass of the index is contained to the right of the value 2, consistent with a model which contains jumps. The difference between the pre-crisis and crisis distributions, is to concentrate the mass of the index around its median (which in the pre-crisis period was 2.095 and in the crisis period 2.119). This is a similar result to that noted for the test statistic \( S_J \), where the increased density of the test statistic above 1 indicated greater certainty that the process was exhibiting jumps as distinct from noise.

Finally Table 2 reports the contributions of the continuous, finite jump and infinite jump components of the process to the quadratic variation experienced during the non-crisis and crisis periods for each maturity, as outlined in equation (9) in Section 2, for the cases of \( \alpha = 2 \). The choice of \( u_n \) is determined as the 5% tail of the pre-crisis period observations. The large jumps are determined as occurring at a threshold greater than 1% of the pre-crisis data. These same thresholds are applied in the pre-crisis and crisis samples. The table clearly shows

\(^7\)We also calculate the activity signature index for differing values of \( k \), and for daily rather than monthly periods. In both cases the results are consistent with those presented here.
that the continuous component accounts for 40% or less of quadratic variation in the pre-crisis period, and this drops to less than 20%, and as low as 15% for the 2 and 5 year bonds, in the crisis period. The proportion of small jumps does not change very much from the pre-crisis to crisis periods, ranging between 18% and 26%, but the proportion of large jumps increases for all maturities in the crisis period. Notably, the shorter maturities show the greatest proportion of large jumps in the crisis at around 65% in the 2 and 5 year bonds, up from 43% and 46% respectively in the pre-crisis period.

Overall, the results from the range of statistics presented in this section strongly support that the form of the data generating process for the US Treasury data is best represented as a continuous process with infinitely active jumps, although the degree of activity in the Treasuries, as given by $\beta_n$, and reported in Table 3, is less than that for equities data reported in Aït-Sahalia and Jacod (2010). Comparing the pre-crisis and crisis data samples, reassuringly confirms the stability of the form of the underlying process (with continuous component and jumps), but allows greater certainty in distinguishing the presence of jumps from mere noise and reveals the increased presence of large jumps during the crisis period. Kolmogorov-Smirnov test statistics reported in Table 4 confirm that the distribution of the $S_W$ and $S_{FA}$ statistics are unable to reject the null hypothesis of no change between the non-crisis and crisis period. However, the formal test rejects the null of no change for the presence of jumps, $S_J$, for the Treasuries data representing an identifiable change in the ability to distinguish jumps in the data during the crisis period.

5 Three Dimensional Representations

The two dimensional representations of the distributions of test results given in Figure 3, contain significant concatenation of information - specifically, by combining results across a range of power functions. This is of some concern due to the visual bias to 1 and skewness which it induces. For example, the $S_J$ statistic converges to 1 in the case of no-jumps, and $k^{p/2-1}$ for jumps. The cases where $p$ is close to 2 will be difficult to discern, and when mixed with cases where $p$ far exceeds 2 will tend to look as though mass is created at 1, supporting the case for jumps. A similar problem arises for $S_W$. As jump activity, $S_J$ provides useful
information on differing behavior in non-crisis and crisis periods we now proceed to examine this in more details.

The two dimensional representation and a three dimensional representation of the test statistic $S_j$ with $p$ providing the additional axis are generated for a number of models. The experiments fall into several simple categories. First we consider basic Brownian motion (Experiment I), Brownian motion with small jumps (Experiment II) and large jumps (Experiment III).\(^8\) To account for the possibility of regime changes in the data set (considering that our data cross a non-crisis and crisis period) we simulate Brownian motion processes with 10 potential regimes (Experiment IV), where there are 5 orders of magnitude between the volatility of the least and most volatile regimes and random switching between the regimes. The model with regimes is also simulated with both small jumps (Experiment V) and large jumps (Experiment VI). The effects of skewness are addressed via a simulated skewed distribution with no jumps, small jumps and large jumps (Experiments VII to IX). The simulation parameters are calibrated to a liquid equity, the same as those used in Aït-Sahalia and Jacod (2009a), which in turn are loosely based on the estimates of the Heston model for S&P500 data in Aït-Sahalia and Kimmel (2007). The sampling frequency used in their calibration (approximately every 5 seconds) is far in excess of the liquidity of the data samples in the applications of this paper, so our calibrations scaled to 201 observations per day.\(^9\) For all processes we simulate 6000 days of observations, producing 60,300 seconds of observations per day from which we extract 201 data points corresponding to 5 minute observations. This gives the equivalent of 100 quarters of observations based on 20 days per month. Across that dataset we randomly assign 10,000 potential jumps, where small jumps are implemented by an additional random element, where that random element is drawn from the random normal distribution and scaled by a factor of 3. Large jumps are implemented by multiplying the random element by 6. The initial price level in the process is set to 1000. Skewed data were

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\(^8\)Results for experiments with a Heston model such as used in the development of the Aït-Sahalia and Jacod (2010) are similar to those with Brownian motion processes.

\(^9\)The calibration to 201 5 minute observations per day represents the trading frequency of overnight equity futures indices for example.
generated from the skewed normal distribution proposed by Azzalini (1985)\textsuperscript{10}, with shape parameter $\alpha = 4$, implying a skewness of 0.78 which is consistent with that observed in financial returns data (see for example Fry, Martin and Tang, 2009).

The results of the Monte Carlo simulations are shown in a series of Figures. Figure 5 gives the 3 dimensional representation of the distribution of test statistics generated by Brownian motion without jumps, with small jumps and with large jumps in its three left hand panels. The vertical axis gives the frequency of occurrence of the value of $S_J$ grouped by the power, $p$, where there were 1000 realizations of each value of $p$. The theoretical result that the mass is centred on the value $k^{p/2-1}$ (where $k = 2$) is clearly evident in the shape of the surface shown, where the range of hills can be viewed as curving away with higher $p$. It is also apparent that as $p$ increases, the concentration of distribution becomes more diffuse, although the greatest mass remains at $k^{p/2-1}$.

The right hand panels of Figure 5 give the corresponding 2 dimensional representations of this statistic when all $p$ are concatenated together, as suggested in Aït-Sahalia and Jacod (2010). It is evident by comparing the two figures, that while the shape of the Brownian motion function is retained, the larger $p$ clearly result in increasingly larger values of $S_J$, due to increased variance in the measure. The choice of $p$ at which to evaluate the statistic will make a difference to how well we are able to identify different outcomes from the 2 dimensional figures.

Experiment II adds small jumps to the Brownian motion, and this and the figures show that this results in a somewhat steeper drop off in the distribution of the mass of the statistic than in the simple Brownian motion case, while in the case of large jumps in Experiment III, the mass is centred clearly on 1 at all

\begin{align*}
\text{A continuous random variable } Z \text{ has a probability density function } f(z) &= 2\phi(z)\Phi(\alpha z) \\
\phi(z) &= \frac{\exp\left(-\frac{z^2}{2}\right)}{\sqrt{2\pi}} \\
\Phi(\alpha z) &= \int_{-\infty}^{\alpha z} \phi(t).
\end{align*}

\textsuperscript{10}A continuous random variable $Z$ has a probability density function

\[ f(z) = 2\phi(z)\Phi(\alpha z) \]

where $\alpha$ is a shape parameter influencing the density function and

\[ \phi(z) = \frac{\exp\left(-\frac{z^2}{2}\right)}{\sqrt{2\pi}} \]

and

\[ \Phi(\alpha z) = \int_{-\infty}^{\alpha z} \phi(t). \]
values of $p$ examined, although again this becomes more diffuse as $p$ increases. The difficulties of visually distinguish between the Brownian motion case and that of Brownian motion with jumps is clearly evident.

Experiments IV to VI involve switching between Brownian motion processes with differing variances. In the case where there are no jumps, the distributions are similar to those of simple Brownian motion, as shown in Figure 7. However, when small or large jumps are introduced the distribution of the statistic is strongly centred on 1, and in the 2 dimensional representation suggests the presence of jumps. Compared with the simulations in the case of Brownian motion and no regimes, the jumps are strongly detected in the small jumps case. Additionally the shape of the 3 dimensional figures does not contain the same extent of widening in variance with increased $p$, as in Figure 5; this feature still exists but is not of commensurate strength as in the benchmark Brownian motion case.

Simple skewed returns without jumps return results shown in Figure 8, which does not looks terribly different to that generated for Brownian motion in Figure 5. The mass is slightly differently distributed, as shall be seen when a direct comparison is made in Figure 11, however if one were to visually assess the 2 dimensional figure it would be difficult to detect any differences. The addition of small jumps in a skewed environment, produces a figure in which it is more difficult to sort out whether jumps or continuous process are dominant. The median in the 2 dimensional figure is clearly above 1, but there is evidence of increasing variance around an increasing mean value at each value of $p$ in the three dimensional representation. Certainly the skewness is sufficient to enable small jumps to be detected in this case, compared with the outcome for Brownian motion with small jumps in Experiment II. This is even more pronounced in Experiment IX with large jumps, where the statistic is centred on 1, but with increasing variance with increasing $p$ in the 3 dimensional representation.

Despite the difficulties in individually assessing differences between the distributions when presented with individual representations, it is possible to see differences between them when a direct comparison can be made as seen in Figure 11. Figure 11 a) clearly shows that there is visually little difference in the no jumps representations between the experiments, although the skewed data will produce more mass closer to $S_J = 1$, consistent with the earlier contention that skewed data
may be influencing the results. In the case of small jumps, shown in Figure 11c), the different outcomes from alternative simulations are readily apparent. Regime switching tends to emphasize the presence of jumps, possibly as regime shifts are being incorporated as jumps. Skewness also increases the mass of the distribution closer to 1 compared with a simple continuous process with small jumps. In the case of large jumps, shown in Figure 11d), all distributions find a clustering around 1, although this is more pronounced with the regime switching experiment than others, and least clear in the case of simple Brownian motion with large jumps. This evidence supports the contention that other features of the data may well be biasing our visual categorization of the process towards accepting the presence of jumps in a given series, particularly in the presence of either skewness or changes in regime, both of which are potentially quite common features of financial market returns data.

6 Application in three dimensional representation

The three dimensional representations of the $S_J$ statistics for the Treasury data during the non-crisis and crisis periods are presented Figure 12. Two aspects of the representations are immediately apparent. The first is that the three dimensional representations are not strikingly similar to any of the simulated series presented in the previous section, despite the seeming correspondence between the two dimensional representations of the simulated data and the application data. The three dimensional data representations have greater mass near the origin - that is at low values of $p$ there is less variance around the mean $S_J$ value at that $p$ than in the simulations. However, as $p$ increases the mass quickly disperses, leaving greater variance at higher $p$ than indicated in the simulations. The differences between the pre-crisis and crisis distributions exacerbates this tendency with even greater clustering of the mass around the mean for low values of $p$, and more dispersion at higher values. The practical implication of this finding is that crisis data can be distinguished from the crisis data by greater evidence of jump type behavior at lower power values. As power increases, there is increased uncertainty about comparisons across the two periods. The suggested value of $p = 4$ in Aït-Sahalia
and Jacod (2009b) is unlikely to yield useful results here, as the actual data has quite a flat distribution in both cases by this point. Lower values of $p$ are likely to lead to more useful distinctions between the distributions. In essence to usefully implement this tool requires an appropriate metric for the choice of $p$.

7 An Indicator of Crisis Behavior

The previous sections emphasize two important points. First, that the tails are important in distinguishing non-crisis and crisis periods - indicated particularly by the results in Section 4 where changes in $S_W$ and $S_{FA}$ which do not contain tail events are statistically insignificant across the two periods, but are statistically significant in $S_J$ which does contain tail events. Second, that concatenating these statistics across different power, $p$, deprives the analyst of potentially useful information.

The changes in tail behavior invite further examination. In particular, the results in Figure 3a) where crises observations in the Treasuries result in a right hand skew to the statistic. The origin of this skew is the extreme observations in the returns subjected to relatively high power in calculating $S_J$. At high power $S_J$ will select large jumps, and in the presence of skewness this will amplify the right tail of the distribution of $S_J$ statistics. Skewness is a standard feature of returns data, and an indicator of change between non-crisis and crisis period data, see Fry, Martin and Tang (2009) for contagion tests based on higher order moments. Unfortunately, the skewness result most often identified between the non-crisis and crisis period is a change of sign, which is not identified using the $S_J$ test which is based on absolute returns.

A simple extension to the suite of statistics proposed by Aït-Sahalia and Jacod (2010) is a statistic describing the tails of the distributions, and given the potential asymmetric nature of the distribution, to construct one for each tail. As these are simply the remnant between the $S_J$ and $S_W$ statistics of Aït-Sahalia and Jacod (2010) they will have equivalent statistical properties. Hence, define, $S_E^+$ as the positive tail statistic and $S_E^-$ as the negative tail statistic as follows:
These statistics can be calculated for any given \( p \). However, the interest in this paper is to compare the tail behavior between two periods. To do this we introduce the \( S \) statistic, which directly compares the tails of the two samples. It has three forms, one for each of the positive and negative tails, \( S^+, S^- \), and one for the combined tail behavior, \( S^{abs} \), as follows:

\[
S^+ = \frac{S^+_{E,\text{crisis}}}{S^+_{E,\text{noncrisis}}}
\]

\[
S^- = \frac{S^-_{E,\text{crisis}}}{S^-_{E,\text{noncrisis}}}
\]

\[
S^{abs} = \left\{ \frac{\left[ \frac{T/\Delta_n}{\sum_{i=1}^{T/\Delta_n} |k\Delta_i^n X| \{k\Delta_i^n X > u_n\} } \right]}{\left[ \frac{T/\Delta_n}{\sum_{i=1}^{T/\Delta_n} |\Delta_i^n X| \{\Delta_i^n X > u_n\} } \right]_{\text{crisis}}} \right\} \left\{ \frac{\left[ \frac{T/\Delta_n}{\sum_{i=1}^{T/\Delta_n} |k\Delta_i^n X| \{k\Delta_i^n X < -u_n\} } \right]}{\left[ \frac{T/\Delta_n}{\sum_{i=1}^{T/\Delta_n} |\Delta_i^n X| \{\Delta_i^n X < -u_n\} } \right]_{\text{noncrisis}}} \right\}
\]

If there is no change in the tail distributions, then each of the statistics will have the value 1, while an increase in the mass in the tails in the crisis period will be indicated by \( S^{abs} > 1 \), and a decrease by \( S^{abs} < 1 \). An increase in the mass in the right hand tail (left hand tail) will be indicated by \( S^+ > 1 \) \( (S^- > 1) \), and a decrease by \( S^+ < 1 \) \( (S^- < 1) \). Applying the \( S \) statistics across incrementing values of \( p \) to the US Treasury data gives the profiles shown in Figure 13 for sampling frequencies \( k = 2, 3 \) in the left and right hand panels of the Figure. Figures 13a)
and b) reveal that $S^{obs}$ is greater than 1 for all values of $p$, indicating an increasing mass in the tails during the crisis period - consistent with increased volatility and fat tails during periods of stress. As expected from the three dimensional results, the outcome varies with $p$, in this case $S^{obs}$ is generally monotonically increasing in $p$, and this is particularly evident for the longer maturities. However, this is not uniformly the case. For $k = 3$ (which compares 15 and 5 minute sampling) the 5 year bonds show a greater deviation between the non-crisis and crisis samples at mid range powers, around $p = 3.3$.

The $S^+$ statistics reported in Figure 13 c) and d) also strongly support greater mass in the right hand tail during the crisis period, consistent with existing results about increasing right hand skewness during crisis periods. There are interesting differences for the alternative sampling frequencies given here - in the case of $k = 2$, the longer maturities show greater evidence of difference between the non-crisis and crisis periods, whereas for $k = 3$ the differences are evident for all maturities.

The final case presented, of $S^-$, provides an interesting contrast. In this case, the long maturities portray definite increases in mass in the negative tails, with this being more apparent for lower power in the 30 year maturities, see Figures 13e) and f). In the case of the short end maturities, however, the mass in the negative tail is reduced. While this may seem counterintuitive it is a reflection of the flight to cash and quality aspect of US Treasury bonds. During the crisis there was high demand for these safe haven assets and they experienced relatively little of the large falls in return experienced in other asset markets, which is precisely what is reflected in the results.

8 Conclusions

The behavior of financial market data during periods of financial stress is of great practical importance to investors, analysts and policy makers alike. Crisis models often assume that some underlying data generating process remains stable across both non-crisis and crisis periods, but is augmented by new or enhanced features during the crisis period. This paper uses a suite of tools developed for characterizing high frequency data to consider which aspects of their data generating process remain stable, and which change during a crisis. Applying these characterizations
for data on US Treasury bonds data reveals that the evidence for Brownian motion and finite or infinite activity jumps is not significantly changed between a pre-crisis period (running from July 1, 2004 to July 16, 2007) and the global financial crisis of July 17, 2007 to December 31, 2008. However, the statistics concerning the presence of jumps differ. The difference between the two periods supports a greater ability to discern jump activity from noise during the crisis period, with the distribution of the jump statistic more concentrated around the critical value associated with jumps and also displaying right skewness.

The right skewness is difficult to interpret. It is at least partly a result of concatenating different powers when evaluating the statistic as proposed in Aït-Sahalia and Jacod (2010) to deal with the common problem of relatively few observations. A series of Monte Carlo experiments demonstrate the potential loss of information from this concatenation, and the importance of the choice of power.

The results in the paper strongly support the contention that changes in the spectrum of financial market returns between non-crisis and crisis periods are evident in tail behavior, and what is more that detection of these changes is affected by the choice of power in calculating summary statistics. Thus, we propose a new statistic based on the behavior of the tails only, treating left and right tails separately, to account for potentially changing skewness patterns identified in Fry, Martin and Tang (2009). This new statistic, specifically compares the truncated tails of the returns across two subsamples, and shows how differential masses in the tail result in deviations of the statistic from 1. In application to the US Treasury market data in the 2007-2008 financial crisis the results show increasing mass in right tails across maturities, but decreasing mass in the left tail of the short end of the US Treasury maturity structure is clearly consistent with flight to quality increasing the desirability of these assets.
References


Figure 1: Average Daily Volume for 2, 5, 10, and 30 year maturity bonds

Figure 2: Realized Variance of 2, 5, 10, and 30 year maturity bonds
a) $S_J$: bonds  
$2.1 \leq p \leq 6, k = 2, 3$

b) $S_{FA}$: bonds  
$2.1 \leq p \leq 6, k = 2, 3$

c) $S_{W}$: bonds  
$1 \leq p \leq 1.75, k = 2, 3$

Figure 3: Two dimensional distributions of statistics in pre-crisis and crisis periods.
Figure 4: Quintile Activity Signature Plot (Monthly)
I: Brownian motion

II: Brownian motion with small jumps

III: Brownian motion with large jumps

Figure 5: Experiments I to III: 2D and 3D representations of the $S_j$ statistics
IV: Brownian motion with regimes

V: Brownian motion with regimes and small jumps

VI: Brownian motion with regimes and large jumps

Figure 6: Experiments IV to VI: 2D and 3D representations of the $S_J$ statistics
Figure 7: Experiments VII to IX: 2D and 3D representations of the $S_J$ statistics
Figure 8: Comparison of 2D representations of the $S_1$ statistics in various experiments.
Figure 9: 3D representations of S1 statistics in pre-crisis and crisis periods.
Figure 10: $S_{\text{absolute}}$, $S^+$, and $S^-$ statistics for 2, 5, 10, and 30 year maturity bonds
$2.1 \leq p \leq 6$, $k = 2, 3$
Table 1:

<table>
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<th>Realized Variance</th>
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<tr>
<td></td>
<td>Pre-Crisis Period</td>
<td>Crisis Period</td>
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<tr>
<td>2 year</td>
<td>0.006</td>
<td>0.006</td>
<td>0.001</td>
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<tr>
<td>5 year</td>
<td>0.034</td>
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<td>10 year</td>
<td>0.095</td>
<td>0.091</td>
<td>0.012</td>
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<tr>
<td>30 year</td>
<td>0.295</td>
<td>0.247</td>
<td>0.043</td>
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Table 2:
The proportion of quadratic variation attributable to the continuous component with a $u_{n}$ equivalent to 5% of the pre-crisis tail, large jumps (assessed as greater than the 1% pre-crisis tail) and small jumps.

<table>
<thead>
<tr>
<th></th>
<th>Pre-Crisis Period</th>
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<tr>
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<td>Continuous Jump</td>
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<td>2 year</td>
<td>0.3730 0.4256 0.2014</td>
<td>0.1593 0.6557 0.1850</td>
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<td>5 year</td>
<td>0.3505 0.4629 0.1866</td>
<td>0.1454 0.6476 0.2070</td>
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<tr>
<td>10 year</td>
<td>0.4034 0.3847 0.2119</td>
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<tr>
<td>30 year</td>
<td>0.4009 0.3755 0.2236</td>
<td>0.1825 0.5705 0.2470</td>
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Table 3:
$\hat{\beta}_{n}$ estimates for different assets during the pre-crisis and crisis periods ($\gamma = 2$).

<table>
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<tr>
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<th>Pre-Crisis Period</th>
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<tbody>
<tr>
<td></td>
<td>$k = 1$</td>
<td>$k = 2$</td>
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<tr>
<td>2 year bond</td>
<td>0.9899 0.6951 0.5493</td>
<td>1.0022 0.7696 0.6496</td>
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<tr>
<td>5 year bond</td>
<td>1.006  0.7665 0.8130</td>
<td>1.0010 0.8106 0.6315</td>
</tr>
<tr>
<td>10 year bond</td>
<td>1.0012 0.8130 0.6687</td>
<td>0.9997 0.4565 0.6359</td>
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<tr>
<td>30 year bond</td>
<td>1.001  0.7795 0.6677</td>
<td>1.0001 0.8102 0.6740</td>
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Table 4:
Kolmogorov-Smirnov (KS) and Cramer-von Mises (CM) test statistics for equality between the pre-crisis and crisis cumulative density functions of the $S_j$, $S_{FA}$ and $S_W$ (critical values at 5% in parentheses). A rejection of the null of equality is indicated by *.

<table>
<thead>
<tr>
<th></th>
<th>$S_j$</th>
<th>$S_{FA}$</th>
<th>$S_W$</th>
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<tbody>
<tr>
<td>Bonds</td>
<td>8.380*</td>
<td>1.186</td>
<td>0.938</td>
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<tr>
<td></td>
<td>(0.000)</td>
<td>(0.120)</td>
<td>(0.343)</td>
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