More on Detection and Attribution

- Levine and Berliner (J. Clim 12, 564–574, 1999)
- Levine, Berliner and Shea (J. Clim 13, 3805–3820, 2000)
Optimal Signal Detection


Suppose the observed climate signal $\Psi$ satisfies

$$\Psi = \Psi^S + \tilde{\Psi}$$

interpreted as “signal+noise”. In practice, we usually assume both $\Psi$ and $\Psi^S$ are in fact *anomalies* from some reference time period. We also assume

- $\Psi^S = \sum_{i=1}^{p} a_i g_i$ where $g_1, ..., g_p$ are $p$ known signal patterns and $a_1, ..., a_p$ are unknown weights; also write $\Psi^S = G \mathbf{a}$.
- $\tilde{\Psi}$ is a vector of “errors” with mean 0 and covariance matrix $\mathbf{C}$. 


Optimal fingerprints (Hasselmann)

- $d_i = f_i^T \Psi$ is “detector” (and $f_i$ is “fingerprint”)

- $d^S = (d_1^S, ..., d_p^S) = (f_1^T \Psi, ..., f_p^T \Psi)$

- Fingerprints $f_i$ constructed to maximize signal to noise ratio

\[ \rho^2(d^S) = (d^S)^T D^{-1} d^S, \]

$(i, j)$ entry of $D$ is $\text{Cov}(\tilde{d}_i, \tilde{d}_j) = \text{Cov}(f_i^T \tilde{\Psi}, f_j^T \tilde{\Psi}) = f_i^T C f_j$.

- This optimization problem leads to $f_i^* = C^{-1} g_i$, and hence $d^* = G^T C^{-1} \Psi$.

- Statistical significance of the signal determined through $\rho^2(d^*)$. 
Alternative formulation (Levine & Berliner)

- Regression equation
  \[ \Psi = Ga + \tilde{\Psi} \]

- GLS estimates \( \hat{a} = (G^TC^{-1}G)^{-1}G^TC^{-1}\Psi \) and hence \( \hat{\Psi}^S = G\hat{a} \).

- Under Gaussian assumptions, \( \hat{a} \sim N[a, (G^T C^{-1}G)^{-1}] \).

- Test \( H_0 : a = 0 \) against \( H_a : a \neq 0 \): UMPI test of level \( \alpha \) rejects \( H_0 \) if
  \[ T = \Psi^T C^{-1}G(G^T C^{-1}G)^{-1}G^TC^{-1}\Psi > \chi_p^2(1 - \alpha). \]

- But \( T = \Psi^T f^*(G^T C^{-1}G)^{-1}(f^*)^T \Psi = \rho^2(d^*) \). Therefore, the two tests are equivalent.
If $C$ is unknown...

Use a sample estimate from historical data, but the $\chi^2_p$ distribution may be a poor approximation unless sample size $n$ is very large.

*Question: Is there an exact result, analogous to Hotelling’s $T^2$?*
Attribution

Suppose we reject the null hypothesis of the previous section, that there is no climate change signal above baseline. We cannot immediately attribute the signal to any of $g_1, ..., g_p$. Instead:

- Two estimates — $\hat{a}_M$ and $\hat{a}_{\text{obs}}$ from climate model and observations

- Assume distributions $\hat{a}_M \sim N[a_M, C_M]$ and $\hat{a}_{\text{obs}} \sim N[a, C_{\text{obs}}]$. (N.B. $C_{\text{obs}} = (G^T C^{-1} G)^{-1}$.)

- Test $H_0: a = a_M$ against $H_0: a \neq a_M$

- Under $H_0$ we have $a_{\text{obs}} - a_M \sim N[0, C_{\text{obs}} + C_M]$.

- Hence reject $H_0$ if

\[ (a_{\text{obs}} - a_M)^T (C_{\text{obs}} + C_M)^{-1} (a_{\text{obs}} - a_M) > \chi^2_p(1 - \alpha). \]
Issues

- $C$ and $C_M$ unknown — estimate from climate model runs, then construct consistency test using Hotelling’s $T^2$ distribution (but not clear exactly how this works)

- Null and alternative hypotheses the wrong way round??? Usually one tries to control the probability of falsely rejecting the null hypothesis, but that suggests the null and alternative should be interchanged

- Test is usually performed *only* if initial “detection” test rejects $a = 0$. But that makes it hard to assess true significance level.
Alternatives

1. Analogy with bioequivalence — test $H_0 : a \neq a_M$ against $H_a : a = a_M$. Idea of Brown, Casella and Hwang (JASA 1995). Define $\theta = a - a_M$, $\hat{\theta} = \hat{a} - \hat{a}_M$, $\Sigma = C_{\text{obs}} + C_M$. If $\hat{\theta} \sim N[\theta, \Sigma]$ then $H_0$ is rejected if and only if the confidence set

$$\left\{ \theta : (\theta^T \Sigma^{-1} \theta)^{1/2} \leq z_\alpha + (\theta^T \Sigma^{-1} \theta)^{-1/2} (\hat{\theta}^T \Sigma^{-1} \theta) \right\}$$

does not contain 0. (Need approximations for $\Sigma$ unknown.)

2. Simple v. simple test: $H_0 : a = 0$ versus $H_a : a = a_M$. For $C$ known, most powerful $\alpha$-level test rejects $H_0$ when

$$T_M = \frac{a_T M C_{\text{obs}}^{-1} \hat{a}_{\text{obs}}}{(a_T M C_{\text{obs}}^{-1} a_M)^{1/2}} > z_\alpha.$$
Other general points about these procedures

- **Covariance Estimation.** Many difficulties in practice — often use long-run control model runs to estimate $C$ but even this may not be adequate. (*Note* — subsequent to this paper, Allen and Tett took this point further by advocating two independent estimates of $C$, one for estimation and the other for model validation.)

- **Correlation-Based Detection.** There is an analogy with methods based solely on correlations, especially when $p = 1$.

- **Statistical v. Practical Significance.** Possible that $a \approx 0$ but $H_0$ still rejected.
Bayesian Climate Change Assessment

Levine, Berliner and Shea (J. Clim 13, 3805–3820, 2000)

Idea: Present and alternative Bayesian viewpoint of detection and attribution procedures
Introduction

Fundamental model of structure

\[ \text{Observations} = \ g \times a + \ \text{noise}. \]

- 1961–1998 observational data (East Anglia) on $5^\circ \times 5^\circ$ grid
- Climate model output (CSM) interpolated to same grid
  - 300-year control run for estimating natural climate variability
  - 120-year CO$_2$ forced run in combination with control run to construct CO$_2$ fingerprint, written $g$.

See Figure 1 for $g$. 
FIG. 1. Fingerprint pattern developed from NCAR CSM control and CO$_2$ runs on a $5^\circ \times 5^\circ$ grid.
Reasons for adopting a Bayesian approach

- Inability to adopt conventional practices for a cause and effect analysis, e.g. randomized trials

- Need to account for uncertainties in observational data as well as in our knowledge of the how the climate system works (though the present discussion is incomplete, e.g. in its treatment of spatial and temporal correlations)

- Standard significance testing does not account for distinction between statistical and practical significance; this is more easily achieved within a Bayesian framework (this has been a theme of much of Jim Berger’s work on Bayesian approaches to hypothesis testing)
Likelihood Function

- Observational data — yearly averages for 38 years, \( n = 2592 \) spatial grid points, do not interpolate missing observations

- \( T_t \) is \( n \)-vector of “true” anomalies in year \( t = 1, ..., m \); \( T \) is \( mn \)-vector formed by concatenating all \( T_t \).

- \( Y_t \) is \( n_t \)-vector of observed anomalies in year \( t \) \( (n_t \leq n \) because of missingness); \( Y \) is concatenated vector.

- Write \( Y_t | T_t, D_t \sim N[X_tT_t, D_t] \) where \( X_t \) is \( n_t \times n \) incidence matrix. Similarly \( Y | T, D \sim N[XT, D] \).

- Suppose \( T_t | a, g, \Sigma \sim N[ga, \Sigma] \).

- Unconditionally, \( Y | a, g, D, \Sigma \sim N[XGa, D + X\Sigma X^T] \) (\( G \) is vector of \( m \) replicates of \( g \); \( \Sigma \) is space-time covariance matrix, assumed separable)
Some simplifications

Do not attempt to model uncertainties in $D$, $\Sigma$ or $g$: concentrate on $a$.

Standard GLS estimator

$$\hat{a}(Y) = \frac{H^T V^{-1} Y}{H^T V^{-1} H}$$

where $H = X G$ and $V = D + X^T \Sigma X$; distribution of $\hat{a}(Y)$ is

$$\hat{a}(Y) \sim N[a, \sigma^2]$$

where $\sigma^2 = (H^T V^{-1} H)^{-1}$. 
Prior distribution for $a$.

Assume a mixture prior:

$$\pi(a) \sim pN[0, \tau^2] + (1 - p)N[\mu_A, \tau^2_A]$$

Estimates of $\tau^2, \tau^2_A, \mu_A$ derived from climate model runs under both control and forced conditions.
Posterior distribution for $a$.

Mixture prior leads to mixture posterior

$$
\pi(a|\hat{a}) = p(\hat{a})N[\mu(a|\hat{a}), \tau^2(a|\hat{a})] + \{1 - p(\hat{a})\}N[\mu_A(a|\hat{a}), \tau^2_A(a|\hat{a})],
$$

$$
\mu(a|\hat{a}) = \frac{\tau^2}{\tau^2 + \sigma^2}\hat{a},
$$

$$
\tau^2(a|\hat{a}) = \frac{\tau^2\sigma^2}{\tau^2 + \sigma^2},
$$

$$
\mu_A(a|\hat{a}) = \frac{\tau^2_A}{\tau^2_A + \sigma^2}\hat{a} + \frac{\sigma^2}{\tau^2_A + \sigma^2}\mu_A,
$$

$$
\tau^2(a|\hat{a}) = \frac{\tau^2_A\sigma^2}{\tau^2_A + \sigma^2},
$$

and $p(\hat{a})$ given by

$$
\left[1 + \left(\frac{1 - p}{p}\right)\sqrt{\frac{\tau^2 + \sigma^2}{\tau^2_A + \sigma^2}}\exp\left\{-\frac{1}{2}\left(\frac{(\hat{a} - \mu_A)^2}{\tau^2_A + \sigma^2} - \frac{\hat{a}^2}{\tau^2 + \sigma^2}\right)\right\}\right]^{-1}.
$$
“Detection and Attribution”

Define “detection set” $D$ (around $a = 0$) and “attribution set” $A$ (around $a = \mu_A$) to represent subsets of physical interest, then

- “Detection” is said to occur if $\Pr\{a \in D|\hat{a}\}$ is small,

- “Attribution” is said to occur if both (i) $\Pr\{a \in A|\hat{a}\}$ is large, and (ii) $1 - p(\hat{a})$ is large.
“Robust Bayesian” viewpoint

Some prior parameters uncertain, especially $\mu_A$, $\tau_A$, so consider families of priors

$$\Gamma = \{ \pi(\alpha) : \mu_\ell \leq \mu_A \leq \mu_u; \ \tau_\ell \leq \tau_A \leq \tau_u \}.$$  

It's possible to calculate the maximum and minimum of $p(\hat{\alpha})$ over these subsets.
Results


Here used prior $p = 0.5$.

Fig. 2: Prior, likelihood and posterior distribution for $a$, for each of four time periods.

Table 1: Numerical summaries of posterior distribution

Conclusion: As $p(a|\hat{a})$ moves closer to 0, stronger evidence that climate model is correct.
Fig. 2. Likelihood function, prior distribution, and posterior distribution of $a$ using our NCAR CSM fingerprint. For each of the time periods (a)–(d): (left) the likelihood (dotted line) and prior distribution components [anthropogenic CO$_2$ forcing (solid line); no anthropogenic impacts (dashed line)], and (right) the posterior mixture distribution (11).
TABLE 1. For each of the four time periods a–d: GLS estimate $\hat{a}$ of $a$ and associated standard deviation $\sigma$; mean and associated standard deviation of $a$ under the first component (no climate change) of the posterior mixture in (11); posterior mean and associated standard deviation under the second component (CO$_2$ forcing) of the posterior mixture in (11); posterior weight, $p(\hat{a})$, assuming prior weight $p = 0.5$, $\tau = 0.02$, $\mu_a = 0.17$, and $\tau_a = 0.05$.

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<td>$p(\hat{a})$</td>
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Fig 3: Posterior probability of $p(\hat{a})$ versus prior probability $p$

Figs 4 and 5: Detection and attribution results based on $D = (-0.05, 0.05)$, $A = (\mu_A - 0.05, \mu_A + 0.05)$.

Table 2: Robust Bayesian results

Even for time period (b), results are much stronger in favor of both detection and attribution than for time period (a). Results become overwhelming for time periods (c) and (d).
Fig. 3. Posterior mixture probability $p(d)$ versus the prior mixture probability $p$ based on two data subsets: (top) 1961–98 and (bottom) 1970–98. The solid line in each denotes the posterior probability computed from the data. The dotted lines denote the upper and lower bounds on the posterior probabilities from the robust Bayesian analysis.

Fig. 4. Posterior probability $Pr(a \in [-0.05, 0.05]|d)$ as a function of the prior probability $p$ based on two time periods: (top) 1961–98 and (bottom) 1970–98. The solid lines in each denotes the posterior probabilities for the primary prior. The dotted lines give the upper and lower bounds on these probabilities from the robust Bayesian analysis.
Fig. 5. Posterior probability $\Pr(a \in [\mu_A - 0.05, \mu_A + 0.05] | d)$ as a function of the prior probability $p$ based on two time periods: (top) 1961–98 and (bottom) 1970–98. The solid lines in each denotes the posterior probabilities for the data. The dotted lines give the upper and lower bounds on these probabilities from the robust Bayesian analysis.
TABLE 2. Upper and lower bounds on data-based contributions to the odds ratio in favor of the CO$_2$ forced posterior model for each of the four data subsets.

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<td>$M$</td>
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<td>6283.24</td>
<td></td>
<td>71 150</td>
<td>43 524</td>
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Results based on alternative fingerprint from paper of Santer et al (1995)

Fig 6: Graphic of $g$ in this case

Table 3: Compare $\hat{a}$ for two models (those for Santer model consistently smaller, but this is not surprising when comparing two different vectors $g$)

Figs. 7–8: Posterior probabilities (less definitive for whole time period, but the Santer fingerprint leads to very strong attribution results in time period (d))
FIG. 6. Fingerprint pattern generated by Santer et al. (1995), after their Fig. 3b, on a $5^\circ \times 5^\circ$ grid.
Table 3. GLS estimates of $\alpha$ and associated standard deviations for each of the four data subsets using the Santer et al. (1995) fingerprint.

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<td>$\sigma$</td>
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<td>$\sigma$</td>
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<td>Santer</td>
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<td>0.017</td>
<td>0.062</td>
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<td>CSM</td>
<td>0.0589</td>
<td>0.022</td>
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Fig. 7. Posterior probability \( \Pr(a \in [-0.05, 0.05]|d) \) as a function of the prior probability \( p \) based on two time periods: (top) 1961–98 and (bottom) 1970–98. The solid lines in each denotes the posterior probabilities for the data. The dotted lines give the upper and lower bounds on these probabilities from the robust Bayesian analysis.

Fig. 8. Posterior probability \( \Pr(a \in [\mu_d - 0.05, \mu_d + 0.05]|d) \) as a function of the prior probability \( p \) based on two time periods: (top) 1961–98 and (bottom) 1970–98. The solid lines in each denotes the posterior probabilities for the data. The dotted lines give the upper and lower bounds on these probabilities from the robust Bayesian analysis.
Comparison with traditional results

See Table 4

Traditional results more likely to reject $H_0$ when $\hat{a}$ very close to 0 (so Bayesian procedure more conservative for detection)

But situation “somewhat reversed” for attribution.
TABLE 4. Significance probabilities for traditional, non-Bayesian detection and attribution test results. Detection is suggested by very small values; attribution may be suggested by moderate or large values.

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<td>0.000</td>
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• Develop statistical data models to combine ensembles from multiple climate models in a fashion that accounts for uncertainty

• “Superensemble experimental design”

  “Superensembles” means ensembles from multiple climate models

• Bayesian climate modeling and forecasting

• Forecasting of future hemispheric monthly-averaged surface temperatures under different SRES scenarios
Outline of approach (univariate)

Suppose we have scalar $X$, and let $\mathbf{Y}_m$ be a vector of $n_m$ model runs for the same variable from model $m$, $m = 1, \ldots, M$.

Construct distribution of $\mathbf{Y}_m$ given $X$ as follows:

- Given $X$, generate $\beta|X \sim N[X, \sigma_\beta^2]$.
- Also generate "model biases" $b_1, \ldots, b_M$ such that $b_m \sim N[\mu_m, \sigma_{b_m}^2]$ (independent for different $m$, though this assumption could be relaxed)
- $\mathbf{Y}_m|\beta, b_m \sim N[(\beta + b_m)1_m, \sigma_{Y_m}^2 \mathbf{I}_{n_m \times n_m}]$
- Hence combine to generate conditional density $[\mathbf{Y}|X]$ in terms of parameters $\sigma_\beta^2, \mu_m, \sigma_{b_m}^2, \sigma_{Y_m}^2, m = 1, \ldots, M$ (prior distributions on all of those)
- Also have a model for evolution of $X$ in time (e.g. autoregressive time series model)
Multiple climate variables

Extend the univariate model using Kronecker products, e.g.

\[
\begin{align*}
\beta | X & \sim N[X, \Sigma \beta], \\
b_m & \sim N[\mu_m, \Sigma b_m], \\
vec(Y_m) | \beta, b_m & \sim N[1_{nm} \otimes (\beta + b_m), I_{nm \times nm} \otimes \Sigma Y_m].
\end{align*}
\]
Outline of Bayesian strategy:

Use superscripts $p$ and $f$ to denote “present” and “future”.

We have models for

- $[X^f | X^p]$ (AR assumption)
- $[Y^p | X^p]$
- $[Y^f | X^f]$

- Assume conditionally independent
- Hence deduce joint $[X^f, Y^p, Y^f | X^p]$
- Hence $[X^f | X^p, Y^p, Y^f]$ by Bayes theorem
Fig. 1. Hemispheric process model means. (a) NH and (b) SH process model mean parameters (denoted by $\alpha$’s in text). One hundred ensembles generated from the posterior distribution are plotted in each of the cases: observation period 1882–2001 (black) and the forecast period 2002–97 for SRES scenarios A1B (red), B1 (blue), and A2 (green), respectively.
FIG. 2. Hemispheric process autoregression parameters. (a) NH and (b) SH process model autoregression coefficients (denoted by $\eta$’s in text). One hundred ensembles generated from the posterior distribution are plotted in each of the cases: observation period 1882–2001 (black) and the forecast period 2002–97 for SRES scenarios A1B (red), B1 (blue), and A2 (green), respectively.
Fig. 3. Hemispheric process variances. (a) NH and (b) SH process variances (denoted by $\sigma^2_i$'s in text). One hundred ensembles generated from the posterior distribution are plotted in each of the cases: observation period 1882–2001 (black) and the forecast period 2002–97 for SRES scenarios A1B (red), B1 (blue), and A2 (green), respectively.
Fig. 4. NH temperature reconstructions and forecasts. One hundred ensembles generated from the posterior distribution are plotted in each of the cases: observation period 1882–2001 (black; observations are also plotted here) and the forecast period 2002–97 for SRES scenarios A1B (red), B1 (blue), and A2 (green), respectively.
Fig. 5. SH temperature reconstructions and forecasts. One hundred ensembles generated from the posterior distribution are plotted in each of the cases: observation period 1882–2001 (black) (observations are also plotted here) and the forecast period 2002–97 for SRES scenarios A1B (red), B1 (blue), and A2 (green), respectively.
Fig. 6. NH temperature reconstructions. Same as in Fig. 4 but based on a model assuming constant biases for all times in the forecast period.
Table 1. Means (std dev) of intercept and slope of regression of means on CO₂ based on 5000 realizations from the posterior distribution.

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<td>10.256 (0.103)</td>
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<td></td>
<td></td>
<td>2092–97</td>
</tr>
</tbody>
</table>
TABLE 3. Means (std dev) of model error covariance matrices based on 5000 realizations from the posterior distribution, where $\Sigma_{y_1}$ is for PCM and $\Sigma_{y_2}$ is for CCSM.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>$\Sigma_{y_1}$</th>
<th>NH variance</th>
<th>SH variance</th>
<th>Covariance</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1B</td>
<td>0.144 (0.003)</td>
<td>0.068 (0.002)</td>
<td>0.012 (0.002)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.338 (0.015)</td>
<td>0.206 (0.010)</td>
<td>0.101 (0.010)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.045 (0.002)</td>
<td>0.032 (0.002)</td>
<td>0.001 (0.002)</td>
<td></td>
</tr>
<tr>
<td>B1</td>
<td>0.174 (0.005)</td>
<td>0.089 (0.003)</td>
<td>0.014 (0.002)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.326 (0.016)</td>
<td>0.153 (0.007)</td>
<td>0.022 (0.006)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.085 (0.004)</td>
<td>0.048 (0.002)</td>
<td>0.001 (0.002)</td>
<td></td>
</tr>
<tr>
<td>A2</td>
<td>0.167 (0.003)</td>
<td>0.085 (0.002)</td>
<td>0.012 (0.003)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.095 (0.007)</td>
<td>0.055 (0.003)</td>
<td>−0.001 (0.003)</td>
<td></td>
</tr>
</tbody>
</table>
Detection and Attribution Analysis for Precipitation

Reference: Zhang et al., Nature 2007 (see earlier detection and attribution notes)
Datasets

Latitude-band averages

- 75 years
- 11 ten-degree latitude bands (from 40°S to 70°N)
- One observational dataset
- Several forced model runs under different forcings (ANT, ALL, NAT)
- Control runs — typically 6 control runs of 1000 years broken up into 75-year segments
Traditional D&A Analysis

1. For each of 11 latitude bands, linear trend is calculated by OLS, including control runs.

2. Covariance matrix from control run used to define the weights for EOF decomposition.

3. Truncate the EOFs, e.g. by Allen-Tett test (anticipate \( \approx 5 \) EOFs retained).

4. Same formation of linear trends and EOFs applied to forced model runs to develop a set of comparison vectors.

5. Optimal weights for D&A defined by regressing observational data EOFs on forcing model EOFs. Calculate standard errors, tests of significance, etc.
Estimation of the Covariance Matrix

The data we have available consists of 17 runs of 75 years’ data from control runs of HADCM3. The data are 11-dimensional because of the 11 latitude bands. The covariance matrix $C$ is defined to be the $11 \times 11$ covariance matrix of estimators of linear trends over the 75 years.

Possible methods of estimating $C$ include:
Method 1

Calculate linear trends for each of the 11 latitude bands and each of the 17 runs; then estimate \( C \) (say, as \( \hat{C}_1 \)) as the sample covariance matrix

\[
\hat{C}_1 = \frac{(B - \bar{B})^T(B - \bar{B})}{16}
\]  

(1)

where \( B \) is the 17 × 11 matrix of linear trend estimates and \( \bar{B} \) is the mean of \( B \) over all rows.
Method 2

Suppose \( Y_{ijk} \) is observation in year \( i \) from latitude band \( j \) and control model run \( k \). Suppose that, for each \( i \) and \( k \), \( \{Y_{ijk}, \ j = 1, \ldots, 11\} \) is an independent vector from \( N[0, V] \), where \( V \) is the \( 11 \times 11 \) covariance matrix associated with a single year of control data. Linear trend estimator for region \( j \) and model run \( k \) given by

\[
B_{jk} = \sum_{i=1}^{75} w_i Y_{ijk} \tag{2}
\]

for weights \( w_i \). Then for any two regions \( j, j' \),

\[
\text{Cov}(B_{jk}, B_{j'k}) = \sum_{i=1}^{75} w_i^2 V_{jj'}. \tag{3}
\]

If \( \tilde{V} \) is sample covariance matrix, estimate

\[
\tilde{C}_2 = (\sum w_i^2)\tilde{V} \tag{4}
\]
**Method 3**

Correct for autocorrelation in method 2.

Assume \( h \)-order autocovariances negligible for \(|h| > H\) some given \(H\). Then we can modify (3) to

\[
\text{Cov}(B_{jk}, B_{j'k}) = \sum_{h=-H}^{H} \sum_{i=1}^{75} w_i w_{i+h} \text{Cov}(Y_{ijk}, Y_{(i+h)j'k}) \tag{5}
\]

where \( w_{i+h} \) is defined to be 0 if \( i + h < 1 \) or \( i + h > 75 \).

If \( \hat{\nu}_{hjj'} \) denotes sample covariance of \( Y_{ijk} \) and \( Y_{(i+h)j'k} \) (computed from all \( 17 \times 75 \) control years of data) then this implies a covariance estimator \( \hat{C}_3 \) with entries

\[
\hat{C}_3(j, j') = \sum_{h=-H}^{H} \sum_{i=1}^{75} w_i w_{i+h} \hat{\nu}_{hjj'}. \tag{6}
\]
Looking for autocorrelations in the control runs

Sample autocorrelations and partial autocorrelations were calculated from the control data. They were considered statistically significant if outside the range $\pm \frac{2}{\sqrt{n}}$.

These results suggest that significant autocorrelations do not persist beyond lag 2. However, as a test, the estimator $\hat{C}_3$ was computed for each of $H = 1, 2, 3, 4, 5$. 
Testing one estimated covariance matrix against another

Advantage of $\hat{C}_1$ — makes no assumption at all about the autocorrelation structure, but it’s based on a small sample (17).

$\hat{C}_2$ or $\hat{C}_3$ — far larger sample size, but maybe incorrect assumptions. Suggests a hypothesis test that of either $H_0: C = \hat{C}_2$ or $H_0: C = \hat{C}_3$, treating $\hat{C}_1$ as a sample covariance matrix.

Korin (1968) proposed likelihood ratio test statistic

$$L = N \left\{ \log \left( \frac{|\Sigma_0|}{|S|} \right) - p + \text{tr}(S\Sigma_0^{-1}) \right\}$$

where $S$ is a $p \times p$ sample covariance matrix with $N$ degrees of freedom, and $\Sigma_0$ is the null hypothesis value of the true covariance matrix. Sampling distributions by Bartlett correction to the $\chi^2$ test, an alternative approximation proposed by Korin, or simulation.
Application

\( \hat{C}_1 \) against \( \hat{C}_2 \) (as null hypothesis): reject under Bartlett’s approximation, accept under Korin’s or simulation (p-values .047, .077, .11) but maybe not a clear-cut result.

However for \( \hat{C}_1 \) against \( \hat{C}_3 \) (as null hypothesis), with any of \( H = 1, 2, 3, 4, 5 \), accept null hypothesis. As a specific example, for \( H = 2 \), Bartlett’s, Korin’s and the simulation test yield \( p \)-values 0.33, 0.42 and 0.47.

**Conclusion:** \( \hat{C}_3 \) seems fine, and the actual value of \( H \) doesn’t matter.
On the other hand, where we see real differences among the estimators is in computing their inverses. As an example, the diagonal entries of $\hat{C}^{-1}_1$ are:

\[
\begin{align*}
202.1 & 	frac{1}{2} 50.5 & 8.1 & 7.5 & 4.9 & 44.4 & 18.4 & 55.3 & 124.0 & 338.0 & 136.3 \\
\end{align*}
\]

The corresponding values for $\hat{C}^{-1}_3$ with $H = 0 - 5$ are:

\[
\begin{align*}
H=0: & \quad 29.9 \quad 20.6 \quad 5.1 \quad 4.9 \quad 5.5 \quad 11.1 \quad 11.7 \quad 27.4 \quad 50.5 \quad 29.7 \quad 24.5 \\
H=1: & \quad 27.8 \quad 19.5 \quad 4.6 \quad 4.5 \quad 5.1 \quad 10.2 \quad 11.7 \quad 28.6 \quad 44.2 \quad 27.3 \quad 22.1 \\
H=2: & \quad 28.7 \quad 20.6 \quad 4.8 \quad 4.6 \quad 5.1 \quad 9.7 \quad 11.5 \quad 31.6 \quad 40.7 \quad 28.4 \quad 24.8 \\
H=3: & \quad 28.5 \quad 19.0 \quad 4.7 \quad 4.5 \quad 5.2 \quad 9.3 \quad 11.0 \quad 33.4 \quad 39.5 \quad 29.1 \quad 26.1 \\
H=4: & \quad 29.1 \quad 20.5 \quad 4.5 \quad 4.4 \quad 5.2 \quad 8.9 \quad 11.1 \quad 33.5 \quad 39.8 \quad 29.2 \quad 25.4 \\
H=5: & \quad 30.7 \quad 20.1 \quad 4.4 \quad 4.4 \quad 4.6 \quad 8.8 \quad 11.4 \quad 31.3 \quad 40.9 \quad 28.6 \quad 25.1 \\
\end{align*}
\]

Evidently, there is much more stability in any of the versions of $\hat{C}^{-1}_3$ (raising the question that maybe no dimension reduction is actually needed??)
Results Applied to D&A
Analysis using full data matrix (11 PCs)

Analyses using $\hat{C}_1, \hat{C}_2, \hat{C}_3 (H = 2)$

<table>
<thead>
<tr>
<th>Models</th>
<th>$\hat{\beta}$</th>
<th>SE</th>
<th>$t$ statistic</th>
<th>P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>ALL</td>
<td>-0.043</td>
<td>0.354</td>
<td>-0.121</td>
<td>0.903</td>
</tr>
<tr>
<td>ANT</td>
<td>0.695</td>
<td>0.318</td>
<td>2.183</td>
<td>0.029</td>
</tr>
<tr>
<td>NAT</td>
<td>-1.701</td>
<td>0.479</td>
<td>-3.550</td>
<td>0.000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Models</th>
<th>$\hat{\beta}$</th>
<th>SE</th>
<th>$t$ statistic</th>
<th>P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>ALL</td>
<td>1.272</td>
<td>0.584</td>
<td>2.179</td>
<td>0.029</td>
</tr>
<tr>
<td>ANT</td>
<td>3.495</td>
<td>0.519</td>
<td>6.730</td>
<td>0.000</td>
</tr>
<tr>
<td>NAT</td>
<td>-3.950</td>
<td>0.736</td>
<td>-5.365</td>
<td>0.000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Models</th>
<th>$\hat{\beta}$</th>
<th>SE</th>
<th>$t$ statistic</th>
<th>P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>ALL</td>
<td>1.730</td>
<td>0.612</td>
<td>2.825</td>
<td>0.005</td>
</tr>
<tr>
<td>ANT</td>
<td>3.470</td>
<td>0.532</td>
<td>6.523</td>
<td>0.000</td>
</tr>
<tr>
<td>NAT</td>
<td>-4.279</td>
<td>0.777</td>
<td>-5.507</td>
<td>0.000</td>
</tr>
</tbody>
</table>
Analysis using 5 PCs

Analyses using $\hat{C}_1, \hat{C}_2, \hat{C}_3 (H = 2)$

<table>
<thead>
<tr>
<th>Models</th>
<th>$\hat{\beta}$</th>
<th>SE</th>
<th>$t$ statistic</th>
<th>P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>ALL</td>
<td>3.888</td>
<td>1.345</td>
<td>2.891</td>
<td>0.004</td>
</tr>
<tr>
<td>ANT</td>
<td>3.370</td>
<td>1.195</td>
<td>2.821</td>
<td>0.005</td>
</tr>
<tr>
<td>NAT</td>
<td>8.084</td>
<td>4.205</td>
<td>1.922</td>
<td>0.055</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Models</th>
<th>$\hat{\beta}$</th>
<th>SE</th>
<th>$t$ statistic</th>
<th>P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>ALL</td>
<td>3.300</td>
<td>0.941</td>
<td>3.508</td>
<td>0.000</td>
</tr>
<tr>
<td>ANT</td>
<td>3.486</td>
<td>0.968</td>
<td>3.601</td>
<td>0.000</td>
</tr>
<tr>
<td>NAT</td>
<td>2.396</td>
<td>1.902</td>
<td>1.259</td>
<td>0.208</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Models</th>
<th>$\hat{\beta}$</th>
<th>SE</th>
<th>$t$ statistic</th>
<th>P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>ALL</td>
<td>2.375</td>
<td>0.990</td>
<td>2.399</td>
<td>0.016</td>
</tr>
<tr>
<td>ANT</td>
<td>3.270</td>
<td>1.252</td>
<td>2.613</td>
<td>0.009</td>
</tr>
<tr>
<td>NAT</td>
<td>2.312</td>
<td>2.199</td>
<td>1.051</td>
<td>0.293</td>
</tr>
</tbody>
</table>
Conclusions (preliminary)

Using the full data matrix, there is a lot of discrepancy between the results based on $\hat{C}_1$ and either of $\hat{C}_2$ or $\hat{C}_3$, but the results comparing $\hat{C}_2$ and $\hat{C}_3$ are quite close.

Using 5 PCs, there is much better agreement between the three covariance models, but results are quite different from those using all 11 PCs.

Based on all 11 PCs with either $\hat{C}_2$ or $\hat{C}_3$, we may claim to have attribution for the all forcings model, but not for either ANT or NAT.