Problem 1: $[2 + 2 + 2 + (2 + 2) = 10 \text{ points}]$ Suppose that $\pi_0 = \pi^*$ and $u_0 = u^*$ are two constants. Consider the following model:

$$\pi_t = \alpha \pi_{t-1} + \beta u_{t-1} + \epsilon_t \text{ where } \epsilon_t \overset{i.i.d.}{\sim} (0, \sigma_{\epsilon\epsilon})$$  \hspace{1cm} (1)

for $t = 1, \ldots, T$. Answer the following questions:

(A) What is the effect of $u_{t-2}$ on $\pi_t$?

(B) Show that the model in equation (1) can be expressed as

$$\pi_t = \alpha^2 \pi_{t-2} + (\beta u_{t-1} + \beta \alpha u_{t-2}) + (\epsilon_t + \alpha \epsilon_{t-1}).$$  \hspace{1cm} (2)

Relate this to your answer in (A).

(C) Show that for any $k = 1, 2, \ldots, t$ the model in equation (1) can be written as

$$\pi_t = \alpha^k \pi_{t-k} + \beta \sum_{j=1}^{k} \alpha^{j-1} u_{t-j} + \sum_{j=1}^{k} \alpha^{j-1} \epsilon_{t-j}. $$  \hspace{1cm} (3)

What is the effect of $u_{t-k}$ on $\pi_t$? With the help of a path-diagram (arrows from one variable to another, as shown in the class), justify the answer that you just got.

(D) Suppose you run a regression of $\pi_t$ on $\pi_{t-2}$, $u_{t-1}$ and $u_{t-2}$. The regressors have true coefficients say $\delta_0, \delta_1$ and $\delta_2$. If the model in equation (1) is correct, what restrictions does that impose on $\delta_0, \delta_1$ and $\delta_2$? Again, if the model in equation (1) is correct, what problems do you expect in testing of hypotheses if you use the command reg $\pi$ L2.$\pi$ L.u L2.u for the above regression?

Answer:
(A) From our discussion in the class we know that the $u_{t-2}$ affects $\pi_t$ along the following route $u_{t-2} \xrightarrow{\beta} \pi_{t-1} \xrightarrow{\alpha} \pi_t$. Therefore, the effect of $u_{t-2}$ on $\pi_t$ is $\beta\alpha$.

(B) Equation (2) is obtained from equation (1) by substitution as follows:

$$
\pi_t = \alpha \pi_{t-1} + \beta u_{t-1} + \epsilon_t
$$

$$
= \alpha (\alpha \pi_{t-2} + \beta u_{t-2} + \epsilon_{t-1}) + \beta u_{t-1} + \epsilon_t
$$

$$
= \alpha^2 \pi_{t-2} + (\beta u_{t-1} + \beta \alpha u_{t-2}) + (\epsilon_t + \alpha \epsilon_{t-1})
$$

Of course, from here it is clear again that the effect of $u_{t-2}$ on $\pi_t$ is $\beta\alpha$.

(C) For $k = 1, 2$ you have the results in (1) and (2). For $k = 3$, by substitution once again, we obtain

$$
\pi_t = \alpha^2 \pi_{t-2} + (\beta u_{t-1} + \beta \alpha u_{t-2}) + (\epsilon_t + \alpha \epsilon_{t-1})
$$

$$
= \alpha^2 (\alpha \pi_{t-3} + \beta u_{t-3} + \epsilon_{t-2}) + (\beta u_{t-1} + \beta \alpha u_{t-2}) + (\epsilon_t + \alpha \epsilon_{t-1})
$$

$$
= \alpha^3 \pi_{t-3} + (\beta u_{t-1} + \beta \alpha u_{t-2} + \beta \alpha^2 u_{t-3}) + (\epsilon_t + \alpha \epsilon_{t-1} + \alpha^2 \epsilon_{t-2})
$$

$$
= \alpha^3 \pi_{t-3} + \beta \sum_{j=1}^{3} \alpha^{j-1} u_{t-j} + \sum_{j=1}^{3} \alpha^{j-1} \epsilon_{t-j}
$$

Similarly, with one more substitution you get the result for $k = 4$ and so on.

(D) If the model in equation (1) is true then by taking $k = 2$ in (3), we know that

$$
\delta_0 = \alpha^2, \ \delta_1 = \beta, \ \delta_2 = \beta \alpha \ \text{and, therefore,} \ \delta_0 = \frac{\delta_2}{\delta_1^2}
$$

is a restriction on the coefficients $\delta_0, \delta_1$ and $\delta_2$. Now about the regression. Note that,
if we denote that error in the model for the $t$th observation by $v_t$ then

$$v_t = \epsilon_t + \alpha \epsilon_{t-1}.$$  

Therefore, $Cov(v_t, v_{t-1}) = Cov(\epsilon_t + \alpha \epsilon_{t-1}, \epsilon_{t-1} + \alpha \epsilon_{t-2}) = \alpha \sigma_{\epsilon \epsilon}$. (Actually \{v_t\} is an MA(1) process.) Therefore, the usual standard errors for the coefficient estimates are going to be incorrect. You need to use the "robust" standard errors.