Achievement & Acription in Educational Attainment:
Genetic & Environmental Influences on Adolescent Schooling

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Derivation of Model
For each sibling $j = 1, 2$ in a pair one can define $3 \times 1$ column vectors
\[ p_j = [p_{1j} \ p_{2j} \ p_{3j}]' \] of observed phenotypes,
\[ a_j = [a_{1j} \ a_{2j} \ a_{3j}]' \] of latent genotypic factors,
\[ c_j = [c_{1j} \ c_{2j} \ c_{3j}]' \] of latent shared environment factors, and
\[ e_j = [e_{1j} \ e_{2j} \ e_{3j}]' \] of latent unshared environment factors.

$X$, $Y$, and $Z$ are $3 \times 3$ lower triangular matrices representing Cholesky factorizations of the genetic, shared environmental, and unshared environmental structure, respectively, so that
\[
X = \begin{bmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{bmatrix}, \quad Y = \begin{bmatrix} y_{11} & 0 & 0 \\ y_{21} & y_{22} & 0 \\ y_{31} & y_{32} & y_{33} \end{bmatrix}, \quad Z = \begin{bmatrix} z_{11} & 0 & 0 \\ z_{21} & z_{22} & 0 \\ z_{31} & z_{32} & z_{33} \end{bmatrix}
\]

$B$ is a $3 \times 3$ sub-diagonal matrix representing the direct paths relating the observed variables, so that
\[
B = \begin{bmatrix} 0 & 0 & 0 \\ b_{21} & 0 & 0 \\ b_{31} & b_{32} & 0 \end{bmatrix}
\]

The model of Figure 2 can then be represented by the equation
\[
p = [p_1 \ p_2] = [B \ 0 \ B] [p_1] + [X \ 0 \ X] [a_1] + [Y \ 0 \ Y] [c_1] + [Z \ 0 \ Z] [e_1] = [p_1 \ p_2] (1)
\]
where \( \mathbf{p} \) is the 6 \( \times \) 1 vector of observations on the siblings. Carrying out the multiplications yields, for each sibling

\[
\begin{align*}
\mathbf{p}_1 &= \mathbf{Bp}_1 + \mathbf{Xa}_1 + \mathbf{YZc}_1 + \mathbf{Ze}_1 \\
\mathbf{p}_2 &= \mathbf{Bp}_2 + \mathbf{Xa}_2 + \mathbf{Yc}_2 + \mathbf{Ze}_2
\end{align*}
\]

(2) (3)

Equations (2) for the first sibling yield

\[
\begin{align*}
\mathbf{p}_1 - \mathbf{Bp}_1 &= \mathbf{Xa}_1 + \mathbf{Yc}_1 + \mathbf{Ze}_1 \\
(\mathbf{I} - \mathbf{B})\mathbf{p}_1 &= \mathbf{Xa}_1 + \mathbf{Yc}_1 + \mathbf{Ze}_1 \\
\mathbf{p}_1 &= (\mathbf{I} - \mathbf{B})^{-1} (\mathbf{Xa}_1 + \mathbf{Yc}_1 + \mathbf{Ze}_1)
\end{align*}
\]

A similar derivation for the second sibling produces

\[
\begin{align*}
\mathbf{p}_2 &= (\mathbf{I} - \mathbf{B})^{-1} (\mathbf{Xa}_2 + \mathbf{Yc}_2 + \mathbf{Ze}_2)
\end{align*}
\]

The expected covariance matrix of the 6 observed variables is

\[
\Sigma = \mathbb{E}\left\{\mathbf{pp}'\right\} = \begin{bmatrix} \mathbb{E}\{\mathbf{p}_1\mathbf{p}_1'\} & \mathbb{E}\{\mathbf{p}_1\mathbf{p}_2'\} \\ \mathbb{E}\{\mathbf{p}_2\mathbf{p}_1'\} & \mathbb{E}\{\mathbf{p}_2\mathbf{p}_2'\} \end{bmatrix} \quad (4)
\]

which contains within-sibling covariances on the diagonal and between-siblings covariances off the diagonal. The within-sibling covariance matrix for sibling 1 is derived as

\[
\Sigma_W = \mathbb{E}\{\mathbf{pp}'\} = \begin{bmatrix} \mathbb{E}\{\mathbf{a}_1\mathbf{a}_1'\} & \mathbb{E}\{\mathbf{a}_1\mathbf{c}_1'\} & \mathbb{E}\{\mathbf{a}_1\mathbf{e}_1'\} \\ \mathbb{E}\{\mathbf{c}_1\mathbf{a}_1'\} & \mathbb{E}\{\mathbf{c}_1\mathbf{c}_1'\} & \mathbb{E}\{\mathbf{c}_1\mathbf{e}_1'\} \\ \mathbb{E}\{\mathbf{e}_1\mathbf{a}_1'\} & \mathbb{E}\{\mathbf{e}_1\mathbf{c}_1'\} & \mathbb{E}\{\mathbf{e}_1\mathbf{e}_1'\} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

which simplifies because the vectors \( \mathbf{a}, \mathbf{c}, \) and \( \mathbf{e} \) are assumed uncorrelated within siblings, so that for sibling 1

\[
\mathbb{E}\{\mathbf{a}_1\mathbf{c}_1'\} = \mathbb{E}\{\mathbf{a}_1\mathbf{e}_1'\} = \mathbb{E}\{\mathbf{c}_1\mathbf{e}_1'\} = 0
\]

and the components of each vector are orthogonal with unit variance so that

\[
\mathbb{E}\{\mathbf{a}_1\mathbf{a}_1'\} = \mathbb{E}\{\mathbf{c}_1\mathbf{c}_1'\} = \mathbb{E}\{\mathbf{e}_1\mathbf{e}_1'\} = 1
\]

Defining matrices \( \mathbf{A} = \mathbf{XX}', \mathbf{C} = \mathbf{YY}', \mathbf{E} = \mathbf{ZZ}', \) and \( \mathbf{W} = (\mathbf{I} - \mathbf{B})^{-1}, \) the covariance matrix for sibling 1 becomes

\[
\Sigma_W = \mathbf{W}(\mathbf{A} + \mathbf{C})\mathbf{W}' \quad (5)
\]

The same result holds for sibling 2.
Because of the names of the matrices involved, the classic behavior genetic
decomposition of the phenotypic covariances embedded within Equation (5) –
with genetic component $A$, shared environment $C$, and unshared environment
$E$ – is traditionally called the ACE model.

A similar derivation can be carried out for each off-diagonal block of $\Sigma$
containing the between-siblings covariances, for example $E\{p_2p_1'\}$. Simplifications
of the vector cross-products in the derivation can be made because by assump-
tion the pattern of covariances of the vectors $a, c, \text{ and } e$ across siblings is such
that
$$E\{a_2c_1'\} = E\{a_2e_1'\} = E\{c_2e_1'\} = 0$$
and
$$E\{a_2a_1'\} = k \otimes I \quad E\{c_2c_1'\} = I \quad E\{e_2e_1'\} = 0$$
The between-siblings covariance can thus be represented by the $3 \times 3$
matrix
$$\Sigma_B = W(k \otimes A + C)W'$$
which contains the covariances of the observed values for sibling 1 (in the
columns) and sibling 2 (in the rows). The expression $k \otimes A$ is the Kronecker
product of $k$ by $A$; since $k$ is a scalar the expression is equivalent to the ordi-
nary multiplication of matrix $A$ by $k$. (Scalar multiplication is implemented as
a Kronecker product in the program Mx.) $k$ represents the degree of genetic
relatedness of siblings. It is set $k = 1$ for MZ twins; assuming random mating
$k = .5$ for DZ twins and full siblings, $k = .25$ for half siblings, $k = .125$ for
cousins, and $k = 0$ for non related siblings. The unshared environmental factors
associated with the matrix $E$ are assumed uncorrelated across siblings and thus
do not contribute to the between-siblings covariances in Equation (6).

The complete $6 \times 6$ matrix $\Sigma$ of expected covariances among all observed
variables, with rows and columns corresponding to the three observations for
sibling 1 followed by the three observations for sibling 2 (in the order viq_1,
gpa_1, cpl_1, viq_2, gpa_2, cpl_2) can be written as the partitioned matrix
$$\Sigma = \begin{bmatrix}
W(A + C + E)W' & W'(k \otimes A + C)W \\
W(k \otimes A + C)W' & W(A + C + E)W'
\end{bmatrix}$$
(7)

With only small notational changes, this is the way the expected covariance
matrices that are fitted to observed covariances $S$ are represented in the Mx
script. The parameters implicit in (7) are estimated by maximum likelihood to
simultaneously fit the observed covariance matrices for the six groups of siblings.
The Mx script used to estimate the parameters of the model and the output are
shown in separate documents.