A lottery is a pair of outcomes and their respective probabilities:

\[ \ell = \left( (x_1, x_2, \ldots, x_n), (p_1, p_2, \ldots, p_n) \right), \]

where \( x_k \in \mathbb{R} \) and \( p_k \geq 0 \) for all \( k = 1, \ldots, n \) and also \( p_1 + p_2 + \ldots + p_n = 1 \).
The lottery that gives outcome $x$ with probability 1 (with certainty) is denoted:

$$\delta_x = ((x), (1)).$$

The expected value of the $\ell_1 = ((x_1, x_2, \ldots, x_n), (p_1, p_2, \ldots, p_n))$ is:

$$E[\ell_1] = p_1 \cdot x_1 + p_2 \cdot x_2 + \ldots p_n \cdot x_n = \sum_{i=1}^{n} p_i \cdot x_i;$$

and variance this lottery is

$$\text{Var}[\ell_1] = p_1 \cdot (x_1 - E[\ell_1])^2 + p_2 \cdot (x_2 - E[\ell_1])^2 + \ldots p_n \cdot (x_n - E[\ell_1])^2 = \sum_{i=1}^{n} p_i \cdot (x_i - E[\ell_1])^2.$$
Composition of Lotteries

Given two lotteries, \( \ell_1 = ((x_1, x_2, \ldots, x_n), (p_1, p_2, \ldots, p_n)) \) and \( \ell_2 = (y_1, y_2, \ldots, y_m), (q_1, q_2, \ldots, q_n) \) and a number \( 0 < \alpha < 1 \), one can create a compound lottery by choosing \( \ell_1 \) with probability \( \alpha \) and \( \ell_2 \) with probability \( 1 - \alpha \).

\[
\ell = \alpha \ell_1 \oplus (1 - \alpha) \ell_2 = \\
= ((x_1, x_2, y_1, y_2), (\alpha p, \alpha (1 - p), (1 - \alpha) q, (1 - \alpha)(1 - q)).
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Composition of Lotteries

The compound lottery $\ell$ plays $\ell_1$ with probability $\alpha$ and $\ell_2$ with probability $\ell_2$:

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Given to lotteries $\ell_a$ and $\ell_b$ such that a decision maker (DM) chooses $\ell_a$ over $\ell_b$,
the following statements are equivalent:

- The DM judges $\ell_a$ no worst than $\ell_b$ (everyday language);
- The DM prefers $\ell_a$ to $\ell_b$ (economics language);
- $\ell_a \succeq_{DM} \ell_b$ (mathematics language).

For simplicity we write:

- $\ell_a > \ell_b$ when $\ell_a \succeq \ell_b$ but $\ell_b \not\succeq \ell_a$ (strict preference);
- $\ell_a \sim \ell_b$ when $\ell_a \succeq \ell_b$ and $\ell_b \succeq \ell_a$ (indifference).
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Preferrences Over Lotteries

- A preference of the DM, $\succeq_{DM}$, over the set of lotteries is just the DM’s ranking of lotteries.
- We wish (for convenience) a numerical score that reflects the DM’s ranking.
von Neuman & Morgenstern’s Assumptions:

**Completeness** For any two lotteries $\ell_1$ and $\ell_2$,

$\ell_1 \succeq \ell_2$ and/or $\ell_2 \succeq \ell_1$.

**Transitivity** For any lotteries $\ell_1$, $\ell_2$ and $\ell_3$,

if $\ell_1 \succeq \ell_2$ and $\ell_2 \succeq \ell_3$ then $\ell_1 \succeq \ell_3$.

**Continuity** If $\ell_1 \succeq \ell_2 \succeq \ell_3$ then exists $p \in [0, 1]$ such that

$\ell_2 \sim p\ell_1 \oplus (1 - p)\ell_3$.

**Independence** If $\ell_1 \succ \ell_2$ then for any $\ell_3$ and any $0 < p < 1$,

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\[ p\ell_1 \oplus (1 - p)\ell_3 > p\ell_2 \oplus (1 - p)\ell_3. \]
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If $\succeq$ satisfy all of the above, there exists $u : \mathbb{R} \to \mathbb{R}$ such that
$$((x_1, x_2, \ldots, x_n), (p_1, p_2, \ldots, p_n)) \succ (y_1, y_2, \ldots, y_m), (q_1, q_2, \ldots, q_n))$$  
if and only if
$$\sum_{k=1}^n u(x_k) \cdot p_k > \sum_{k=1}^m u(y_k) \cdot q_k.$$
We write:

$$U(\ell_1) = u(x_1) \cdot p_1 + \ldots + u(x_n) \cdot p_n$$

and refer to $U$ as the expected utility and to $u$ as the:

- utility for money
- Bernoulli utility
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Recapitulating so far:
1. We made assumptions about the agents’ preferences over lotteries so we can represent his/her preferences by *an expected utility*.
2. The agent will choose the lottery that delivers the highest expected utility.

Sometimes, in real life applications, the word "lottery" describes the flow (variation of wealth) and not the final wealth.

To compute the expected utility of a "lottery" $(X, p)$, where $X = (x_1, \ldots, x_n)$ and $p = (p_1, \ldots, p_n)$ and $x_k$ denotes the wealth flow (income/loss) if the event $k$ happens – we need to know the value of the initial wealth $\omega_0$. In this case the expected utility is:

$$U(X, p; \omega_0) = \sum_{s=1}^{n} u(\omega_0 + x_s) \cdot p_s$$

Remark 1: $U(X, p)$ is the expected utility of lottery $(X, p)$.
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Computing Expected Utility

Examples with zero initial wealth (or lottery already gives final wealth).

1 Lotteries: \( A = ((1000, 0), (1/2, 1/2)) \) and \( B = ((500, 0), (1, 0)) \). vN-M utility: \( u(x) = \sqrt{x} \). Then \( U(A) = \frac{1}{2} \sqrt{1000} + \frac{1}{2} \sqrt{0} = 5\sqrt{10} \) (see the prob. tree) and \( U(B) = 1 \cdot \sqrt{500} = \sqrt{500} = 10\sqrt{5} \) (prob. tree).

2 Lotteries: \( C = ((4, 0), (2/3, 1/3)) \) and \( D = ((2.66, 0), (1, 0)) \). vN-M utility: \( u(x) = x^2 \). Then \( U(C) = \frac{2}{3} \cdot 16 + \frac{1}{3} \cdot 0 = \frac{32}{3} = 10.66 \) and \( U(D) = 7.11 \).

3 Lotteries: \( A = ((1000, 0), (1/2, 1/2)) \) and \( B = ((500, 0), (1, 0)) \) and vN-M utility: \( u(x) = 2^x \). Then \( U(A) = \frac{1}{2} 2^{1000} + \frac{1}{2} 2^0 = 2^{999} + \frac{1}{2} \) and \( U(B) = 2^{500} \).

4 Lotteries: \( C = ((4, 0), (2/3, 1/3)) \) and \( D = ((2.66, 0), (1, 0)) \). vN-M utility: \( u(x) = \ln(x) \). Then \( U(C) = \frac{2}{3} \ln(4) + \frac{1}{3} \ln(0) = -\infty \) and \( U(D) = \ln(2.66) \).
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Computing Expected Utility
Examples with initial wealth equal to $\omega_s = 8$

1. Lottery $A = ((1000, 0), (1/2, 1/2))$, lottery $B = ((500, 0), (1, 0))$, and $u(x) = \sqrt{x}$. Then

$U(A) = \frac{1}{2}\sqrt{1000} + \frac{1}{2}\sqrt{0} = 5\sqrt{10}$ (see the prob. tree) and

$U(B) = 1 \cdot \sqrt{500} = \sqrt{500} = 10\sqrt{5}$ (prob. tree).

2. The vN-M utility is $u(x) = -\exp(-x)$ and a coin if flipped twice, the lottery pays $10$ if HH, $5$ if HT, $0$ if TH and -$3$ if TT. The agent must pay 1 for the lottery and her initial wealth is $8$. Please see the prob. tree for how to compute her expected utility.
Computing Expected Utility

Examples with initial wealth equal to $\omega_s = 8$

Lottery $A = ((1000, 0), (1/2, 1/2))$, lottery $B = ((500, 0), (1, 0))$, and $u(x) = \sqrt{x}$. Then $U(A) = \frac{1}{2}\sqrt{1000} + \frac{1}{2}\sqrt{0} = 5\sqrt{10}$ (see the prob. tree) and $U(B) = 1\cdot \sqrt{500} = \sqrt{500} = 10\sqrt{5}$ (prob. tree).

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Lottery A, initial wealth $\omega = 0$ and $u(x) = \sqrt{x}$.
Lottery B, initial wealth $\omega = 0$ and $u(x) = \sqrt{x}$
Lottery A, initial wealth \( \omega = 8 \)
Lottery B, initial wealth $\omega = 8$
Decision/Probability Tree
Two coins example, $u(x) = -\exp(-x)$.

$$E[U(X)] = \frac{-\exp(-17)}{4} + \frac{-\exp(-12)}{4} + \frac{-\exp(-7)}{4} + \frac{-\exp(-4)}{4}.$$
Extracting $u$ from $\geq$
Extracting $u$ from $\succeq$
Extracting $u$ from $\succeq$
Extracting $u$ from $\succeq$
A Behavioral Look at Choice

- Anchoring
- Availability
- Representativeness
- Optimism and over confidence
- Gains and losses
- Status Quo Bias
- Framming
Let’s go back to expected utility theory, consider the two lotteries:

\[ \ell_1 = ((100, 200), \left(\frac{1}{2}, \frac{1}{2}\right)) \]

and

\[ \delta_{150} = ((150), (1)) \]

We have

\[ U(\ell_1) = u(150 - 50) \cdot \frac{1}{2} + u(150 + 50) \cdot \frac{1}{2}, \text{ and} \]

\[ U(\delta_{150}) = u(150) \cdot 1. \]
Risk Aversion

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Risk Aversion

\[ U(\ell_1) - U(\delta_{150}) = \left[ \frac{u(200) - u(150)}{50} - \frac{u(150) - u(100)}{50} \right] \cdot \frac{50}{2} \]
Risk Aversion

\[ U(\ell_1) - U(\delta_{150}) = \left[ \frac{u(200) - u(150)}{50} \right] \approx Mu(150) - \left[ \frac{u(150) - u(100)}{50} \right] \approx Mu(100) \cdot \frac{50}{2} \]
Risk Aversion

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Expected Utility Theory
Attitudes Towards Risk

1. Diminishing marginal utility, $u$ is concave, $u'' < 0 \Rightarrow$, the consumer is risk-averse.

   $$U(X) < u(E[X]) \text{ for all } X$$

2. Increasing marginal utility, $u$ is convex, $u'' > 0 \Rightarrow$, the consumer is risk-loving.

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Measuring the Degree of Risk-Aversion

The Arrow-Pratt or Absolute Measure of Risk Aversion

**Definition**

The **Arrow-Pratt** absolute measure of risk-aversion of an agent with VN-M utility $u$ at wealth level $w$ is:

$$\rho_u(w) = \frac{-u''(w)}{u'(w)}.$$ 

If for two individuals with VN-M utilities $u$ and $\tilde{u}$ we have that $\rho_u(w) > \rho_{\tilde{u}}(w)$ for all wealth levels $w$ then we say that the agent with utility $u$ is more risk-averse than the agent with utility $\tilde{u}$. 

Measuring the Degree of Risk-Aversion
The Arrow-Pratt or Absolute Measure of Risk Aversion

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We are not covering this material, please skip this slide...

**Definition**

The *relative* absolute measure of risk-aversion of an agent with VN-M utility $u$ at wealth level $w$ is:

$$r_u(w) = \frac{-u''(w) w}{u'(w)}.$$
Often we wish to evaluate the marginal impact of ONE given variable on some function of several variables.

\[ M_Y f = \frac{\partial}{\partial y} f(x, y, z) = \lim_{\Delta \to 0} \frac{f(x, y + \Delta, z), z - f(x, y, z)}{\Delta} \]

We call \( M_Y \):

1. The *marginal* change \( f \) with respect to \( y \).
2. The *partial derivative* of \( f \) wrt \( y \).
3. The *slope* of \( f \) wrt \( y \).

To compute partial derivatives, we use the exact same rules of derivation of calculus with one variable and we treat all the other variables that are not of interest (in the example above, \( x \) and \( z \)) as constants.
Math. Review

The Chain Rule

If \( h(x) = f(g(x)) \) then

\[
h'(x) = f'(g(x)) \cdot g'(x)
\]

The chain rule is one of the most useful calculus tricks we have.
For each of the composite functions below tell us, what are the corresponding $f$ and $g$ and compute $h'$.

1. $h(x) = \sqrt{2x}$.
2. $h(x) = -\exp(-\rho \cdot x)$
3. $h(x) = (4 + x^\sigma)^{\frac{1}{\sigma}}$.

If $k(x) = f(g(h(x)))$ is a composition of three functions, instead of two, apply the chain rule twice to compute $k'(x)$. 
Consider a function of one variable defined on the real line, \( f : \mathbb{R} \to \mathbb{R} \). If \( f \) is differentiable, we write the first order Taylor approximation:

\[
 f(x + h) - f(x) \simeq f'(x) \cdot h
\]

The approximation works well only if \(|h|\) is "small".

For a function of two variables and \( h = (h_1, h_2) \) we have a similar expression:

\[
 f(x + h_1, y + h_2) - f(x, y) \simeq \frac{\partial}{\partial x} f(x, y) \cdot h_1 + \frac{\partial}{\partial y} f(x, y) \cdot h_2
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\]
Math. Review
Marginal Utility \& Taylor's Approximation

\[ \Delta U = U(x + \Delta x, y + \Delta y) - U(x, y) \]

\[ = MU_x \cdot \Delta x + MU_y \cdot \Delta y \]
Maximizing a function of one variable defined on the real line, \( f : \mathbb{R} \rightarrow \mathbb{R} \).

**Maximization Problem**

\[
\max_{x \in \mathbb{R}} f(x) \quad (P)
\]

**First order condition**

\[
f'(x) = 0 \quad (FOC)
\]

**Second order condition**

\[
f''(x) \leq 0 \quad (SOC)
\]

Any point \( x \) satisfying FOC and SOC is a candidate for an *interior solution*. 
Maximizing a function of one variable defined on an interval, 
\( f : [a, b] \rightarrow \mathbb{R} \). As before,

Maximization Problem 
\[
\max_{b \geq x \geq a} f(x) \quad (P)
\]

First order condition 
\[
f'(x) = 0 \quad \text{(FOC)}
\]

Second order condition 
\[
f''(x) \leq 0 \quad \text{(SOC)}
\]

Any point \( x \) satisfying FOC and SOC is a candidate for an interior solution and now,

- \( x = a \) is a candidate for a corner solution if \( f'(a) \leq 0 \).
- \( x = b \) is a candidate for a corner solution if \( f'(b) \geq 0 \).
Consider any function $f : \mathbb{R}^k \to \mathbb{R}$.

**Definition:** $f$ is **concave** if and only if, for all $\alpha \in [0, 1]$, and any two points $x, y \in \mathbb{R}^k$, we have

$$f(\alpha x + (1 - \alpha) y) \geq \alpha f(x) + (1 - \alpha) f(y).$$

Another definition: We say that $f$ is **convex** if $-f$ is concave.
Proposition. Assume $f$ is concave and also assume that $x$ satisfy the FOC then $x$ is a solution to the maximization problem (i.e. $x$ is a global maximum).
Math. Review
The implicit Function Theorem

Let $f(x, y)$ be a real-valued function of two variables and let $g(x)$ be a real-valued function of one-variable such that if we set $y = g(x)$ we have that $f$ remains a constant when we change $x$. That is, we have

$$f(x, g(x)) = c \quad \text{for all } x, \quad \text{where } c \text{ is a constant.}$$

Then,

$$g'(x) = -\frac{\partial}{\partial x} \frac{\partial}{\partial y} f(x, g(x)).$$
Math. Review
The implicit Function Theorem

Proof: Taking the total derivative of $f$ with respect to $x$,

$$
\frac{d}{dx} f(x, g(x)) = \frac{\partial}{\partial x} f(x, g(x)) \cdot \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} f(x, g(x)) \cdot \frac{\partial}{\partial x} g(x) \]

$$

$$
= \frac{\partial}{\partial x} f(x, g(x)) + \frac{\partial}{\partial y} f(x, g(x)) \cdot g'(x) \Rightarrow
$$

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$$

As $f$ is constant along the curve $g(x)$, we call $g$ an iso-curve of $f$. Familiar iso-curves examples are: indifference curves and iso-cost curves.
Intertemporal Consumption

Key concepts:

1. Present Value
2. Arbitrage
3. Intertemporal Marginal Rate of Substitution - MRIS

Learning Goals:

1. Be able to compute PV.
2. Solve for the optimal consumption bundle.
3. Be able to justify the PV by arbitrage arguments.
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Intertemporal Model (no uncertainty)

- \( t = 0, 1, \ldots, T \) periods.

- one good at each period, \( c_t \) consumption at period \( t \)
- \( \pi_t = 1 \) is the spot price for all \( t \) (pay at the "spot")
- \( p_t \) is the forward price (contingent price) (pay today)

**Definition:** A forward contract is a non-standardized contract between two parties to buy or to sell an asset at a specified future time at a price agreed upon today.
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<table>
<thead>
<tr>
<th>( t=0 )</th>
<th>( p_0 = \pi_0 )</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>( p_1 )</td>
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<td></td>
<td>( p_2 )</td>
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<td>( \vdots )</td>
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<td>( p_T )</td>
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\[ \begin{array}{c}
\text{t}=1 \\
\pi_1 \\
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\pi_T \\
\end{array} \]

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**Definition:** A forward contract is a non-standardized contract between two parties to buy or to sell an asset at a specified future time at a price agreed upon today.
In practice, forward contracts are over-the-counter (OTC) - bilateral contracts between two parties that are customized as opposed to standard contracts that are traded in markets. Here, however, we assume forward contracts are traded in a competitive market. As a result, by arbitrage, we must have:

\[ p_t = \frac{\pi_t}{(1 + r)^t}. \]

Can you explain why?
Present Value

- $I_t$ cash-flow in period $t$
- $\nu$ interest rate period $t$ to $t + 1$ (constant)

Present value formula:

$$PV(I_0, I_1, I_2, \ldots, I_T) = I_0 + \frac{I_1}{1 + \nu} + \frac{I_2}{(1 + \nu)^2} + \cdots + \frac{I_T}{(1 + \nu)^T}$$

$$= \sum_{t=0}^{T} \frac{I_t}{(1 + \nu)^t}$$

(PV)
Present Value

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(PV)
Inter-temporal Consumption

2-Period ($T = 2$) Consumer Problem

\[ \max_{c_1, c_2} U(c_0, c_1) \]

subject to

\[ c_0 + \frac{1}{1 + \iota} c_1 \leq Y_0 + \frac{1}{1 + \iota} Y_1 \]

\[ c_0 \geq 0 \text{ and } c_1 \geq 0 \]
Inter-temporal Consumption

2-Period \((T = 2)\) Consumer Problem

\[
\begin{align*}
\max_{c_1, c_2} & \quad U(c_0, c_1) \\
\text{subject to} & \quad c_0 + \frac{1}{(1+i)} c_1 \leq Y_0 + \frac{1}{(1+i)} Y_1 \\
& \quad c_0 \geq 0 \quad \text{and} \quad c_1 \geq 0
\end{align*}
\]
Inter-temporal Consumption
2-Period ($T = 2$) Consumer Problem

$$\max_{c_1, c_2} U(c_0, c_1)$$
subject to

$$c_0 + \frac{1}{(1+i)} c_1 \leq Y_0 + \frac{1}{(1+i)} Y_1$$
$$c_0 \geq 0 \text{ and } c_1 \geq 0$$
The idea of arbitrage

The A’s front office realized right away, of course, that they couldn’t replace Jason Giambi with another first baseman just like him. There wasn’t another first baseman just like him and if there were they couldn’t have afforded him and in any case that’s not how they thought about the holes they had to fill. "The important thing is not to recreate the individual," Billy Beane would later say. "The important thing is to recreate the aggregate." He couldn’t and wouldn’t find another Jason Giambi; but he could find the pieces of Giambi he could least afford to be without, and buy them for a tiny fraction of the cost of Giambi himself. – Moneyball by Micheal Lewis, p. 103
The A’s front office had broken down Giambi into his obvious offensive statistics: walks, singles, doubles, home runs along with his less obvious ones: pitches seen per plate appearance, walk to strikeout ratio and asked: which can we afford to replace? And they realized that they could afford, in a roundabout way, to replace his most critical offensive trait, his on-base percentage, along with several less obvious ones. The previous season Giambi’s on-base percentage had been .477, the highest in the American League by 50 points. (Seattle’s Edgar Martinez had been second at .423; the average American League on-base percentage was .334.) There was no one player who got on base half the time he came to bat that the A’s could afford; – *Moneyball by Micheal Lewis, p. 103*
on the other hand, Jason Giambi wasn’t the only player in the Oakland A’s lineup who needed replacing. Johnny Damon (onbase percentage .324) was gone from center field, and the designated hitter Olmedo Saenz (.291) was headed for the bench. The average on-base percentage of those three players (.364) was what Billy and Paul had set out to replace. They went looking for three players who could play, between them, first base, outfield, and DH, and who shared an ability to get on base at a rate thirty points higher than the average big league player. – *Moneyball* by Micheal Lewis, p. 103
What is the value today of $Y_1$ dollars tomorrow?

1. What should we do if someone else thinks that $x$ dollars tomorrow are worth less than $\frac{x}{1+i}$ today?

2. Or alternatively, believes $x$ dollars tomorrow are worth more than $\frac{x}{1+i}$ dollars today?
Understanding The Solution to the Consumer Problem

MRIS \equiv \frac{MU_0}{MU_1} = 1 + \nu

c_0 + \frac{1}{(1 + \nu)}c_1 = PV(Y_0, Y_1)

To understand the first equation: MRIS is how many units of consumption tomorrow are equal to one unit of consumption today for the consumer and $1 + \nu$ is how many units of consumption tomorrow are equal to one unit of consumption today for the market. In equilibrium they ought to be equal.

The second equation just says the consumer expends her income during her lifetime.
Financial Instruments:
Options

A stock option is an option (not an obligation) to buy (call option) or to sell (put option) some specified number of shares of the stock at price (per-share) $K$ (the strike price) at expire date $T$ (European option) or, alternatively at any point in time before $T$ (American option).

Example: Two periods: at $t = 0$ the price of the stock is 27 and at $t = 1$ the it is 28 with prob. $\frac{2}{3}$ or 26 with prob. $\frac{1}{3}$. The option is an European (you can use it only at the expire date $T = 1$) call (gives you the right to buy 1 stock) option with strike price $K = 26.5$.

Note: the strike price is not the price of the option (in practice, the price of the option is called *premium*).
A Call Option Example

continuation

The payoffs associated to this call option are:

\[ t = 0 \]

\[ \$ - 26.5 + 28 = 1.5 \]
If a DM with utility for money $u$ and initial wealth $36$ buys the option paying $P$ at $t = 0$, his/her expected utility is:

$$u(37.5 - P) = \frac{2u(37.5 - P)}{3} + \frac{u(36 - P)}{3}$$
The expected utility of not buying the call is

$$U(\text{not buy}) = u(36).$$

The expected utility of the call is

$$U(\text{buy}) = \frac{2u(37.5 - P)}{3} + \frac{u(36 - P)}{3}.$$ 

If $P = 0$ then $U(\text{not buy}) = u(36) < U(\text{buy}) = \frac{2u(37.5)}{3} + \frac{u(36)}{3}$. If $P = 1.5$ then $U(\text{not buy}) = u(36) > U(\text{buy}) = \frac{2u(36)}{3} + \frac{u(35.5)}{3}$. As $P \nearrow$ we have $U(\text{buy}) \searrow$ and $U(\text{not buy}) = \text{cte.}$

There exists $P_{\text{max}}$ such that $U(\text{not buy}) = U(\text{buy})$. 
A Call Option Example

1. \( u(x) = x \implies P_{\text{max}} = 1. \)

2. \( u(x) = x^2 \implies P_{\text{max}} = 1.0069. \) Making the DM indifferent, we get \( 73.5 - 74P + P^2 = 0 \) so 
   \[ P = \frac{74 - \sqrt{74^2 - 4(73.5)}}{2} \approx 1.0069. \]

3. \( u(x) = \sqrt{x} \implies P_{\text{max}} = 0.9965. \) Making the DM indifferent, we get 
   \[ 2\sqrt{36 - P} + \frac{3}{2} + \sqrt{36 - P} = 3 \cdot 6. \] The "trick" is to call \( y = \sqrt{36 - P} \) and remove the square root in 
   \[ 2\sqrt{y^2 - \frac{3}{2}} + y = 18 \] to get a quadratic equation in \( y \). We solve it for \( y \) and set \( P = 36 - y^2. \).
Now there is a bond that costs $1 at $t = 0$ and pays $1 + \iota$ at $t = 1$. If we buy/sell $s$ shares of the stock and $b$ bonds such that:

$$
s \begin{pmatrix} 27 \\ 28 \\ 26 \end{pmatrix} + b \begin{pmatrix} 1 \\ 1 + \iota \\ 1 + \iota \end{pmatrix} = \begin{pmatrix} P \\ 1.5 \\ 0 \end{pmatrix}
$$

It must be that $2s = 1.5$ so $s = 0.75$ and $b = -\frac{0.75 \cdot 26}{1 + \iota}$ As a result we can figure out the price of the call:

$$P = 0.75 \cdot 27 - \frac{0.75 \cdot 26}{1 + \iota}.$$

If $0 \leq \iota < +\infty$ then $0.75 \leq P < 20.25$. 
A simple model of portfolio choice, there is one investor and:

- Two periods $t = 0, 1$ and two states at $t = 1$ (H or L).
- Investor has wealth only at $t = 0$, $w_0 > 1$ and $w_1 = 0$.
- Investor uses assets to transfer wealth across periods.
- There are two assets: the riskless and the risky one.
- The riskless asset’s rate of return is $(1 + \iota)$ in both states.
- The risky asset returns $R_H$ in state H and $R_L$ in state L.
- No asset is dominated that is, $R_H > 1 + \iota > R_L$.
- The fraction (of $w_0$) invested in the riskless asset is $\alpha$.
- The probability of state H is $p$ and the prob. of L is $1 - p$. 
The problem of the investor is to choose \( \alpha \in [0, 1] \) to maximize her expected utility:

\[
U(\alpha) = p \cdot u \left( (\alpha(1 + \gamma) + (1 - \alpha)R_H)w_0 \right) + \\
+ (1 - p) \cdot u \left( (\alpha(1 + \gamma) + (1 - \alpha)R_L)w_0 \right) .
\]
The problem of the investor is to choose $\alpha \in [0, 1]$ to maximize her expected utility:

$$U(\alpha) = p \cdot u \left( (\alpha(1 + \iota) + (1 - \alpha)R_H)w_0 \right) +$$

$$+ (1 - p) \cdot u \left( (\alpha(1 + \iota) + (1 - \alpha)R_L)w_0 \right).$$

wealth when state $H$ happens
The problem of the investor is to choose $\alpha \in [0, 1]$ to maximize her expected utility:

$$U(\alpha) = p \cdot u ((\alpha(1 + \nu) + (1 - \alpha)R_H)w_0) +$$

$$+(1 - p) \cdot u ((\alpha(1 + \nu) + (1 - \alpha)R_L)w_0).$$

wealth when state L happens
The corresponding first-order condition for a maximum is:

\[ U'(\alpha) = 0 \text{ or equivalently,} \]

\[
p \cdot u' \left( (\alpha(1 + \iota) + (1 - \alpha)R_H)w_0 \right) \cdot (1 + \iota - R_H) \cdot w_0 + \\
\text{expected MU}
\]

\[
\text{MC of riskless asset}
\]

\[
(1 - p) \cdot u' \left( (\alpha(1 + \iota) + (1 - \alpha)R_L)w_0 \right) \cdot (1 + \iota - R_L)w_0 = 0
\]

\[
\text{expected MU}
\]

\[
\text{MB of riskless asset}
\]
Intertemporal Choice (recap.)

\[
\begin{align*}
\max_{c_1, c_2} & \quad u(c_0) + \delta u(c_1) \\
\text{subject to} & \quad c_0 + \frac{1}{(1+\nu)} c_1 \leq Y_0 + \frac{1}{(1+\nu)} Y_1 \\
& \quad c_0 \geq 0 \text{ and } c_1 \geq 0
\end{align*}
\]

Indifference curve with utility $\bar{u}$

\[
c_1(c_0) = u^{-1}(\bar{u} - u(c_0)/\delta)
\]

\[
c'_1(c_0) = -\text{MRIS}
\]
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\[
\begin{align*}
& \max_{c_1, c_2} \quad u(c_0) + \delta u(c_1) \\
& \text{subject to} \\
& c_0 + \frac{1}{(1 + \iota)} c_1 \leq Y_0 + \frac{1}{(1 + \iota)} Y_1 \\
& c_0 \geq 0 \quad \text{and} \quad c_1 \geq 0
\end{align*}
\]

Indifference curve with utility \( \tilde{u} \)

\[
c_1(c_0) = u^{-1}(\tilde{u} - u(c_0)/\delta)
\]

\[
c_1'(c_0) = -\text{MRIS}
\]

\[
U(c_0, c_1) = 5
\]

\[
Y_0 + \frac{Y_1}{1 + \iota}
\]
Intertemporal Choice (recap.)

\[
\max_{c_1, c_2} u(c_0) + \delta u(c_1)
\]

subject to

\[
c_0 + \frac{1}{(1+\nu)} c_1 \leq Y_0 + \frac{1}{(1+\nu)} Y_1
\]

\(c_0 \geq 0 \text{ and } c_1 \geq 0\)

Indifference curve with utility \(\bar{u}\)

\[
c_1(c_0) = u^{-1}(\bar{u} - u(c_0)/\delta)
\]

\[
c'_1(c_0) = -\text{MRIS}
\]

\[
U(c_0, c_1) = 6
\]
Intertemporal Choice (recap.)

\[
\max_{c_1, c_2} u(c_0) + \delta u(c_1)
\]
subject to
\[
c_0 + \frac{1}{(1+\iota)} c_1 \leq Y_0 + \frac{1}{(1+\iota)} Y_1
\]
\[
c_0 \geq 0 \quad \text{and} \quad c_1 \geq 0
\]

Indifference curve with utility \( \bar{u} \)

\[
c_1(c_0) = u^{-1}(\bar{u} - u(c_0)/\delta)
\]
\[
c'_1(c_0) = -\text{MRIS}
\]

\[
U(c_0, c_1) = 5.9
\]
General Equilibrium

Consider a $2 \times 2$ pure trade economy (2 consumers, A & B; 2 goods $c_1$ & $c_2$, no firms).

**Endowment** The initial basket that a consumer owns:

$e^A = (e^A_1, e^A_2)$ and $e^B = (e^B_1, e^B_2)$.

**Feasible Allocation** Division of the total endowment between the consumers: $(c^A, c^B)$ such that

$c^A_1 + c^B_1 = e^A_1 + e^B_1$ and $c^A_2 + c^B_2 = e^A_2 + e^B_2$.

**Efficient Allocation** An allocation where trade can never benefit all consumers: $MRS^A = MRS^B$ in the case that $c^A > 0$ and $c^B > 0$ and MRSs are decreasing.

**Individual Rational Allocations** Given consumers higher or equal utility than if they just consume their endowments.

**Contract Curve** Set of allocations that are individually rational and efficient.
General Equilibrium

Consider a $2 \times 2$ pure trade economy (2 consumers, A & B; 2 goods $c_1$ & $c_2$, no firms).

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General Equilibrium
A's endowment of $c_0$
B's endowment of $c_0$
General Equilibrium

A's endowment of $c_0$

B's endowment of $c_0$

endowment
General Equilibrium

A's endowment of $c_0$

B's endowment of $c_0$

A's endowment of $c_1$

B's endowment of $c_1$
General Equilibrium

A's endowment of $c_0$

B's endowment of $c_0$

A's endowment of $c_1$

B's endowment of $c_1$
General Equilibrium

A's endowment of $c_0$

B's endowment of $c_0$

B's endowment of $c_1$

A's endowment of $c_1$
utility of A better than endowment
General Equilibrium

The image shows a graph with axes labeled $c_0^A$, $c_1^A$, $c_0^B$, and $c_1^B$. The graph illustrates the concept of general equilibrium in economics.
utility of B better than endowment
Both are better off.
Both are better off when \( MRS_A = MRS_B \) and the allocations are efficient.
In general equilibrium, the marginal rate of substitution (MRS) between two goods for two consumers, $A$ and $B$, must be equal at the efficient allocations. This is because efficient allocations are characterized by the condition that the indifference curves of both consumers are tangent at the point of equilibrium, ensuring that the marginal rate of substitution is equal to the price ratio of the two goods. The diagram illustrates this with the equality $MRS_A = MRS_B$ at the equilibrium point.
General Equilibrium

efficiency: indifference curves are tangent

$\text{MRS}_A = \text{MRS}_B$

efficient allocations
General Equilibrium

MRS_A = MRS_B

efficient allocations

efficiency: indiff. curves are tangent
General Equilibrium

Efficient allocations

\[ MRS_A = MRS_B \]
General Equilibrium

Efficient allocations $MRS_A = MRS_B$
General Equilibrium

Efficient allocations

\[ MRS_A = MRS_B \]
General Equilibrium

\[ c_1^A = c_1^B \]

Efficient allocations

\[ MRS_A = MRS_B \]
General Equilibrium

\[ c_1^A = c_1^B \]

Efficient allocations

\[ MRS_A = MRS_B \]
General Equilibrium

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General Equilibrium

$c_1^A$  $c_0^B$

$c_0^A$  $c_1^B$

efficient allocations

$MRS_A = MRS_B$
General Equilibrium

$MRS_A = MRS_B$

efficient allocations
General Equilibrium

Efficient allocations

\[ \text{MRS}_A = \text{MRS}_B \]
**Definition:** An *Arrow-Debreu* good is defined by (at least) 4 dimensions:

1. Its **physical properties** and qualities.
2. The geographic **location** where it is available.
3. The **time** when it is available for consumption.
4. The **state of the world** where it is available for consumption.

Say there are $K$ distinct physical attributes, $L$ locations, $T$ time periods and $S$ states.

The total number of goods is $n = K \cdot L \cdot T \cdot S$. That is, we have $n$ prices and $n$ markets.
Uncertainty
Arrow-Debreu goods, an example

Assume that:

- $K \in \{\text{umbrella, parasol}\}$,
- $K \in \{\text{Hillsborough, Chicago}\}$,
- $T \in \{\text{today, tomorrow}\}$, and
- $S \in \{\text{sun, rain}\}$.

Then we have 16 goods! 16 markets! 16 prices!

For instance, the first good is an umbrella in Hillsborough available today provided today is a sunny day, the second good is an umbrella in Hillsborough available today if it is rainy, the third good is an umbrella in Hillsborough available tomorrow if tomorrow is sunny, ..., the last good is a parasol available tomorrow in Chicago if it rains.
Complete Markets

When all the markets for Arrow-Debreu goods exists we call it the case of complete markets.

The assumption of complete markets may appear too extreme and unrealistic, however there are more markets out there than meets the eye:

- Weather Markets
- A weather contract
- Events Markets
Complete Markets

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1 General v. Partial Equilibrium.
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   - What is another name for the optimal basket/bundle of goods chosen by a given consumer?
   - What is income? How do we calculate it?
General Equilibrium

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The consumer problem revisited.

- What is another name for the optimal basket/bundle of goods chosen by a given consumer?
- What is income? How do we calculate it?
Income is the value of the consumer endowment. It may include:

- Value of number of hours of labor sold (wages = wage rate \times hours).
- Value of the amount of capital lend (interest rate \times capital).
- Share of the profits of firms.
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An equilibrium is a vector of prices $\mathbf{p}$ such that:

- Consumers maximize their utility $\Rightarrow$ individual demands.
- Firms maximize their profits $\Rightarrow$ supply output and input demand.
- Markets clear: for any good $j$, either $D_j(p) = S_j(p)$ or
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GE: Example 1

One consumer (Chuck), two goods (X and Y) and no firms.

Chuck’s preferences are: \( u(x, y) = \ln(x) + 2 \ln(y) \); his endowment is: \( e = (3, 4) \); and so his budget is:
\[
p_X \cdot x + p_Y \cdot y \leq p_X \cdot 3 + p_Y \cdot 4.
\]
Solving for his optimal basket, we find his demand functions:
\[
X(p_x, p_y) = \frac{p_X \cdot 3 + p_Y \cdot 4}{3p_x} \quad \text{and} \quad Y(p_x, p_y) = \frac{2}{3} \cdot \frac{p_X \cdot 3 + p_Y \cdot 4}{p_Y}.
\]

Market demand of good X is
\[
D_X(p_X, p_Y) = \frac{p_X \cdot 3 + p_Y \cdot 4}{3p_X},
\]
which coincides with Chuck’s demand, and the market supply of good X, which coincides with Chuck’s endowment of X, is \( S_X(p_X, p_Y) = 3 \) (perfectly inelastic). Equilibrium prices must clear the market:
\[
D_X(p_X, p_Y) = S_X(p_X, p_Y) \iff \frac{p_X \cdot 3 + p_Y \cdot 4}{3p_X} = 3 \Rightarrow \frac{p_X}{p_Y} = \frac{4}{9}.
\]
GE: Example 2

Two consumers (Chuck & Bob), two goods (X and Y) and no firms.

Chuck’s preference and endowments are as in example 1, so his demand remains the same. Bob’s preferences are 

\[ u^B(x, y) = 2\ln(x) + \ln(y); \]

his endowment is \( e^B = (1, 1) \); so his budget is: 

\[ p_X \cdot x + p_Y \cdot y \leq p_X \cdot 1 + p_Y \cdot 1 \]

Solving for Bob’s optimal basket, we find his demand functions:

\[ X^B(p_x, p_y) = \frac{2}{3} \frac{p_X + p_Y}{p_x} \]

and 

\[ Y^B(p_x, p_y) = \frac{p_X + p_Y}{3p_Y} \] 

The market demand of good X is the sum of Chuck’s and Bob’s demand for X:

\[ D_X(p_X, p_Y) = \frac{3p_X + 4p_Y}{3p_X} + \frac{2p_X + p_Y}{3p_X} = \frac{5p_X + 6p_Y}{3p_X} \]
We found the market (aggregated) demand for X was:

\[
D_X(p_X, p_Y) = \frac{3p_X + 4p_Y}{3p_X} + \frac{2}{3} \frac{p_X + p_Y}{p_X} = \frac{5p_X + 6p_Y}{3p_X}.
\]

The total supply of X is the amount of X available in the economy (sum of the amounts of X consumers were endowed with):

\[
S_X(p_X, p_Y) = e_X^C + e_X^B = 3 + 1 = 4.
\]

For prices to clear the market:

\[
\frac{5p_X + 6p_Y}{3p_X} = 4 \Rightarrow \frac{p_X}{p_Y} = \frac{6}{7}.
\]
GE: comparing the economies of examples 1 & 2

1. What happened with the supply of good X when you moved to example 2?

2. What would have been the market demand in example 2 at the equilibrium prices of example 1?

3. If the quantity demanded was as in item 2 above, in the economy of example 2, the market for good X would have excess demand or excess supply?

4. What happened with the equilibrium price (ratio) when you moved from example 1 to example 2? Increased? Decreased? Remained constant? In economy 2 is X cheaper? or more expensive?

5. Can you explain (please try): Why the prices moved in this direction?
Assume two consumers (Telma & Louise) with utilities for present and future consumption give by:

\[ u^T(c_0, c_1) = \ln(c_0) + \frac{9}{10} \ln(c_1) \] and \[ u^L(c_0, c_1) = \ln(c_0) + \frac{8}{10} \ln(c_1) \]

Assume their endowments are

\[ e^T = (6, 6) \] and \[ e^L = (3, 3). \]

1. Write down T and L’s respective budget constraints.
2. Find the demand function of T and of L for present consumption \((c_0)\).
3. Find the market demand for present consumption \((c_0)\).
4. Find the total supply for present consumption.
5. Find the equilibrium price (ratio): \( \frac{p_0}{p_1} \).
Consider an economy with 3 periods $t = 0, 1$ and $2$. There is one consumer whose (ex-post) utility is

$$U(c) = s \cdot \sqrt{c_1} + (1 - s) \sqrt{c_2},$$

where $c_1$ is the amount of consumption in period 1 and $c_2$ is the amount of consumption in period 2 (he does not value consumption in period zero).
At $t = 0$, the value of the state $s$ variable is not known but the probabilities $\Pr[s = 0] = p$ and $\Pr[s = 1] = 1 - p$ are common-knowledge.

At $t = 1$, the realization (value) of $s$ is publicly observed.
GE: Example 4
Production

- The consumer is endowed with: one unit of consumption at \( t = 0 \); nothing at \( t = 1, 2 \); and ownership of firms \( S \) and \( L \).

- Firms operate in competitive markets (price takers).

- Firm \( S \) uses \( q_S \) units of consumption good as input at \( t = 0 \) to produce \( q_S \) units of consumption good at \( t = 1 \).

- Firm \( L \) uses \( q_L \) units of consumption good at \( t = 0 \) to produce \( R \cdot q \) units of output at \( t = 2 \).


\[ E_S[U_D(c)] = (1 - p) \cdot \sqrt{c_1} + p \cdot \sqrt{c_2} \]

The are 5 goods in this economy:

1. consumption contingent on \( t = 0 \), \( c_0 \)
2. consumption contingent on \( t = 1 \) and \( s = 0 \), \( c_{1,0} \).
3. consumption contingent on \( t = 1 \) and \( s = 1 \), \( c_{1,1} \).
4. consumption contingent on \( t = 2 \) and \( s = 0 \), \( c_{2,0} \).
5. consumption contingent on \( t = 2 \) and \( s = 1 \), \( c_{2,1} \).

With this notation, expected utility becomes

\[ (1 - p) \cdot \sqrt{c_{1,1}} + p \cdot \sqrt{c_{2,0}}. \]
The budget constraint is:

\[ p_0 c_0 + p_{1,0} c_{1,0} + p_{1,1} c_{1,1} + p_{2,0} c_{2,0} + p_{2,1} c_{2,1} \leq p_0 \cdot 1 + \pi_S + \pi_L \]

\[ \pi_S = p_{1,0} q_S + p_{1,1} q_S - p_0 q_S \]

\[ \pi_L = p_{2,0} R q_L + p_{2,1} R q_L - p_0 q_L \]

where \( \pi_S \) and \( \pi_L \) are the profits of the firms \( S \) and \( L \).
GE: Example 4
Production

FOC

\[
\text{MRS}_{(1,1),(2,0)} = \frac{(1 - p)}{\frac{2\sqrt{c_{1,1}}}{p}} = \frac{p_{1,1}}{p_{2,0}}
\]

\[
\pi'_S = p_{1,0} q_S + p_{1,1} q_S - p_0 q_S \Rightarrow p_{1,0} + p_{1,1} - p_0 = 0.
\]

\[
\pi'_L = p_{2,0} R q_L + p_{2,1} R q_L - p_0 q_L \Rightarrow R(p_{2,0} + p_{2,1}) - p_0 = 0.
\]
Market clearing conditions:

\[ c_0 + q_S + q_L = 1, \]
\[ c_{1,0} = q_S, \]
\[ c_{1,1} = q_S, \]
\[ c_{2,0} = Rq_L, \]
\[ c_{2,1} = Rq_L. \]
The Consumer Problem under Complete Markets

Two States and One Good

max \begin{array}{c} c_L, c_H \\ \pi_L u(c_L) + \pi_H u(c_H). \end{array}

\text{st.}

p_L c_L + p_H c_H \leq p_L Y_L + p_H Y_H

\mathcal{L}(c_L, c_H, \lambda) = \pi_L u(c_L) + \pi_H u(c_H) - \lambda (p_L Y_L + p_H Y_H - p_L c_L - p_H c_H)

\frac{\partial}{\partial c_L} \mathcal{L}(c_L, c_H, \lambda) = \pi_L u'(c_L) + \lambda p_L = 0 \quad \text{(FOC}_{c_L})

\frac{\partial}{\partial c_H} \mathcal{L}(c_L, c_H, \lambda) = \pi_H u'(c_H) + \lambda p_H = 0 \quad \text{(FOC}_{c_H})

\frac{\partial}{\partial \lambda} \mathcal{L}(c_L, c_H, \lambda) = p_L Y_L + p_H Y_H - p_L c_L - p_H c_H = 0 \quad \text{(FOC}_{\lambda})

Two states: \( H \) with probability of \( \pi_H \) and \( L \) with prob. \( \pi_L \).

Endowment: \( Y = (Y_H, Y_L) \).

Income (value of endowment): \( I = p_H Y_H + p_L Y_L \).

Price of unit of good delivered if \( H \) (\( L \)) happens: \( p_H \) (\( p_L \)).
Equilibrium Prices under Complete Markets

Two States, Two Consumers and One Good

- Consumers A and B with expected utilities:
  \[ \pi_L u^A(c_L) + \pi_H u^A(c_H) \text{ and } \pi_L u^B(c_L) + \pi_H u^B(c_H). \]
- Their endowments are given, \( Y^A = (Y^A_H, Y^A_L) \) and \( Y^B = (Y^B_H, Y^B_L) \).
- Solve the consumer problem to find their individual demands (see previous page). Notice \( c^A_L \) and \( c^A_H \) depend only the probabilities, the prices and A’s endowment and likewise, \( c^B_L \) and \( c^B_H \) depend only the probabilities, the prices and B’s endowment.
- To find the price ratio:
  \[ \Rightarrow \text{Equate total supply with total demand,} \]
  \[ Y^A_H + Y^B_H = c^A_H(p_H, p_L) + c^B_H(p_H, p_L). \]

- Important: we can always normalize one price to one. If we set \( p_H = 1 \) and then solve the above equation for \( p_L \) then we actually get the value of \( \frac{p_L}{p_H} \).
- Important: See Mathematica file on General Equilibrium.
But even if markets are not complete we can “complete” the missing markets if we have Arrow securities.

An Arrow-security is a financial instrument that pays $1 unit of accounting in a given location, date $t$, and state and it is traded in a market at date $t - 1$. Thus at each point in time we need only $L \cdot S$ markets.

Also even if we do not have Arrow securities we can complete the markets if we have enough financial instruments (as we did in class).
Financial Market Eq. with Arrow Securities

Only two states, \( s \in \{L, H\} \), and consumption takes place only at date \( T = 1 \). But consumer makes decisions and markets operate at date \( T = 0 \). The consumer problem with complete markets is

\[
\max_{c_L, c_H} U(c_L, c_H).
\]

\[
\text{st.}
\]

\[
p_L c_L + p_H c_H \leq p_L Y_L + p_H Y_H
\]

and with Arrow-securities is

\[
\max_{c_L, c_H, z_L, z_H} U(c_L, c_H).
\]

\[
\text{st.}
\]

\[
q_L z_L + q_H z_H \leq 0
\]

\[
\hat{p}_L c_L \leq \hat{p}_L Y_L + z_L
\]

\[
\hat{p}_H c_H \leq \hat{p}_H Y_H + z_H
\]
Financial Market Eq. with Arrow Securities

Only two states, \( s \in \{L, H\} \), and consumption takes place only at date \( T = 1 \). But consumer makes decisions and markets operate at date \( T = 0 \). The consumer problem with with Arrow-securities is:

\[
\max_{c_L,c_H,z_L,z_H} U(c_L, c_H).
\]

subject to

\[
q_L z_L + q_H z_H \leq 0
\]

\[
\hat{p}_L c_L \leq \hat{p}_L Y_L + z_L
\]

\[
\hat{p}_H c_H \leq \hat{p}_H Y_H + z_H
\]

where:

- \( q_s \) is the price of one unit of the security \( s \).
- \( z_s \) is the amount of securities \( s \) the consumer buys (negative if he or she sells).
The consumer problem under uncertainty with Arrow securities

\[
\begin{align*}
\max_{c_L, c_H} & \quad \pi_L u(c_L) + \pi_H u(c_H). \quad \text{(CP - Arrow securities)} \\
\text{st.} & \quad q_L z_L + q_H z_H \leq 0 \\
& \quad \hat{p}_L c_L \leq \hat{p}_L Y_L + z_L \\
& \quad \hat{p}_H c_H \leq \hat{p}_H Y_H + z_H \\
\mathcal{L}(c_L, c_H, \lambda) & = \pi_L u(c_L) + \pi_H u(c_H) - \lambda_1 (q_L z_L + q_H z_H) + \\
& - \lambda_2 (\hat{p}_L c_L - \hat{p}_L Y_L - z_L) - \lambda_3 (\hat{p}_H c_H - \hat{p}_H Y_H - z_H) \\
\frac{\partial}{\partial c_L} \mathcal{L}(c_L, c_H, \lambda) & = \pi_L u'(c_L) + \lambda_2 \hat{p}_L = 0 \quad \text{(FOC}_{c_L} \text{)} \\
\frac{\partial}{\partial c_H} \mathcal{L}(c_L, c_H, \lambda) & = \pi_H u'(c_U) + \lambda_3 \hat{p}_H = 0 \quad \text{(FOC}_{c_H} \text{)} \\
\frac{\partial}{\partial \lambda_1} \mathcal{L}(c_L, c_H, \lambda) & = q_L z_L + q_H z_H = 0 \quad \text{(FOC}_{\lambda_1} \text{)} \\
\frac{\partial}{\partial \lambda_2} \mathcal{L}(c_L, c_H, \lambda) & = \hat{p}_L c_L - \hat{p}_L Y_L - z_L = 0 \quad \text{(FOC}_{\lambda_2} \text{)} \\
\frac{\partial}{\partial \lambda_3} \mathcal{L}(c_L, c_H, \lambda) & = \hat{p}_H c_H - \hat{p}_H Y_H - z_H = 0 \quad \text{(FOC}_{\lambda_3} \text{)}
\end{align*}
\]
Let’s assume:

- two consumers (A and B)
- complete markets (with Arrow-securities we will obtain identical results).
- total endowment constant across states, 

\[ Y = Y^A_L + Y^B_L \quad \text{and} \quad Y = Y^A_H + Y^B_H. \]

From the first-order condition, we have that:

\[
\frac{u'_A(c^A_L)}{u'_A(c^A_H)} = \frac{u'_B(c^B_L)}{u'_B(c^B_H)} \quad \Rightarrow \quad c^A_L > c^A_H \iff c^B_L > c^B_H
\]

\[
\frac{u'_A(c^A_L)}{u'_A(c^A_H)} = \frac{u'_B(Y - c^A_L)}{u'_B(Y - c^A_H)} \quad \Rightarrow \quad c^A_L > c^A_H \iff Y - c^A_L > Y - c^A_H
\]

\[ \iff c^A_L < c^A_H \quad \text{But this is a contradiction!} \]
Portfolio Choice

\[
\text{max}_{\theta, c_L, c_H} \quad \pi_L u(c_L) + \pi_H u(c_H),
\]

\[
\text{st.} \quad 0 \leq \theta \leq 1,
\]

\[
c_L = \theta R_L W_0 + (1 - \theta) W_0
\]

\[
c_H = \theta R_H W_0 + (1 - \theta) W_0
\]

(CP - portfolio choice)

\[
\mathcal{L}(\theta, c_L, c_H, \lambda_1, \lambda_2) = \pi_L u(c_L) + \pi_H u(c_H) +
\]

\[
- \lambda_1 \left( c_L - \theta R_L W_0 - (1 - \theta) W_0 \right) - \lambda_2 \left( c_H - \theta R_H W_0 - (1 - \theta) W_0 \right)
\]

\[
\frac{\partial}{\partial \theta} \mathcal{L}(\theta, c_L, c_H, \lambda_1, \lambda_2) = \lambda_1 (R_L - 1) W_0 + \lambda_2 (R_H 1) W_0 = 0
\]

\[
\frac{\partial}{\partial c_L} \mathcal{L}(\theta, c_L, c_H, \lambda_1, \lambda_2) = \pi_L u'(c_L) - \lambda_1 = 0
\]

\[
\frac{\partial}{\partial c_H} \mathcal{L}(\theta, c_L, c_H, \lambda_1, \lambda_2) = \pi_H u'(c_H) + \lambda_2 = 0
\]

\[
\frac{\partial}{\partial \lambda_1} \mathcal{L}(\theta, c_L, c_H, \lambda_1, \lambda_2) = c_L - \theta R_L W_0 - (1 - \theta) W_0 = 0
\]
If we have complete-markets and we know the equilibrium price vector, $p = (p_s)$. We can price ANY financial asset/security. A financial asset that pays $f_s$ at state $s$ must value $\sum_{s \in S} p_s \cdot f_s$ where $S$ is the set of all states (including time periods).
Say we have one state today and two tomorrow, $S = \{s_0, s_{1H}, s_{2L}\}$ and the price of delivery of one unit of consumption if the state $s$ happens is $p_0$, $p_{1H}$ and $p_{2H}$ for the respective states.

1. A financial security that always pays 1 in period 1 (a risk-less bond) and zero today must be worth (today) $0 \cdot p_0 + 1 \cdot p_{1H} + 1 \cdot p_{2H}$.

2. A bet that pays 1 if $H$ happens and $-1$ if $L$ happens and nothing today is worth $p_{1H} - p_{1L}$ today.
Let’s assume:

- Three dates (0, 1 and 2) and one consumer.
- Investment occurs at dates 0 and 1.
- Consumption occurs at dates 1 or 2.
- With prob. $\pi_1$ consumption takes place only at date 1.
- With prob. $\pi_2 = 1 - \pi_1$ consumption takes place at date 2.
- Safe (short asset) investment of $x$ yields $x$ at next date.
- Risky (long asset) investment of $x$ at date 0 yields $Rx$ at date 2 where $R > 1$. The long asset is *illiquid* at date 1.
- Initial wealth: $W_0 = 1$.
- The fraction of wealth in short asset is $\beta$.

$$\max_{\beta \text{ st. } 0 \leq \beta \leq 1} \pi_1 u(\beta) + \pi_2 u(\beta + (1 - \beta)R) \quad (2.1)$$
Liquidity Shocks
An Example

- Dates: \( t = 0, 1, 2 \).
- Consumer with utility \( u(c) = \log(c) \).
- Safe (short asset): investment of \( x \) yields \( x \) at next date.
- Risky (long asset): invest. \( x \) at \( t = 0 \) yields \( 4x \) at \( t = 2 \).
- Long asset is illiquid at \( t = 1 \).
- Initial wealth: \( W_0 = 10 \).
- Investment at \( t = 0, 1 \).
- Consumption at \( t = 1, 2 \) but not both.
- With prob. \( \frac{1}{2} \) consumption takes place only at date 1.
- The fraction of wealth in short asset is \( \beta \).

\[
\max_{\beta} \quad \frac{1}{2} \log(\beta 10) + \frac{1}{2} \log(\beta 10 + (1 - \beta)40) \\
\text{st.} \quad 0 \leq \beta \leq 1
\]
Liquidity Shocks
An Example

\[
\max_{\beta} \frac{1}{2} \log (\beta \cdot 10) + \frac{1}{2} \log (\beta \cdot 10 + (1 - \beta) \cdot 40)
\]

\[
0 \leq \beta \leq 1
\]

\[
\frac{10}{2 (\beta \cdot 10)} + \frac{10 - 40}{2 (\beta \cdot 10 + (1 - \beta) \cdot 40)} = 0
\quad \text{(FOC)}
\]

\[
\beta = \frac{2}{3}
\]
Liquidity with risk-pooling

- Two agents, $i = 1, 2$.
- Initial individual wealth, $W_0^i = \omega$.
- Short asset with rate of return $r$, $1 \leq r < R$.
- Long asset with rate of return $R$.
- $\beta$ fraction of wealth invest in short-asset.
- $\pi$ prob. of liquidity shock (independent across agents).
- $d$ amount of short-asset promised to an agent who had a liquidity shock if the other agent did not suffer a liquidity shock.
If agents do not pool their resources, each agent choose its own $\beta$ to maximize:

$$\max_{\beta} \quad \pi u (r \beta \omega) + (1 - \pi) u (r^2 \beta \omega + (1 - \beta) R \omega)$$

subject to

$$0 \leq \beta \leq 1$$

(INDIVIDUAL)

If agents pool their resources, each agent choose its own $\beta$ to maximize:

$$\max_{\beta} \quad \pi^2 u (r \beta \omega) + \pi (1 - \pi) u (d) +$$

$$+ (1 - \pi) \pi u (r (r \beta 2 \omega - d) + (1 - \beta) R 2 \omega) +$$

$$+ (1 - \pi)^2 u (r^2 \beta \omega + (1 - \beta) R \omega)$$

subject to

$$0 \leq \beta \leq 1$$
$$0 \leq d \leq 2 r \beta \omega$$

(POOL)
Liquidity
with risk-pooling (continuation)

To understand the payoffs in the previous page, let’s look at the following tables:

Without pooling:

<table>
<thead>
<tr>
<th>State</th>
<th>Probability</th>
<th>Agent 1’s consumption</th>
</tr>
</thead>
<tbody>
<tr>
<td>All suffer the shock</td>
<td>$\pi^2$</td>
<td>$r \beta \omega$</td>
</tr>
<tr>
<td>Only 1 suffers</td>
<td>$\pi(1-\pi)$</td>
<td>$r \beta \omega$</td>
</tr>
<tr>
<td>Only 2 suffers</td>
<td>$(1-\pi)\pi$</td>
<td>$r^2 \beta \omega + R (1-\beta) \omega$</td>
</tr>
<tr>
<td>None suffers</td>
<td>$(1-\pi)^2$</td>
<td>$r^2 \beta \omega + R (1-\beta) \omega$</td>
</tr>
</tbody>
</table>

With pooling:

<table>
<thead>
<tr>
<th>State</th>
<th>Probability</th>
<th>Agent 1’s consumption</th>
</tr>
</thead>
<tbody>
<tr>
<td>All suffer the shock</td>
<td>$\pi^2$</td>
<td>$\beta \omega$</td>
</tr>
<tr>
<td>Only 1 suffers</td>
<td>$\pi(1-\pi)$</td>
<td>$d$</td>
</tr>
<tr>
<td>Only 2 suffers</td>
<td>$(1-\pi)\pi$</td>
<td>$r (r \beta 2 \omega - d) + R (1-\beta)2 \omega$</td>
</tr>
<tr>
<td>None suffers</td>
<td>$(1-\pi)^2$</td>
<td>$r^2 \beta \omega + R (1-\beta) \omega$</td>
</tr>
</tbody>
</table>
Remark 1: The agents are always better-off by pooling their resources. The optimal \((\beta^*, d^*)\) that solves the maximization problem (POOL) always delivers a higher utility than the \(\beta^{**}\) that solves the maximization problem (INDIVIDUAL).

Remark 2: For risk-pooling to occur is crucial that not all agents receive liquidity shocks at the same time. If they face aggregate uncertainty, they can not (fully) diversify their risks. Individual (idiosyncratic uncertainty) can be diversified.
Risk Pooling

Let’s assume:

- Three dates (0, 1 and 2).
- Infinitely many consumers $i \in [0, 1]$, each one with $W_i = 1$ and same preferences $u_i = u$.
- Investment opportunities and consumption are as before.
- Probabilities of liquidity shocks are independent.
- $\pi_1$ prob. of a ‘bad’ liquidity shock or fraction of consumers who suffer a ‘bad’ shock.
- Company decides on investment decision for the pool of consumers, it promises $c_1$ to early consumers and $c_2$ to late consumers.
- Company faces no risk (Law of Large Numbers), its plans are feasible if $\pi_1 c_1 = \beta$ and $\pi_2 c_2 = (1 - \pi_1) c_2 = (1 - \beta) R$.

$$\max_{\beta} \pi_1 u(c_1) + (1 - \pi_1) u(c_2) \quad (2.2)$$

\begin{align*}
\text{st.} \\
0 \leq \beta \leq 1 \\
\pi_1 c_1 = \beta \\
\pi_2 c_2 = (1 - \beta) R
\end{align*}
We saw before that under risk-pooling, the company promises \( c_1 = \beta / \pi_1 \) to each depositor who needs to consume in period 1. Let’s assume we have a finite number of consumers, \( N \).

- Clearly this promise can be carried out if the number of agents who do suffer a liquidity shock, \( \tilde{N} \), is \textit{less or equal} than the average number of consumers who suffer a liquidity shock \( \pi_1 N \).

- However, if \( \tilde{N} > \pi_1 N \), the company can not honor its promises.
Law of Large Numbers
eliminating uncertainty thru averages, side comment

Assume that instead of promising the maximum possible, 
\( c_1 = \frac{\beta}{\pi_1} \) for some small \( \varepsilon > 0 \), the company promises to pay
\( \hat{c}_1 = (1 - \varepsilon) c_1 \).

What is the probability that the company can honor its promises?

\[
\Pr \left[ \tilde{N} \cdot \hat{c}_1 \leq N \cdot \beta \right] = \Pr \left[ \tilde{N} \cdot (1 - \varepsilon) \beta / \pi_1 \leq N \cdot \beta \right]
\]

\[
= \Pr \left[ \tilde{N} \leq \frac{\pi_1 N}{(1 - \varepsilon \pi_1)} \right] = \sum_{k=0}^{\left\lfloor \frac{\pi_1 N}{(1 - \varepsilon)} \right\rfloor} \frac{N!}{k!(N-k)!} \pi^k (1 - \pi_1)^{N-k}
\]

where \([x]\) is the floor function, it gives the largest integer below \(x\).
Law of Large Numbers
eliminating uncertainty thru averages, side comment

What is the probability that the company can honor its promises?

\[
\Pr \left( \tilde{N} \leq \frac{\pi_1 N}{(1 - \varepsilon)} \right) = \sum_{k=0}^{\pi_1 N \frac{N!}{(1-\varepsilon)}} \frac{N!}{k!(N-k)!} \pi^k (1 - \pi_1)^{N-k}
\]

Assume \( \pi = \frac{1}{3} \) then:

<table>
<thead>
<tr>
<th>( \varepsilon )</th>
<th>( N )</th>
<th>2</th>
<th>10</th>
<th>100</th>
<th>1,000</th>
<th>9,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>.1</td>
<td>.1</td>
<td>0.444444</td>
<td>0.559264</td>
<td>0.812311</td>
<td>0.993344</td>
<td>1</td>
</tr>
<tr>
<td>.01</td>
<td>.01</td>
<td>0.444444</td>
<td>0.559264</td>
<td>0.518803</td>
<td>0.585493</td>
<td>0.75259</td>
</tr>
<tr>
<td>.005</td>
<td>.005</td>
<td>0.444444</td>
<td>0.559264</td>
<td>0.518803</td>
<td>0.559216</td>
<td>0.63596</td>
</tr>
<tr>
<td>0</td>
<td>.005</td>
<td>0.444444</td>
<td>0.559264</td>
<td>0.518803</td>
<td>0.505947</td>
<td>0.504956</td>
</tr>
</tbody>
</table>

Please see the Mathematica file largenumbers.nb
The model is the same as before in the risk-pooling case with the exception that consumers investment decisions are again individual and there is a market at $t = 1$ for the long-asset.

- $0 \leq x$ fraction of wealth invested in the long-asset,
- $0 \leq y$ fraction of wealth invested in the short-asset.
- $P$ is the price of the long-asset at date $t = 1$.
- It must be that $P \leq R$ in eq.
- $\lambda$ prob. investor wants to consume only at $t = 1$.

$$\max_{x,y \atop x+y \leq 1} \lambda u(y + Px) + (1 - \lambda) u \left((x + \frac{y}{P})R\right)$$

In eq. we have that

$$P = 1 \Rightarrow c_1 = 1 \& c_2 = R \Rightarrow u^* = \lambda u(1) + (1 - \lambda) u( R)$$
A deposit contract is a pair of consumption promises \((c_1, c_2)\) such that if a consumer deposits his entire wealth \(W_0 = 1\) at the bank at \(t = 1\), the consumer has the right to withdraw \(c_1\) at date 1 and \(c_2\) at date 2.

The optimal deposit contract is the one that solves the risk-pooling problem.

Liquidation technology: long asset is worth \(r \leq 1\) units of the good at date \(t = 1\).

If \(c_1 > rx + y\) the bank will no be able to honor the deposit contract if all consumers ‘run’.

If \(c_2 > c_1 > rx + y\) we may or may not have a bank-run (it depends on the consumers/investors) expectations.
In the previous analysis, we assumed that the deposit contract was the same regardless if the Bank expected a bank run or not. But if the Bank expects a bank-run with certainty, the bank will offer a deposit contract \((c_1, c_2) = (1, 1)\) and so late consumers are indifferent between withdrawing at \(t = 1\) or at \(t = 2\).
To simplify the model, we shall assume that the liquidation technology is perfect: that is $r = 1$ so there is no penalty in liquidating the long-asset at $t = 1$. The bank can transform the long-asset into the good or cash or short-asset in a 1-to-1 ratio. In this case the long-asset dominates the short-asset as there is no reason for the bank to hold the short-asset. If the bank needs cash to pay depositors at $t = 1$ the bank can just liquidate the long-asset.
At $t = 0$, the total wealth of the bank is $N \cdot W_0$ where $W_0$ is the initial wealth and $N$ is the number of depositors.

At $t = 1$, the total wealth of the bank is reduced by the withdraws which amount: $\lambda \cdot N \cdot c_1$ where $c_1$ is the amount promised by the deposit contract and $\lambda$ is the fraction of early consumers.

At $t = 2$, the total wealth of the bank is $R \cdot N \cdot (W_0 - \lambda c_1)$ which must cover the withdraws by late consumers which amount to $(1 - \lambda) \cdot N \cdot c_2$.

Assume for simplicity that $W_0 = 1$ and so the budget constraint for any feasible deposit contract must be:

$$0 \leq c_1 \leq \frac{1}{\lambda} \quad \text{and} \quad 0 \leq c_2 \leq \frac{R(1 - \lambda c_1)}{1 - \lambda}$$

Notice that $N$ cancels out so this is the budget constraint per capita.
Avoiding a run

**Important remark:** If $c_1 \leq 1$ the bank will be solvent to pay any late consumers at $t = 2$ even if a bank-run takes place. So if $c_1 \leq 1$ and $c_1 < c_2$ no late consumer will ever want to join a bank-run.

Thus if the bank wants to avoid a run, it can choose $c_1 \leq 1$. In this case, the bank maximizes:

$$
\max_{0 \leq c_1 \leq 1} \max_{0 \leq c_2 \leq \frac{R(1-\lambda c_1)}{1-\lambda}} \lambda u(c_1) + (1 - \lambda) u(c_2) \quad (P_{NR})
$$

To solve this we substitute $\frac{R(1-\lambda c_1)}{1-\lambda}$ for $c_2$ and take derivative with respect to $c_1$, equate to zero, and solve for $c_1$. If the solution satisfies $c_1 \leq 1$ we are done but if the ‘solution’ has $c_1 > 1$ then we are violating the constraint $c_1 \leq 1$. In this case the true solution must be $c_1 = 1$ and so $c_2 = \frac{R(1-\lambda)}{1-\lambda} = \frac{R(1-\lambda)}{1-\lambda} = R$. 

Not avoiding a run

Assume that in the previous problem $P_{NR}$ the optimal deposit contract that avoids a run is $(1, R)$ that is, ideally the bank would like to have $c_1 > 1$ but that may cause a run so the bank sets $c_1 = 1$. In this event, what will be the optimal deposit contract if the bank does not avoid a run. In the case the bank expects a run with probability $\pi$, the bank maximizes:

$$\max_{0 \leq c_1} \pi u(1) + (1 - \pi) [\lambda u(c_1) + (1 - \lambda) u(c_2)] \quad (P_N)$$

To solve this we again substitute $\frac{R(1-\lambda c_1)}{1-\lambda}$ for $c_2$ and take derivative with respect to $c_1$, equate to zero, and solve for $c_1$. The optimal solution does not depend on the value of $\pi$!
Which is the optimal deposit contract? We saw the optimal deposit contract when the bank wants to avoid a run and when it does not. The overall optimal deposit contract is the one that yields the greatest utility. The bank will avoid a run if and only if:

\[
\lambda u(1) + (1 - \lambda) u(R) = \max_{0 \leq c_1 \leq 1} \lambda u(c_1) + (1 - \lambda) u(c_2) > \\
> \max_{0 \leq c_1 \leq \frac{R(1 - \lambda)}{1 - \lambda}} \frac{R(1 - \lambda)}{1 - \lambda} \pi u(1) + (1 - \pi) \left[ \lambda u(c_1) + (1 - \lambda) u(c_2) \right] = \\
= \pi u(1) + (1 - \pi) \max_{0 \leq c_1 \leq \frac{R(1 - \lambda)}{1 - \lambda}} \lambda u(c_1) + (1 - \lambda) u(c_2)
\]

that is, only if \( \pi \) is sufficiently large.
The model is similar the the ones we saw before. But now there is uncertainty regarding the fraction of the population who suffers a liquidity shock: it could be high or low, $\lambda_H$ (in state $H$ that happens with prob. $\pi$) or $\lambda_L$ (in state $L$ that happens with prob. $1 - \pi$).

- Dates: $t = 0, 1, 2$. States: $s = H, L$.
- At date 0: consumers make deposit decisions and banks offer deposit contracts and then make portfolio decisions (how much to invest in the short-asset and how much to invest in the long-asset).
- At the start of date 1: all learn what is the current state and mkts. open for trade.
- At $t = 1$ there are two markets: the good market (where $c_2$ is exchanged for $c_1$) and the asset market (where the long-asset yields a payoff (rate of return $R > 1$)).
Let $p_s$ be the price of $c_2$ (in terms of $c_1$) and $P_s$ the price of the long-asset (in terms of the short-asset). (Notice we can always convert $c_1$ into the short-asset and vice-versa).

- Banks are profit maximizing and they compete to attract depositors.
- At $t = 1$, consumers can not trade in the asset market nor in the forward good market.
- At $t = 1$, banks can trade in both markets.
- Consumers place all their wealth in one bank.
Banks are profit maximizing and compete to attract depositors.

Consumers can not trade in the asset market nor in the forward (good) market.

\[ P_s = R \cdot p_s \]

\[ p_s \leq 1 \]
Let’s assume for now that consumers do their own investment: \( y \) in the short-asset and \( x \) in the long asset, \( x + y = 1 \).

**Behavior in the consumption good market**

Obviously their consumption will be lower since there is no risk-pooling:

- \( c_{1s} = y \) where \( s = H, L \) and
- \( c_{2s} = R \cdot \left( \frac{y}{P_s} + (1 - y) \right) \) since in equilibrium \( R/P_s \leq 1 \).
- The consumer takes \( P = (P_H, P_L) \) as given and chooses \( y \) to maximize the expected utility

\[
U(c_{1H}, c_{2H}, c_{1L}, c_{2L}) = \pi \cdot \lambda_H \cdot u(c_{1H}) + \pi \cdot (1 - \lambda_H) \cdot u(c_{2H}) + (1 - \pi) \cdot \lambda_L \cdot u(c_{1L}) + (1 - \pi) \cdot (1 - \lambda_L) \cdot u(c_{2L})
\]
**Behavior in the asset market**

In the good market we took as given the price of the assets $P = (P_H, P_L)$. In the asset market, we take as given the consumption decisions and solve for the asset prices that equate supply and demand.

- $S_s(P_s) = (1 - y)\lambda_s$, supply of long-asset (inelastic, early consumers sell).

- $D_s(P_s) = \begin{cases} 
  (1 - \lambda_s)y & \text{if } P_s < R \\
  \frac{P_s}{R} & \text{if } P_s = R \\
  0 & \text{if } P_s > R 
\end{cases}$, demand of long-asset (late consumers buy as long as price is not too high, $P_s \leq R$)

We solve supply equal demand to find $P_s$ for $s = H, L$. 

With financial intermediaries (banks) consumers are able to consume more:

- \( c_{1s} = \begin{cases} 
  d & \text{if incentive constraint holds} \\
  y + (1 - y)P_s & \text{otherwise}
\end{cases} \)

- \( c_{2s} = \begin{cases} 
  y + (1 - y)P_s - \lambda_s d & \text{if incentive constraint holds} \\
  y + (1 - y)P_s & \text{otherwise}
\end{cases} \)

- The bank takes \( P = (P_H, P_L) \) as given and chooses \((d, y)\) to maximize the expected utility \( U(c_{1H}, c_{2H}, c_{1L}, c_{2L}) = \)

  \[
  \pi \cdot \lambda_H \cdot u(c_{1H}) + \pi \cdot (1 - \lambda_H) \cdot u(c_{2H}) +
  (1 - \pi) \cdot \lambda_L \cdot u(c_{1L}) + (1 - \pi) \cdot (1 - \lambda_L) \cdot u(c_{2L})
  \]
In state $s$ banks will be able to avoid a bank-run if and only if late consumers lack the incentive to withdraw earlier (at $t = 1$), this happens only when

$$y + (1 - y)P_s \geq \lambda_s d + p_s(1 - \lambda_s)d,$$

or equivalently

$$y + (1 - y)P_s \geq \lambda_s d + \frac{P_s}{R}(1 - \lambda_s)d.$$

When we consider the equilibrium, there are essentially two cases: the incentive condition always holds in both states ("no default or crises") and the incentive condition fails to hold in the "bad" state ($s = H$).
If no bank ever defaults in period 1, then given \((P_H, P_L)\), all banks are maximizing the same function (the expected utility of a depositor). An important consequence of this fact is that:

*All banks will choose the same deposit contract \((d, y)\).*
Behavior in the consumption good market:
This is as before, the banks take $P_H, P_L$ as given and choose $d$ and $y$ to maximize the consumer’s expected utility.

Behavior in the asset market:
In state $s$, banks have to pay $d$ to early consumers and they hold $y$ as cash. So they have to cover the difference (if positive) by selling some amount of the long-asset. If they have more cash than they need, they must be willing to hold cash until $t = 2$. As the supply of cash is $S_s(P_s) = y$ and the demand is $D_s(P_s) = \lambda_s \cdot d$, we have:

$$y > \lambda_s \cdot d \Rightarrow P_s = R$$

$$y \leq \lambda_s \cdot d \Rightarrow P_s \leq R$$
Because (in the no default case) banks are choose the same strategy \((d, y)\). No banks can ever be short of cash at state \(s\) because if one bank has to liquidate (forced to sell) some of the long asset, then all banks will be selling (and none will be buying) so the price would fall to \(P_s = 0\). But if the price of the long-asset is zero in any state, then in period \(t = 0\), any bank should invest all in the short-asset and buy an infinite amount of the long-asset in period \(t = 1\) in the state where \(P_s = 0\). But all banks doing this cannot be part of an equilibrium...
Behavior in the asset market
no default, continuation

So we have that $y > \lambda_L \cdot d \Rightarrow P_L = R$ and $y = \lambda_H \cdot d$.

Why $y = \lambda_H \cdot d$? Because if $y > \lambda_H \cdot d$ then we would have excess liquidity in both states but then the bank would be able to reduce its position in the short-asset without compromising its ability to pay depositors. Alternatively if $y < \lambda_H \cdot d$ then the bank would need to (partially) divest from the long-asset but because all banks are using the same strategy this can not happen as we pointed before.

Now remember that when we solved for $y$ and $d$ in bank problem we take $P_H$ and $P_L$ as given so both $y$ and $d$ are functions of $(P_H, P_L) = (P_H, R)$. We can use this to find the value of $P_H$ by solving:

$$y(P_H, R) = \lambda_H \cdot d(P_H, R).$$
The Optimal Deposit Contract
no default

The expected utility of the representative consumer/depositor is:

\[
U(d, y) = (\pi \cdot \lambda_H + (1 - \pi) \cdot \lambda_L) \cdot u(d) + \\
+ \pi \cdot (1 - \lambda_H) \cdot u\left(\frac{y + (1 - y)P_H - \lambda_H d}{(1 - \lambda_H) \cdot p_H}\right) + \\
+ (1 - \pi) \cdot (1 - \lambda_L) \cdot u\left(\frac{y + (1 - y)R - \lambda_L d}{(1 - \lambda_L) \cdot 1}\right)
\]
The Optimal Deposit Contract

no default

We know that $d = y/\lambda_H$ and $P_L = R$ and there is no default, so we can write the expected utility of the depositor as a function of $y$ only (of course it also depends of $P_H$ but the bank takes it as given).

$$U\left(\frac{y}{\lambda_H}, y\right) = (\pi \cdot \lambda_H + (1 - \pi) \cdot \lambda_L) \cdot u\left(\frac{y}{\lambda_H}\right) +$$

$$+ \pi \cdot (1 - \lambda_H) \cdot u\left(\frac{y + (1 - y)P_H - \lambda_H \frac{y}{\lambda_H}}{(1 - \lambda_H) \cdot P_H}\right) +$$

$$+ (1 - \pi) \cdot (1 - \lambda_L) \cdot u\left(\frac{y + (1 - y)R - \lambda_L \frac{y}{\lambda_H}}{(1 - \lambda_L) \cdot 1}\right)$$
The Optimal Deposit Contract
no default, continuation

Remember that $P_s = R \cdot p_s$ so we can simplify:

$$U\left(\frac{y}{\lambda_H}, y\right) = (\pi \cdot \lambda_H + (1 - \pi) \cdot \lambda_L) \cdot u\left(\frac{y}{\lambda_H}\right) +$$

$$+ \pi \cdot (1 - \lambda_H) \cdot u\left(\frac{(1 - y) R}{1 - \lambda_H}\right) +$$

$$+ (1 - \pi) \cdot (1 - \lambda_L) \cdot u\left(\frac{(1 - \frac{\lambda_L}{\lambda_H}) y + (1 - y) R}{(1 - \lambda_L)}\right)$$

We now solve $\frac{d}{dy} U = \frac{\partial}{\partial d} U \cdot \frac{\partial}{\partial y} \frac{y}{\lambda_H} + \frac{\partial}{\partial y} U = 0$ for $y$ ...
Notice that we solved for $y$ and $d$ without using the value of $P_H$! It cancelled because we assumed $d = \frac{y}{\lambda_H}$ ...

How should we obtain the equilibrium value of $P_H$ then? The price $P_H$ must be such that the bank chooses $y = \lambda_H \cdot d$ ...

$$U(d, y) = (\pi \cdot \lambda_H + (1 - \pi) \cdot \lambda_L) \cdot u(d) +$$

$$+ \pi \cdot (1 - \lambda_H) \cdot u \left( \frac{y + (1 - y)P_H - \lambda_H \cdot d}{(1 - \lambda_H)P_H/R} \right) +$$

$$+ (1 - \pi) \cdot (1 - \lambda_L) \cdot u \left( \frac{y + (1 - y)R - \lambda_L \cdot d}{(1 - \lambda_L)} \right)$$
The price $P_H$ must be such that the bank chooses $y = \lambda_H \cdot d$.

$$\frac{\partial}{\partial y} U(d, y) = 0 +$$

$$\pi \cdot (1 - \lambda_H) \cdot u' \left( \frac{y + (1 - y)P_H - \lambda_H \cdot d}{(1 - \lambda_H)P_H / R} \right) \cdot \frac{1 - P_H}{(1 - \lambda_H)P_H / R} +$$

$$(1 - \pi) \cdot (1 - \lambda_L) \cdot u' \left( \frac{y + (1 - y)R - \lambda_L \cdot d}{(1 - \lambda_L)} \right) \cdot \frac{1 - R}{(1 - \lambda_L)}$$

Simplifying and using $d = \frac{y}{\lambda_H}$:

$$\frac{\partial}{\partial y} U(d, y) = \pi \cdot u' \left( \frac{(1 - y)R}{(1 - \lambda_H)} \right) \cdot \frac{(1 - P_H) \cdot R}{P_H} +$$

$$\left(1 - \pi \right) \cdot u' \left( \frac{(1 - \frac{\lambda_L}{\lambda_H})y + (1 - y)R}{(1 - \lambda_L)} \right) \cdot (1 - R) = 0$$

Notice this is a linear equation in $P_H$! We solve it for $P_H$. Notice that in the solution we must have $P_H < 1$! This implies the short-asset is not dominated at $t = 0$.

At $t = 1$ and $s = H$, because $P_H < 1 < R$, banks lose money if they sell the long-asset so no wants to sell it and also no bank has cash to buy it.
An Example

Let \( u(x) = \ln(x) \) then solving \( U'(y) = 0 \) for \( y \) gives us:
Default Scenario

Remember that under no default, no banks ever sold amounts of the long-asset. They carried enough cash to pay depositors. But if default occurs (say in the $H$ state), banks need to sell the long-asset. But in this case we can not have the all banks in default otherwise $P_H = 0$ (all want to sell the long-asset and none wants to buy) which is not compatible with equilibrium. The only way that some banks escape default is if they use a more conservative strategy (they hold more cash and promise lower payments at date 1):

- safe banks’ choice $(y^B, d^A)$
- risky banks’ choice $(y^R, d^A)$
- where $d^B > y^R$ and $d^B < d^A$
- $\rho$ is the fraction of risky banks ($A=$active)
- $1 - \rho$ is the fraction of safe banks ($B=$boring)
We are going to construct examples of equilibrium where:

- The risky banks always sell long-asset to the safe banks at $t = 1$.
- In the state $H$, there will be no excess liquidity and the price of the long-asset will be less than one, $P_H < 1$.
- In the state $H$, as the price of the long-asset is too low, the risky banks are insolvent. They go bankrupt.
The banks’ objectives

\[ U^B(d^B, y^B) = (\pi \cdot \lambda_H + (1 - \pi) \cdot \lambda_L) \cdot u\left(d^B\right) + \]
\[ + \pi \cdot (1 - \lambda_H) \cdot u\left(\frac{y^B + (1 - y^B)P_H - \lambda_H d^B}{(1 - \lambda_H) \cdot P_H/R}\right) + \]
\[ + (1 - \pi) \cdot (1 - \lambda_L) \cdot u\left(\frac{y^B + (1 - y^B)P_H - \lambda_L d^B}{(1 - \lambda_L) \cdot P_L/R}\right) \]

\[ U^A(d^A, y^A) = (1 - \pi) \cdot \lambda_L \cdot u\left(d^A\right) + \]
\[ + \pi \cdot u\left(y^A + (1 - y^A)P_H\right) + \]
\[ + (1 - \pi) \cdot (1 - \lambda_L) \cdot u\left(\frac{y^A + (1 - y^A)P_L - \lambda_L d^A}{(1 - \lambda_L) \cdot P_L/R}\right) \]
Market Clearing Conditions

\[ \rho c_1(s, d^A, y^A) + (1 - \rho)c_1(s, d^B, y^B) \begin{cases} = \rho y^A + (1 - \rho)y^B & \text{if } p_s < 1, \\ \leq \rho y^A + (1 - \rho)y^B & \text{if } p_s = 1 \end{cases} \]

\[ \rho C(s, d^A, y^A) + (1 - \rho)C(s, d, y^B) = \rho(y^A + R(1 - y^A)) + (1 - \rho)(y^B + R(1 - y^B)), \]

where \( C = c_1 + c_2 \).

If \( p_s < 1 \) banks are not willing to carry cash from date 1 into date 2.
Other Eq. Conditions

- Consumers utility of putting money on A or B bank is the same.