1. Consider a parameterized curve

\[ x = f(v) \quad z = g(v) \quad v \in (a, b) \]

in the \(xz\)-plane, with \(f(v) > 0\) everywhere. Let \(S\) be the surface of revolution obtained by revolving the curve around the \(z\)-axis. If \(u \in [0, 2\pi]\) is the angular coordinate, calculate the induced metric on \(S \subset \mathbb{R}^3\) in local coordinates \((u, v)\).

2. Consider the rotationally symmetric metric

\[ dr^2 + (\phi(r))^2 d\theta^2 \]

where \(\phi : [0, b) \to [0, \infty)\) is an analytic function with \(\phi(0) = 0\) and \(\phi(r) > 0\) for \(r > 0\). To check for smoothness of the metric at the origin we need to change from polar to Cartesian coordinates:

\[ x = r \cos \theta \quad y = r \sin \theta \]

Write the metric in \((x, y)\)-coordinates and show that the components \(g_{xx}, g_{xy} = g_{yx},\) and \(g_{yy}\) are smooth if and only if

\[ \phi'(0) = 1 \quad \text{and} \quad \phi^{(even)}(0) = 0. \]

**Hint:** Write \(\phi(r) = r \psi(r)\). Calculate \(g_{xx}, \) etc., in terms of \(\psi(r)\). Show that smoothness at \(r = 0\) implies \(\psi(0) = 1\) and \(\psi^{(odd)}(0) = 0\).

3. Setting \(\phi(r) = \sinh(r)\) in the previous problem, with \(r \in [0, \infty)\), gives a rotationally symmetric metric on \(\mathbb{R}^2\). Show that this is isometric to the Poincaré disc, i.e., the (open) unit disc in \(\mathbb{R}^2\) with the metric

\[ g = \frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2}. \]

4. A **proper affine function** is a map \(f : \mathbb{R} \to \mathbb{R}\) given by \(f(t) = bt + a\) for \(a \in \mathbb{R}\) and \(b \in \mathbb{R}_{>0}\). The set of proper affine functions forms a Lie group: as a smooth manifold it is just the upper half plane \(\{(a, b) \in \mathbb{R}^2 | b > 0\}\), and the group operation is given by composition.

a) Find a formula for the product \((a, b)(c, d)\).

b) The identity element is \(e = (0, 1)\). Find a formula for the left-invariant metric which takes the value

\[ g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

at \(e\).