The Binomial Distribution

Example. Suppose we have a large number of cards each marked with one of the letters A,B,C,D,E. I draw five cards (independently) from this deck and ask someone to guess which letter is on each of them. What is the probability that they get exactly two of them correct?

Context: This is a model for an ESP experiment (p. 292 of text). If someone guesses the right answer more often than they should do by chance, this is sometimes taken as evidence of ESP. However, maybe they just got lucky — we can assess this better if we know the probabilities of different outcomes that might occur by chance.
In this case let's label the possible outcomes of the individual experiments either S (success) or F (failure).

How many ways can we do this five times and get exactly two S’s?

SSFFF  SFSFF
SFFSF  SFFFS
FSSFF  FSFSF
FSFFS  FFSSF
FFSFS  FFFSS

There are 10 ways to do it. Each of these has the same probability, which is \(0.2 \times 0.2 \times 0.8 \times 0.8 \times 0.8 = .02048\).

Therefore the answer is \(10 \times .02048 = .2048\).
This is not an especially small probability. But suppose the person got it right 4 times out of 5. In this case the possible orders are

SSSSF
SSSFS
SSFSS
SFSSS
FSSSS

Five possible outcomes, each with a probability $0.2 \times 0.2 \times 0.2 \times 0.2 \times 0.8 = .00128$.

The overall probability is $5 \times .00128 = .0064$.

This seems small enough to be suspicious — maybe he or she really does have ESP!
The Math

Suppose there are $n$ experiments, and the probability that someone gets the right answer on any given experiment is $p$. So in the first example above, $n = 5$ and $p = 0.2$. Let $X$ be the number of correct results — this is a random variable (discrete). The formula (page 293 of the text) says

$$P(x) = \frac{n!}{x!(n-x)!} p^x (1 - p)^{n-x}.$$ 

Here $n!$ is a special symbol known as factorial $n$. Factorial means you multiply together all the numbers from 1 to $n$. So

$$5! = 1 \times 2 \times 3 \times 4 \times 5 = 120.$$
\[
P(x) = \frac{n!}{x!(n - x)!} p^x (1 - p)^{n-x}.
\]

The second part of the formula gives the probability of a specific sequence of S’s and F’s — so if there are exactly \(x\) S’s and \(n - x\) F’s, we calculate the probability as \(p \times p \times p \times \ldots \times p\) (\(x\) times) multiplied by \((1 - p) \times (1 - p) \times (1 - p) \times \ldots \times (1 - p)\) (\(n - x\) times) for an answer of \(p^x (1 - p)^{n-x}\).
\[ P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}. \]

The first part of the formula is the number of possible ways of arranging the S’s and F’s. For example, with \( n = 5 \) and \( x = 2 \) we find

\[
\frac{n!}{x!(n-x)!} = \frac{5!}{2!3!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 3 \times 2 \times 1} = \frac{120}{12} = 10
\]

exactly as we got before by counting. However if \( n \) and \( x \) were large it would be very tedious to write out all the possible combinations of S’s and F’s, whereas this formula is relatively easy to apply.
So in the case $n = 5$, $x = 2$, $p = 0.2$ we have

$$P(x) = \frac{5!}{2!3!} \times 0.2^2 \times 0.8^3$$

$$= 10 \times 0.04 \times 0.512$$

$$= 0.2048.$$ 

Or with $n = 5$, $x = 4$, $p = 0.2$ we have

$$P(x) = \frac{5!}{4!1!} \times 0.2^4 \times 0.8^1$$

$$= \frac{5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1 \times 1} \times 0.0016 \times 0.8$$

$$= 5 \times 0.0016 \times 0.8$$

$$= .0064.$$