Outline

Road Map
Decision Problems
Static Games
Nash Equilibrium
Pareto Efficiency
Constrained Optimization
Applications of Nash Equilibrium
Other Solution Concepts
Mixed Strategies
Dynamic Games
  Bargaining
Bayesian and Extensive Games of Incomplete Info
  Insurance
Repeated Games
Cooperative Game Theory
Can players enter into bidding agreements?
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- **Cooperative GT.**
  - yes

- **Non-Cooperative GT.**
  - no
<table>
<thead>
<tr>
<th>Uncertainty</th>
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<tbody>
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<td>Dynamic</td>
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## Non-Cooperative Games

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1. normal form games
Non-Cooperative Games

Uncertainty

Dynamic

No Uncertainty

Static

1. normal form games

2. extensive games with perfect info.
Non-Cooperative Games

Uncertainty

Dynamic

1. normal form games

Static

2. extensive games with perfect info.

3. Bayesian games.

No Uncertainty

1. normal form games
Non-Cooperative Games

Dynamic

Uncertainty

4. extensive games with imperfect info.

No Uncertainty

2. extensive games with perfect info.

Static

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1. normal form games.
Strategic versus Decision Problems

Decision Problems (studied by Operations Research)

- Consumer
- Competitive Firm

Engineering Problems

Strategic Problems (studied by Game Theory)

- Agents care about the others decisions because ...
- their decisions affect the agents’ utility/profit/payoff.
- put simply, there are externalities
Strategic versus Decision Problems

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Caveats

The issue of whether a problem is strategic or non-strategic is a *modeling choice*

- Sometimes a seemingly non-strategic problem, like an engineering problem, may require strategic considerations.
- The Millennium Bridge wobbling.
- The Millennium Bridge (a possible explanation).
- Other times, a seemingly strategic problem, can be treated as a non-strategic one (no need to over-analyze).
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Other times, a seemingly strategic problem, can be treated as a non-strategic one (no need to over-analyze).
“It is not from the benevolence of the butcher, the brewer, or the baker, that we expect our dinner, but from their regard to their own interest.”

“People of the same trade seldom meet together, even for merriment and diversion, but the conversation ends in a conspiracy against the public, or in some contrivance to raise prices. It is impossible indeed to prevent such meetings, by any law which either could be executed, or would be consistent with liberty and justice. But though the law cannot hinder people of the same trade from sometimes assembling together, it ought to do nothing to facilitate such assemblies; much less to render them necessary.”
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A lottery is a pair of outcomes and their respective probabilities:

\[ \ell = ((x_1, x_2, \ldots, x_n), (p_1, p_2, \ldots, p_n)) \]

where \( x_k \in \mathbb{R} \) and \( p_k \geq 0 \) for all \( k = 1, \ldots, n \) and also \( p_1 + p_2 + \ldots + p_n = 1 \).
The Certain Lottery, Expectation and Variance

The lottery that gives outcome $x$ with probability 1 (with certainty) is denoted:

$$\delta_x = ((x), (1)).$$

The expected value of the $\ell_1 = ((x_1, x_2, \ldots, x_n), (p_1, p_2, \ldots, p_n))$ is:

$$\mathbb{E}[\ell_1] = p_1 \cdot x_1 + p_2 \cdot x_2 + \ldots p_n \cdot x_n = \sum_{i=1}^{n} p_i \cdot x_i;$$

and variance this lottery is

$$\text{Var}[\ell_1] = p_1 \cdot (x_1 - \mathbb{E}[\ell_1])^2 + p_2 \cdot (x_2 - \mathbb{E}[\ell_1])^2 + \ldots p_n \cdot (x_n - \mathbb{E}[\ell_1])^2 =$$

$$= \sum_{i=1}^{n} p_i \cdot (x_i - \mathbb{E}[\ell_1])^2.$$
Composition of Lotteries

Given two lotteries, $\ell_1 = ((x_1, x_2, \ldots, x_n), (p_1, p_2, \ldots, p_n))$ and $\ell_2 = (y_1, y_2, \ldots, y_m), (q_1, q_2, \ldots, q_n)$ and a number $0 < \alpha < 1$, one can create a compound lottery by choosing $\ell_1$ with probability $\alpha$ and $\ell_2$ with probability $1 - \alpha$. 
The compound lottery $\ell$ plays $\ell_1$ with probability $\alpha$ and $\ell_2$ with probability $\ell_2$:

$$\ell = \alpha \ell_1 \oplus (1 - \alpha) \ell_2 = ((x_1, x_2, y_1, y_2), (\alpha p, \alpha (1 - p), (1 - \alpha) q, (1 - \alpha)(1 - q)).$$
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Preferences Over Lotteries

- Preferences or choices?
- Individual autonomy?
- Neurology?
Given to lotteries \( \ell_a \) and \( \ell_b \) such that a decision maker (DM) chooses \( \ell_a \) over \( \ell_b \),

the following statements are equivalent:

- The DM judges \( \ell_a \) no worst than \( \ell_b \) (everyday language);
- The DM prefers \( \ell_a \) to \( \ell_b \) (economics language);
- \( \ell_a \succeq \ell_b \) (mathematics language).

For simplicity we write:

- \( \ell_a \succ \ell_b \) when \( \ell_a \succeq \ell_b \) but \( \ell_b \nless \ell_a \) (strict preference);
- \( \ell_a \sim \ell_b \) when \( \ell_a \succeq \ell_b \) and \( \ell_b \succeq \ell_a \) (indifference).
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A preference of the DM, $\succeq$, over the set of lotteries is just the DM’s ranking of lotteries.

We wish to have a **numerical score** that reflects the DM’s ranking.
von Neuman & Morgenstern’s Assumptions:

**Completeness** For any two lotteries \( \ell_1 \) and \( \ell_2 \),
\[ \ell_1 \succeq \ell_2 \text{ and/or } \ell_2 \succeq \ell_1. \]

**Transitivity** For any lotteries \( \ell_1, \ell_2, \) and \( \ell_3 \),
if \( \ell_1 \succeq \ell_2 \) and \( \ell_2 \succeq \ell_3 \) then \( \ell_1 \succeq \ell_3 \).

**Continuity** If \( \ell_1 \succeq \ell_2 \succeq \ell_3 \) then exists \( p \in [0, 1] \) such that
\[ \ell_2 \sim p\ell_1 + (1 - p)\ell_3. \]

**Independence** If \( \ell_1 \succ \ell_2 \) then for any \( \ell_3 \) and any \( 0 < p < 1 \),
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If $\succeq$ satisfy all of the above, there exists $u : \mathbb{R} \to \mathbb{R}$ such that 

$$((x_1, x_2, \ldots, x_n), (p_1, p_2, \ldots, p_n)) \succ (y_1, y_2, \ldots, y_m), (q_1, q_2, \ldots, q_n))$$

if and only if 

$$\sum_{k=1}^{n} u(x_k) \cdot p_k > \sum_{k=1}^{m} u(y_k) \cdot q_k.$$
Expected Utility

We write:

$$U(\ell_1) = u(x_1) \cdot p_1 + \ldots + u(x_n) \cdot p_n$$

and refer to $U$ as the expected utility and to $u$ as the:

- utility for money
- Bernoulli utility
- von Neumann-Morgenstern utility (vN-M)
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Extracting $u$ from $\geq$
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Extracting $u$ from $\succeq$
Extracting $u$ from $\succeq$
A Behavioral Look at Choice

- Anchoring
- Availability
- Representativeness
- Optimism and over confidence
- Gains and losses
- Status Quo Bias
- Framming
Risk Aversion

Let’s go back to expected utility theory, consider the two lotteries:

\[ \ell_1 = ((100, 200), (\frac{1}{2}, \frac{1}{2}) \]

and

\[ \delta_{150} = ((150), (1)) \]

We have

\[ U(\ell_1) = u(150 - 50) \cdot \frac{1}{2} + u(150 + 50) \cdot \frac{1}{2} \quad \text{and} \]

\[ U(\delta_{150}) = u(150) \cdot 1. \]
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Risk Aversion

\[ U(\ell_1) - U(\delta_{150}) = \left[ \frac{u(200) - u(150)}{50} - \frac{u(150) - u(100)}{50} \right] \cdot \frac{50}{2} \]
Risk Aversion

\[ U(\ell_1) - U(\delta_{150}) = \begin{bmatrix} \frac{u(200) - u(150)}{50} \\ \simeq M u(150) \end{bmatrix} - \begin{bmatrix} \frac{u(150) - u(100)}{50} \\ \simeq M u(100) \end{bmatrix} \cdot \frac{50}{2} \]
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Risk Aversion

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**Expected Utility Theory**

**Attitudes Towards Risk**

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Attitudes Towards Risk

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\[ U(X) < u(E[X]) \text{ for all } X \]

2. Increasing marginal utility, \( u \) is convex, \( u'' > 0 \), the consumer is risk-loving.

\[ U(X) > u(E[X]) \text{ for all } X \]

3. Constant marginal utility, \( u \) is affine (linear plus a constant), \( u'' = 0 \), the consumer is risk-neutral.

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**Expected Utility Theory**

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Measuring the Degree of Risk-Aversion
The Arrow-Pratt or Absolute Measure of

**Definition**

The *Arrow-Pratt* absolute measure of risk-aversion of an agent with VN-M utility $u$ at wealth level $w$ is:

$$\rho_u(w) = \frac{-u''(w)}{u'(w)}.$$

If for two individual with VN-M utilities $u$ and $\tilde{u}$ we have that $\rho_u(w) > \rho_{\tilde{u}}(w)$ for all wealth levels $w$ then we say that the agent with utility $u$ is more risk-averse than the agent with utility $\tilde{u}$. 
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Definition

The relative absolute measure of risk-aversion of an agent with VN-M utility $u$ at wealth level $w$ is:

$$r_u(w) = \frac{-u''(w) w}{u'(w)}.$$
Definition: A strategic game $\Gamma$ (normal form game) is a triple:

$$\Gamma = \left( I, \bigtimes_{i \in I} S_i, \left( u_i : \bigtimes_{i \in I} S_i \to \mathbb{R} \right)_{i \in I} \right)$$

- $I$ is the set of players. Who plays the game? e.g. $I = \{1, 2, \ldots, n\}$ or $I = [0, 1]$

- $S_i$ is the strategy set of player $i$.

- $S = \bigtimes_{i \in I} S_i$ is the set of strategy profiles or outcomes.

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Strategic Problems
Static Games: The Normal Form

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A strategic game $\Gamma = (I, S, u)$ is a *non-cooperative game*.

We assume players are not able to communicate or engage in bidding agreements *unless* – the communication and/or agreements are modeled explicitly as strategies in $S = \times_{i \in I} S_i$.

In the case of two players ($\#I = 2$), the normal form representation of the game is referred as a bi-matrix game. When asked "model this as a strategic game", you are being asked to describe $I$, $S_i$ and $u_i$ for all $i$ in $I$. 
Say we have a game with three players, each player can choose between A, B or C.

Consider the strategy profile $s = (A, B, C)$ where: player 1 chooses $A$, $s_1 = A$; player 2 chooses $B$, $s_2 = B$; and player 3 chooses $C$, $s_3 = C$.

Very often we may want to distinguish a player.

For example, if we want to take the point of view of player 2 we write the previous strategy profile $s = (A, B, C)$ as $s = (B, s_2) = (B, (A, C))$.

In sum, $s_2 = (s_1, s_3)$ is
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Profile Notation

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For example, if we want to take the point of view of player 2 we write the previous strategy profile $s = (A, B, C)$ as

$s = (B, s_{-2}) = (B, (A, C))$.

In sum, $s_{-2} = (s_1, s_3)$ is the strategy profile of players distinct from player 2.
Examples of Strategic Games

1. The Battle of Sexes game.
2. Prisoners’ Dilemma
3. Cournot Duopoly
4. Bertrand Duopoly
5. The Stag-Hunt game (with 3 players)
The Battle of Sexes

The strategic game (normal form)

\[ I = \{1, 2\} \]

- \( S_1 = S_2 = \{B, S\} \),
- \( S = S_1 \times S_2 = \{(B, B), (B, S), (S, B), (S, S)\} \),
- \( u_1(B, B) = u_2(S, S) = 3 \),
- \( u_1(S, S) = u_2(B, B) = 2 \),
- \( u_1(B, S) = u_1(S, B) = 0 \),
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The Battle of Sexes

The strategic game (normal form)

\[
\begin{array}{c|cc}
 & B & S \\
\hline
B & 3 & 0 \\
S & 0 & 2 \\
\end{array}
\]

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The Battle of Sexes

The strategic game (normal form)

\[
\begin{array}{cc}
\text{Player 2} & \text{B} & \text{S} \\
\text{B} & 3 & 0 \\
\text{S} & 0 & 2 \\
\end{array}
\]

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The strategic game (normal form)

$\begin{array}{ccc}
\text{Player 1} & \text{B} & \text{S} \\
\hline
\text{B} & 3 & 0 \\
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\end{array}$

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### The Prisoners’ Dilemma

<table>
<thead>
<tr>
<th>Player 2</th>
<th>Effort</th>
<th>Shirk</th>
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<tbody>
<tr>
<td><strong>Effort</strong></td>
<td>4 4</td>
<td>-1 8</td>
</tr>
<tr>
<td><strong>Shirk</strong></td>
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Golden Balls

Player 1

<table>
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<tr>
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Radio Lab podcast

Golden Balls YouTube video
### The Stag Hunt

<table>
<thead>
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<th></th>
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<tbody>
<tr>
<td><strong>Player 2</strong></td>
<td><strong>Stag</strong></td>
<td><strong>Hare</strong></td>
</tr>
<tr>
<td><strong>Stag</strong></td>
<td>4</td>
<td>0</td>
</tr>
<tr>
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“Or, to change the metaphor slightly, professional investment may be likened to those newspaper competitions in which the competitors have to pick out the six prettiest faces from a hundred photographs, the prize being awarded to the competitor whose choice most nearly corresponds to the average preferences of the competitors as a whole; so that each competitor has to pick, not those faces which he himself finds prettiest, but those which he thinks likeliest to catch the fancy of the other competitors, all of whom are looking at the problem from the same point of view. It is not a case of choosing those which, to the best of one’s judgment, are really the prettiest, nor even those which average opinion genuinely thinks the prettiest. We have reached the third degree where we devote our intelligences to anticipating what average opinion expects the average opinion to be. And there are some, I believe, who practise the fourth, fifth and higher degrees.” – Keynes, The General Theory of Employment, Interest, and Money.
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**Definition:** The strategy profile \( a^* = (a_1^*, \ldots, a_i^*) \) \( \in S \) is a **Nash Equilibrium**, if and only if,

\[ u_i(a_i^*, a_{-i}^*) \leq u_i(a_i, a_{-i}^*), \]

for all \( i \) in \( I \) and for all \( a_i \) in \( S_i \).
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for all $i$ in $I$ and for all $a_i$ in $S_i$. 
A wrong definition (or suggestion) regarding what Nash equilibrium is (about).

**LBD Exercise**

1. Model as a strategic game the matching problem described in the bar scene of the movie "A Beautiful Mind".
2. Compute all the Nash equilibrium of the game.
3. Explain why the behavior recommendation suggested by the character "nash" in the scene fails to be a Nash equilibrium.
### More Examples

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Player 1 | Player 2
More Examples

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Player 2

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More Examples

\[
\begin{array}{ccc}
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|---|-----|-----|-----|
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\end{array}
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More Examples
More Examples

Player 2

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Player 1's choices are $s_1$ and $s_2$, while Player 2's choices are $t_1$, $t_2$, and $t_3$. The payoffs are given for each combination of choices.
More Examples

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More Examples

Player 2

Player 1

\[
\begin{array}{ccc}
 s_1 & t_1 & t_2 & t_3 \\
 4,3 & 2,7 & 0,4 \\
 s_2 & 5,5 & 5,-1 & -4,-2 \\
\end{array}
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### More Examples

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Player 1 | Player 2
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More Examples

Player 2

\[ \begin{array}{c|c|c}
   s_1 & t_1 & t_2 & t_3 \\
   \hline
   s_1 & 4,3 & 2,7 & 0,4 \\
   s_2 & 5,5 & 5,-1 & -4,-2 \\
\end{array} \]
More Examples

\[
\begin{array}{ccc}
\text{Player 2} & t_1 & t_2 & t_3 \\
\hline
\text{Player 1} & \begin{array}{c}
s_1 \\
\end{array} & \begin{array}{c}
4,3 \\
5,5 \\
\end{array} & \begin{array}{c}
2,7 \\
5,-1 \\
\end{array} & \begin{array}{c}
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-4,-2 \\
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Player 2

Player 1
Nash equilibrium is \((s_2, t_1)\) and not \((5,5)\)!
Definition: The strategy profile \( a^* = (a_1^*, \ldots, a_I^*) \in S \) is a **efficient**, if and only if, \( u_i(a^*) < u_i(a) \) for some player \( i \) and some alternative profile \( a \) then exists another player \( j \in I \) such that \( u_j(a^*) > u_j(a) \).

Remark: A Nash equilibrium may or may not be efficient. Most often, it will fail to be efficient because of the externalities in the strategic environment.
Cooperative Games: Pareto Efficiency

**Definition:** The strategy profile \( a^* = (a_1^*, \ldots, a_I^*) \in S \) is a **efficient**, if and only if, \( u_i(a^*) < u_i(a) \) for some player \( i \) and some alternative profile \( a \) then exists another player \( j \in I \) such that \( u_j(a^*) > u_j(a) \).

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**Remark:** A Nash equilibrium may or may not be **efficient**. Most often, it will fail to be efficient because of the externalities in the strategic environment.
**Definition:** A welfare function is an increasing map from the player’s payoffs into the real line,

\[ W : \mathbb{R} \rightarrow \mathbb{R} \]

\[ (u_1, u_2, \ldots, u_I) \mapsto W(u_1, u_2, \ldots, u_I) \]

\[ \frac{\partial}{\partial u_i} W \geq 0 \text{ for every player } i = 1, 2, \ldots, I. \]

An iso-welfare curve is a region in the utility space such that the welfare function is constant, \( W(u_1, u_2, \ldots, u_I) = \text{cte.} \)
**Definition:** A welfare function is an increasing map from the player’s payoffs into the real line,

\[
W : \mathbb{R}^I \longrightarrow \mathbb{R}
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\[(u_1, u_2, \ldots, u_I) \longmapsto W(u_1, u_2, \ldots, u_I)\]

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An iso-welfare curve is a region in the utility space such that the welfare function is constant, \(W(u_1, u_2, \ldots, u_I) = \text{cte.}\)
Pareto Efficiency and Welfare in payoff space

\[ u_1 \quad u_2 \]

\begin{array}{c|c|c|c|c|c}
\hline
& \text{Player 1} & \text{Player 2} \\
\hline
\text{Effort} & 9 & 1 \\
\text{Shirk} & 10 & 2 \\
\hline
\end{array}

\text{Effort} \quad \text{Shirk}
Pareto Efficiency and Welfare in payoff space

\[ W(u_1, u_2) = u_1 + u_2 \]
Pareto Efficiency and Welfare in payoff space

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Pareto Efficiency and Welfare in payoff space

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Pareto Efficiency and Welfare in payoff space

\[ W(u_1, u_2) = u_1 + u_2 \]
Pareto Efficiency and Welfare
in payoff space
Pareto Efficiency and Welfare in payoff space

not efficient!
makes all better-off
Pareto Efficiency and Welfare in payoff space
Pareto Efficiency and Welfare in payoff space

no alternative makes all better-off

efficient!
Pareto Efficiency and Welfare in payoff space
Pareto Efficiency and Welfare in payoff space

Efficient! no alternative makes all better-off
Pareto Efficiency and Welfare in payoff space

$u_1$ $u_2$

$1$ $2$ $9$ $10$
Pareto Efficiency and Welfare
in payoff space

\begin{align*}
\text{no alternative makes all better-off}
\end{align*}
Pareto Efficiency & Welfare Functions

If $s = (s_1, \ldots, s_n) \in S$ maximizes $W(u(s))$ then $s$ is an efficient outcome and $u(s) = (u_1(s), \ldots, u_n(s))$ is an efficient payoff.

Conversely if the set of feasible payoffs $\mathcal{U} = \{v \in \mathbb{R}^n : \text{exists } s \text{ such that } u(s) = v\}$ is convex then any efficient payoff maximizes a linear welfare function.

A set $A$ in $\mathbb{R}^n$ is convex if the line joining any two elements of the set is inside the set, that is for any $a$ and $b$ in $A$, we must have that $\alpha \cdot a + (1 - \alpha) \cdot b$ is in $A$ for all $\alpha \in [0, 1]$.

We call a weighted average of the vectors $a$ and $b$, which is also a vector, $\alpha \cdot a + (1 - \alpha) \cdot b$, a "convex combination of $a$ and $b$".

Example: $a = (5, 7)$ and $b = (-1, 0)$ then $\alpha \cdot a + (1 - \alpha) \cdot b = (\alpha \cdot 5 + (1 - \alpha) \cdot (-1), \alpha \cdot 7 + (1 - \alpha) \cdot 0)(6\alpha - 1, 7\alpha)$. 
Pareto Efficiency & Welfare Functions

If \( s = (s_1, \ldots, s_n) \in S \) maximizes \( W(u(s)) \) then \( s \) is an efficient outcome and \( u(s) = (u_1(s), \ldots, u_n(s)) \) is an efficient payoff.

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\]
Profit functions are

\[ u_1(q_1, q_2) = (1 - q_1 - q_2 - c_1) \cdot q_1 \] and
\[ u_2(q_1, q_2) = (1 - q_1 - q_2 - c_2) \cdot q_2 \]

Claim: Let \((v_1, v_2)\) be such that

\[ v_1 \in \left[ 0, \left( \frac{1 - c_1}{2} \right)^2 \right] \]

\[ v_2 = \max_{q_1, q_2} u_2(q_1, q_2) \]

subject to
\[ u_1(q_1, q_2) = v_1 \]

then \((v_1, v_2)\) is an efficient payoff.
Finding the efficient frontier for a Cournot duopoly

\[ u_1(q_1, q_2) = v_1 \Rightarrow -2q_1^2 + (1 - c_1 - q_2)q_1 - v_1 = 0 \]

\[ u_1(q_1, q_2) = v_1 \Rightarrow q_1 = \frac{-(1 - c_1 - q_2) + \sqrt{(1 - c_1 - q_2)^2 - 4(-2)(-v_1)}}{2 \cdot (-2)} \]

\[ q_1 = \frac{(1 - c_1 - q_2) - \sqrt{(1 - c_1 - q_2)^2 - 8v_1}}{4} \]
Nash Equilibrium
Examples

1. Battle of Sexes
2. Cournot Duopoly
3. Stag-Hunt (with 3 players)
4. Prisoner’s Dilemma
5. Coordination Games
Two firms produce an homogeneous good at unit cost $c$. The market demand is $P = \alpha - \beta Q$ where $Q = q_1 + q_2$. Firms simultaneously chose their output quantities. Payoffs are:

$$u_1(q_1, q_2) = (\alpha - \beta q_1 - \beta q_2 - c) \cdot q_1 \quad \text{and} \quad u_2(q_1, q_2) = (\alpha - \beta q_1 - \beta q_2 - c) \cdot q_2$$

Claim: $(q_1, q_2) = \left(\frac{\alpha - c}{3\beta}, \frac{\alpha - c}{3\beta}\right)$ is the unique Nash equilibrium.
Cournot Duoploy

Two firms produce an homogeneous good at unit cost $c$. The market demand is $P = \alpha - \beta Q$ where $Q = q_1 + q_2$. Firms simultaneously chose their output quantities. Payoffs are:

$$u_1(q_1, q_2) = (\alpha - \beta q_1 - \beta q_2 - c) \cdot q_1$$ and $$u_2(q_1, q_2) = (\alpha - \beta q_1 - \beta q_2 - c) \cdot q_2$$

Claim:

$$(q_1, q_2) = \left( \frac{\alpha - c}{3\beta}, \frac{\alpha - c}{3\beta} \right)$$ is the unique Nash equilibrium.
Proving that \((q_1, q_2) = \left(\frac{\alpha - c}{3\beta}, \frac{\alpha - c}{3\beta}\right)\) is the unique Nash equilibrium...

\[ u_1(q_1, \frac{\alpha - c}{3\beta}) = (\alpha - \beta q_1 - \beta \frac{\alpha - c}{3\beta} - c) \cdot q_1 = \left(\frac{2}{3}(\alpha - c) - \beta q_1\right) \cdot q_1 \]
**BEST RESPONSE FUNCTION**

**Notation:**
For a set $Z$, we write $2^Z$ to denote set of all parts of the set $Z$.

**Definition:** The best response of player $i$ to a given action profile of the other players is denoted by $BR_i : S_{-i} \rightarrow 2^{S_i}$ and defined by:

$$BR_i(a_{-i}) = \arg \max_{a_i \in S_i} u_i(a_i, a_{-i}),$$

In another words,
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In another words,

$$a_i \in BR_i(a_{-i}) \text{ if and only if } \text{ for any } \hat{a}_i \in S_i$$

$$\Rightarrow u_i(a_i, a_{-i}) \geq u_i(\hat{a}_i, a_{-i}).$$
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In another words,

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It follows (almost immediately) from the definitions:

The strategy profile $a^* = (a_1^*, \ldots, a_I^*)$ is a Nash Equilibrium, if and only if,

$a_i^* \in Br_i(a_{-i}^*)$ for all $i \in I$. 
It follows (almost immediately) from the definitions:

The strategy profile \( a^* = (a_1^*, \ldots, a_I^*) \) is a Nash Equilibrium, if and only if, \( a_i^* \in BR_i(a_{-i}^*) \) for all \( i \in I \).
Finding Nash Equilibrium by Means of Best Responses

For simplicity assume that for any player $i$ and any action profile of the other players $a_{-i}$, $i$’s best response is unique.

In this case, the Nash equilibria are the solutions of the system of $I$ equations:

$$a_i^* = BR_i(a_{-i}^*) \text{ for all } i \in I.$$
For simplicity assume that for any player $i$ and any action profile of the other players $a_{-i}$, $i$’s best response is unique.

In this case, the Nash equilibria are the solutions of the system of $I$ equations:

$$a_i^* = BR_i(a_{-i}^*) \text{ for all } i \in I.$$
Best Responses

Legend: \( BR_1 = \bigcirc \)
\( BR_2 = \bigstar \)

\[
\begin{array}{cc|cc}
\text{Player 2} & \text{Stag} & \text{Hare} \\
\hline
\text{Stag} & 4 & 0 \\
\text{Hare} & 1 & 1 \\
\end{array}
\]
Best Responses

Legend:

\[ BR_1 = \circ \]

\[ BR_2 = \star \]

\[
\begin{array}{c|c}
\text{Stag} & \text{Hare} \\
\hline
4 & 0 \\
4 & 1 \\
\end{array}
\]

\[
\begin{array}{c|c}
\text{Hare} & \text{Hare} \\
\hline
1 & 0 \\
1 & 1 \\
\end{array}
\]
Best Responses

Legend: $BR_1 = \circ$
$BR_2 = \star$

<table>
<thead>
<tr>
<th></th>
<th>Stag</th>
<th>Hare</th>
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<tbody>
<tr>
<td>Stag</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>Hare</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Player 2

Player 1

$S_2$

$H$

$S$

$S_1$
Best Responses

Legend:

$BR_1 = \circ$

$BR_2 = \star$

Game:

<table>
<thead>
<tr>
<th></th>
<th>Stag</th>
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<tbody>
<tr>
<td>Stag</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>Hare</td>
<td>1</td>
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Player 1

Player 2
Best Responses

Legend: $BR_1 = \circ$

$BR_2 = \star$

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Nash eq. = all players are best responding, best response curves cross.
Best Responses

Legend: $BR_1 = \bigcirc$

$BR_2 = \bigstar$

Player 1

Player 2

<table>
<thead>
<tr>
<th>Effort</th>
<th>Shirk</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>10</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>
Best Responses

Legend: \( BR_1 = \circ \)
\( BR_2 = \star \)

\[
\begin{array}{c|c|c}
\text{Effort} & \text{Shirk} \\
\hline
9 & 1 \\
9 & 10 \\
10 & 2 \\
1 & 2 \\
\end{array}
\]
Best Responses

Legend: \( BR_1 = \circ \)
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</tr>
</tbody>
</table>

Graph showing Best Responses with points labeled E and S.
Best Responses

Legend: \( BR_1 = \bigcirc \)
\( BR_2 = \bigstar \)

\[
\begin{array}{cccc}
\text{Effort} & \text{Shirk} \\
9 & 1 \\
9 & 10 \\
10 & 2 \\
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\]
Best Responses

Legend: \( BR_1 = \circ \)
\( BR_2 = \star \)

Nash eq. = all players are best responding, best response curves cross.
Computing best responses

How to find $BR_i(s_{-i})$.

1. If $S_i$ is finite we just pick all the $a_i$ that deliver the highest value of $u_i(\cdot, s_{-i})$, where $s_{-i}$ is given, and the set of such $s_i$ is the best response of player $i$ to $s_{-i}$.

2. If $S_i$ is a infinite and more exactly a subset of $\mathbb{R}^k$ we can use calculus provided $u_i$ is differentiable. For simplicity assume $S_i = [a, b]$ and $u_i$ differentiable in this case:
   
   1. We call all $x$ such that satisfy the first-order condition for an interior maximum (FOC), $\frac{\partial}{\partial s_i} u_i(x, s_{-i}) = 0$, the candidates for a maximum of $u_i(\cdot, s_{-i})$.
   
   2. If we have, the corner condition $\frac{\partial}{\partial s_i} u_i(x, a) \leq 0$, we also say that $a$ is a (corner) candidate for a maximum.
   
   3. If we have, the corner condition $\frac{\partial}{\partial s_i} u_i(x, b) \geq 0$, we also say that $b$ is a (corner) candidate for a maximum.
   
   4. Amongst all the candidates for a maximum identified in a,b and c, we pick the one(s) that yield the highest value for $u_i(\cdot, s_{-i})$. 
Consider a Cournot duopoly (two firms) game where firm 1 and 2 have cost function $c_i(s_i) = 3s_i$ where $i = 1, 2$; and the market demand function is $P = 4 - s_1 - s_2$. Quantities are non-negative so $s_i \in [0, +\infty)$. So firm $i$'s profits is $u_i(s_i, s_{-i}) = (1 - s_i - s_{-i})s_i$. The FOC for an interior solution is

$$1 - 2s_i - s_{-i} = 0 \Rightarrow$$

$$s_i = \frac{1 - s_{-i}}{2}.$$

Clearly for this $s_i$ to be interior we must have $s_{-i} < 1$. Otherwise, if $s_{-i} \geq 1$ we have that $\frac{\partial}{\partial s_i} u_i(0, s_{-i}) \leq 0$ and so the maximum is a corner solution.

$$s_i = \begin{cases} 
\frac{1 - s_{-i}}{2} & \text{if } s_{-i} < 1 \\
0 & \text{otherwise.}
\end{cases}$$
In the Cournot games, the best response was *single-valued* but in general the best response is *set-valued*. We may have several (or even infinite) number of best responses to $s_{-i}$. It is even possible the best response to $s_{-i}$ may fail to exist!

The next notes shows this is possible.
There are no fixed costs and the marginal cost is constant, $c$. The market demand is $P = \alpha - \beta Q$ but firms compete in prices. The firm charging the lowest price captures the entire market. If both firms charge the same, they split the market.

Profits are:

$$u_1(p_1, p_2) = \begin{cases} (p_1 - c) \cdot \left( \frac{\alpha - p_1}{\beta} \right) & \text{if } p_1 < p_2, \\ (p_1 - c) \cdot \left( \frac{\alpha - p_1}{\beta} \right) & \text{if } p_1 = p_2 \text{ and } \\ 0 & \text{if } p_1 > p_2. \end{cases}$$

For player 2, $u_2(p_1, p_2) = u_1(p_2, p_1)$. 
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For player 2, $u_2(p_1, p_2) = u_1(p_2, p_1)$. 
Remark: $u_1(p_1, p_2) > 0$ if and only if $c < p_1 \leq p_2$.

$$BR_1(p_1, p_2) = \begin{cases} \emptyset & \text{if } p_2 > c, \\ [c, +\infty) & \text{if } p_2 = c \text{ and} \\ (p_1, +\infty) & \text{if } p_2 < c. \end{cases}$$
Remark: $u_1(p_1, p_2) > 0$ if and only if $c < p_1 \leq p_2$.

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Nash eq. at $(c, c)$, best responses overlap
$N$ farmers choose their labor/effort levels $l_i$ for $i = 1, \ldots, N$. The production function of the $i$th farm is $f(l_i)$; the cost of effort is $c(l_i)$; and the price of the agricultural produce is $p$. Farms are private owned so each farmer maximizes:

$$p \cdot f(l_i) - c(l_i).$$

We assume that

$f' > 0, f'' < 0, f'(0) > 0, c' > 0, c'' > 0, c(0) = c'(0) = 0$.

Then each farmer best response is the $l_i$ that solves

$$p f'(l_i) - c'(l_i) = 0.$$
Computing best responses
Examples: Collective Action and Free riding

$N$ farmers choose their labor/effort levels $l_i$ for $i = 1, \ldots, N$. The production function of the $i$th farm is $f(l_i)$; the cost of effort is $c(l_i)$; and the price of the agricultural produce is $p$. Farms are private owned so each farmer maximizes:

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Assume that now farms are collectivized and so each farmer’s profit is a $\frac{1}{N}$ share of each farm profit. Now each farmer maximizes:

$$\frac{p}{N} \sum_{j=1}^{N} \cdot f(l_j) - c(l_i).$$

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**Definition:** A two-person game is a symmetric game if:

1. the strategy sets of both players are identical and,
2. for all strategies, $s_1$ of player 1 and all strategies, $s_2$, of player 2, we have that “if player 1 and player 2 switch/swap their actions, their payoff also switches/swaps” or formally:

   $$u_1(s_1, s_2) = u_2(s_2, s_1).$$

**Definition:** A Nash equilibrium $s^* = (s_1^*, s_2^*)$ in a symmetric game is symmetric if and only if $s_1^* = s_2^*$. 
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Symmetric Games

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\[
u_1(s_1, s_2) = u_2(s_2, s_1).
\]

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$$u_1(s_1, s_2) = u_2(s_2, s_1).$$

Definition: A Nash equilibrium $s^* = (s_1^*, s_2^*)$ in a symmetric game is symmetric if and only if $s_1^* = s_2^*$. 
Consider a two-person game where $S_1 = S_2 = [0, 1]$ and

$$u_i(s_1, s_2) = \max(s_1, s_2) - \min(s_1, s_2).$$

1. Is this game symmetric?
2. Compute the best response of 1 to $s_2 > \frac{1}{2}$.
3. Compute the best response of 1 to $s_2 < \frac{1}{2}$.
4. Compute the best response set of 1 to $s_2 = \frac{1}{2}$.
5. Compute all Nash equilibria of this game.
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5. Compute all Nash equilibria of this game.
Let $f : [a, b]^n \to [a, b]^n$ a mapping from the the $n$-dimensional cuboid (i.e. box) to itself.

**Definition:** $x$ is a fixed point of $f$ if and only if $x = f(x)$.

**Theorem (Brower):** if $f : [a, b]^n \to [a, b]^n$ is continuous then $f$ has a fixed-point.

**Assumption:** Let the game $\Gamma = (\{1, 2\}, [0, 1]^2, u_1, u_2)$ be such that the best-response set of a player $BR_i(s_-; i)$ is a singleton set (i.e. a set with only one element). Also assume $BR_i(\cdot)$ is a continuous function.
Define the function $f : [0, 1] \longrightarrow [0, 1]$ by

$$f(x) = BR_1(BR_2(x)).$$

Notice that $f$ is continuous given our assumptions. **Question 1:** Read page 505 of the textbook, Mathematical Appendix, Proofs 17.7 and using the definition of Nash Equilibrium on page 22 (you may **not** use Proposition 36.1 directly), prove that if $x^*$ is a fixed point of $f$ then there is a Nash equilibrium where player 1 plays $x^*$.

**Question 2:** For some given $x_0$ define the infinite sequence $x_1 = f(x_0), x_2 = f(x_1), \ldots, x_k = f(x_{k-1}), \ldots$ If this sequence has a limit, $\hat{x} = \lim_{k \to +\infty} x_k$, prove that $\hat{x}$ must be a fixed point of $f$. 
**Definition:** The function $f : [0, 1] \rightarrow [0, 1]$ is a $\lambda$-contraction where $0 < \lambda < 1$ if for all $x \in [0, 1]$ we have that $f'(x) < \lambda$.

**Theorem:** If $f$ is a $\lambda$-contraction then it has a fixed point.

**Question 3:** For any $x$ and $y$ show that if $f$ is a $\lambda$-contraction then $|f(x) - f(y)| < \lambda |x - y|$.

**Question 4:** For any the sequence in question 2, show that for every $j = 1, 2, \ldots, k$ we have:

$$|f(x_{k+1}) - f(x_k)| < \lambda^j |x_{k+1-j} - x_{k-j}|.$$
\[
\max_{s.t.} \sqrt{x} + \sqrt{y}
\]

\[
x + y \leq 10
\]
\[
2x + 10 \leq 14
\]

\[S_2 = \{ s_2 : s_2 = (\lambda_1, \lambda_2) \in [0, 1000]^2 \}\]

\[S_1 = \{ s_1 : s_1 = (x, y) \in \mathbb{R}_+^2 \}\]

\[u_1(s_1, s_2) = \sqrt{x} + \sqrt{y} - \lambda_1 \cdot (x + y - 10) - \lambda_2 \cdot (2x + y - 14)
\]

\[u_2(s_1, s_2) = -u_1(s_1, s_2)
\]

\[x + y - 10 > 0 \quad \text{BPR}_2 \quad \lambda_1 = 1000
\]

\[2x + y - 14 > 0 \quad \text{BPR}_2 \quad \lambda_2 = 1000
\]

\[\lambda_1 = 1000 \quad \text{BPR}_1 \quad x + y < 10 \quad \text{Why?}
\]

\[\lambda_2 = 1000 \quad \text{BPR}_2 \quad 2x + y < 14 \quad \text{Why?}
\]
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\[ x + y - 10 > 0 \xrightarrow{\text{BR}_2} \lambda_1 = 1000 \]
\[ 2x + y - 14 > 0 \xrightarrow{\text{BR}_2} \lambda_2 = 1000 \]
\[ \lambda_1 = 1000 \xrightarrow{\text{BR}_1} x + y < 10 \text{ Why?} \]
\[ \lambda_2 = 1000 \xrightarrow{\text{BR}_2} 2x + y < 14 \text{ Why?} \]
\[
\max_{s.t.} \sqrt{x} + \sqrt{y} \\
x + y \leq 10 \\
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\[x + y - 10 > 0 \quad \xrightarrow{BR_2} \quad \lambda_1 = 1000 \]

\[2x + y - 14 > 0 \quad \xrightarrow{BR_2} \quad \lambda_2 = 1000 \]

\[\lambda_1 = 1000 \quad \xrightarrow{BR_1} \quad x + y < 10 \quad \text{Why?} \]

\[\lambda_2 = 1000 \quad \xrightarrow{BR_3} \quad 2x + y < 14 \quad \text{Why?} \]
\[
\max_{s.t.} \sqrt{x} + \sqrt{y} \\
x + y \leq 10 \\
2x + 10 \leq 14
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\[ u_1(s_1, s_2) = \sqrt{x} + \sqrt{y} - \lambda_1 \cdot (x + y - 10) - \lambda_2 \cdot (2x + y - 14) \]

\[ u_2(s_1, s_2) = -u_1(s_1, s_2) \]

\[ x + y - 10 > 0 \quad \Rightarrow \quad \lambda_1 = 1000 \]

\[ 2x + y - 14 > 0 \quad \Rightarrow \quad \lambda_2 = 1000 \]

\[ \lambda_1 = 1000 \quad \Rightarrow \quad x + y < 10 \quad \text{Why?} \]

\[ \lambda_2 = 1000 \quad \Rightarrow \quad 2x + y < 14 \quad \text{Why?} \]
Constrained Optimization

The cook-book recipe:

1. Write down the maximization (or minimization) problem. Do not forget to write all constraints in the form \( g_i(x) \leq c_i \! \).
2. Write the corresponding Lagrangian.
3. Obtain the first-order-conditions (FOC).
4. Write the constraints for the choice variables.
5. Write the constraints for the Lagrange multipliers.
6. Write down the complementary slackness conditions.
7. List all possible cases (when a constraint binds or not) in a table the multiplies in the first row, and each additional row has either with a plus sign \( + \) or 0 do denote whether the constraint is active or not.
8. Mix ingredients in steps 3, 4, 5, 6, and 7 together to either find a solution for a case or to conclude it is inconsistent.
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   - List all possible cases (when a constraint binds or not) in a table the multiplies in the first row, and each additional row has either with a plus sign $+$ or $0$ do denote whether the constraint is active or not.
   - Mix ingredients in steps 3, 4, 5, 6, and 7 together to either find a solution for a case or to conclude it is inconsistent.
Constrained Optimization

The cook-book recipe:

1. Write down the maximization (or minimization) problem. Do not forget to write all constraints in the form $g_i(x) \leq c_i$!

2. Write the corresponding Lagrangian.

3. Obtain the first-order-conditions (FOC).

4. Write the constraints for the choice variables.

5. Write the constraints for the Lagrange multipliers.

6. Write down the complementary slackness conditions.

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The Economics of Torts (Accidents)

1. BP’s oil spill in the Gulf of Mexico
2. Melamine, China Tainted Baby Formula Scandal
3. Securities Fraud
Player 1, a (potential) injurer chooses his/her level of precautionary effort $e_1 \in [0, 1]$. Player 2 is a (potential) victim who chooses his/her level of precautionary effort $e_2 \in [0, 1]$.

With probability $(1 - e_1)(1 - e_2)$ an accident occurs with resulting loss of $L > 0$; and with probability $1 - (1 - e_1)(1 - e_2)$ no accident occurs so the resulting loss is zero.

Effort is costly: the cost of precautionary effort for the injurer is $c_1(e) = e^2$ and for the victim is $c_2(e) = e^2$, where $e$ is that amount of effort chosen by the respective player.
If any loss is borne by the victim (no liability regime) the payoffs are:

\[ u_1(e_1, e_2) = -e_1^2 \quad \text{and} \quad u_2(e_1, e_2) = -(1 - e_1)(1 - e_2)L - e_2^2. \]

We previously derived the following best responses:

\[ BR_1(e_2) = 0 \quad \text{and} \quad BR_2(e_1) = \begin{cases} \frac{(1 - e_1)L}{2} & \text{if } \frac{(1-e_1)L}{2} \leq 1 \text{ and} \\ 1 & \text{otherwise.} \end{cases} \]

So the Nash equilibrium is

\[ \begin{cases} (0, \frac{L}{2}) & \text{if } L \leq 2 \text{ and} \\ (0, 1) & \text{otherwise.} \end{cases} \]
Tort Law
Case: any loss is borne by the injurer (strict liability regime)

If any loss is borne by the victiminjurer (strict liability regime) the payoffs are:

\[ u_1(e_1, e_2) = -(1 - e_1)(1 - e_2)L - e_1^2 \quad \text{and} \quad u_2(e_1, e_2) = -e_2^2. \]

The best responses are similar to the no liability regime (because precautionary costs are identical for the victim and injurer):

\[ BR_1(e_2) = \begin{cases} \frac{(1 - e_2)L}{2} & \text{if } \frac{(1-e_2)L}{2} \leq 1 \text{ and} \quad \text{and } BR_2(e_1) = 0 \\ 1 & \text{otherwise.} \end{cases} \]

So the Nash equilibrium is

\[ \begin{cases} (\frac{L}{2}, 0) & \text{if } L \leq 2 \text{ and} \\ (1, 0) & \text{otherwise.} \end{cases} \]
Notice that despite its “undesirable” outcomes, the no liability and strict liability regimes are both efficient.

One however may be interested in other efficient outcomes distinct from the no liability regime’s outcome.

For example, one may be interested in the outcome that minimizes the sum of the expected loss and precautionary costs.
Notice that despite its “undesirable” outcomes, the no liability and strict liability regimes are both efficient.

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For example, one may be interested in the outcome that minimizes the sum of the expected loss and precautionary costs.
Tort Law
Minimizing the sum of the expected loss and precautionary costs

\[ \mathcal{L} = -(1 - e_1)(1 - e_2) - e_1^2 - e_2^2 + \lambda_1 e_1 - \lambda_2 e_1 + \lambda_3 e_2 - \lambda_4 e_1 \]

\[
\begin{align*}
\lambda_1 e_1 & = 0 \\
\lambda_3 e_2 & = 0
\end{align*}
\]

\[
\begin{align*}
\lambda_2 (1 - e_1) & = 0 \\
\lambda_4 (1 - e_2) & = 0
\end{align*}
\]

\[
\begin{align*}
\frac{\partial}{\partial e_1} \mathcal{L} & = (1 - e_2)L - 2e_1 + \lambda_1 - \lambda_2 = 0 \\
\frac{\partial}{\partial e_2} \mathcal{L} & = (1 - e_1)L - 2e_2 + \lambda_3 - \lambda_4 = 0
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\]
Case \( \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0 \Rightarrow \)

\[
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\]

Solving the system:

\[
(e_1, e_2) = \left( \frac{L}{L + 2}, \frac{L}{L + 2} \right)
\]

As \( 0 \leq \frac{L}{L + 2} \leq 1 \) this is indeed a solution. Actually it is the unique solution of the problem (as the other cases have no solutions satisfying all constraints).
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Tort Law
Case: any loss is borne by the injurer provided the injurer was negligent.

The injurer is negligent if $e_1 < e$ where $e$ is a legal standard.

Under the negligence rule payoffs are:

$$u_1(e_1,e_2) = \begin{cases} -(1-e_1)(1-e_2)L - e_1^2 & \text{if } e_1 < e \text{ and } e_2 \leq 1 \text{ and } L \leq 1 \\ -e_1^2 & \text{otherwise} \end{cases}$$

$$u_2(e_1,e_2) = \begin{cases} -e_2^2 & \text{if } e_1 < e \text{ and } e_1(1-e_2)L - e_2^2 & \text{otherwise} \end{cases}$$

The best responses are slightly ‘trickier’ to obtain in this case because our recipe for solving constrained optimization problems (the Khun-Tucker method) requires payoffs to be differentiable and here the payoff of player 1 is not even continuous in $e_1$. But not everything is lost because 1’s payoff is piecewise differentiable.
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-e_1^2 & \text{otherwise}
\end{cases}
\]

\[
u_2(e_1, e_2) = \begin{cases} 
-e_2^2 & \text{if } e_1 < \underline{e} \text{ and } e_2 < 1 \\
-(1 - e_1)(1 - e_2)L - e_2^2 & \text{otherwise}
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Negligence Rule
The victim’s best response

Given the choice of \( e_1 \), the payoff of the victim (player 2) is always continuous in the choice variable so we can always use the recipe. But we must solve two problems because the victim’s payoff changes depending whether \( e_1 < e \) or \( e_1 \geq e \): For \( e_1 < e \), we solve

\[
\max_{e_2} -e_2^2
\]

subject to

\[
0 \leq e_2 \leq 1
\]

and the solution is \( e_2 = 0 \). And for \( e_1 \geq e \), we solve

\[
\max_{e_2} -(1 - e_1)(1 - e_2)L - e_2^2.
\]

And the best response is,

\[
BR_2(e_1) = \begin{cases} 
0 & \text{if } e_1 < e \\
\frac{(1-e_1)L}{2} & \text{if } e_1 \geq e \text{ and } (1 - e_1)L \leq 2, \\
1 & \text{if } e_1 \geq e \text{ and } (1 - e_1)L > 2
\end{cases}
\]
Negligence Rule
The injures’ best response

We have to solve the two problems,

\[
\max_{e_1} -(1 - e_1)(1 - e_2)L - e_1^2 \quad \text{and} \quad \max_{e_1} -e_1^2.
\]

subject to

\[
0 \leq e_1 \leq 1
\]

Clearly the solution of the second problem is \( e_1 = e \) and this gives a payoff of \(-e^2\).

If the first problem has an interior solution then it must be that \( e_1 = \frac{(1-e_2)L}{2} \) and this interior solution gives utility,

\[
-(1 - \frac{(1 - e_2)L}{2})(1 - e_2)L - \left(\frac{(1 - e_2)L}{2}\right)^2 =
\]

\[
(1 - e_2)L \left(\frac{(1 - e_2)L}{4} - 1\right)
\]
Negligence Rule
cont.

In the first problem, the solution is a corner solution, \( e_1 = e \), when \( e \geq \frac{(1-e_2)L}{2} \). Thus, the best response is:

\[
BR_1(e_2) = \begin{cases} 
\frac{e}{(1-e_2)L} & \text{if } -e^2 \geq (1 - e_2)L \left( \frac{(1-e_2)L}{4} - 1 \right) \text{ or } e \leq \frac{(1-e_2)}{2} \\
\frac{1}{2} & \text{otherwise}
\end{cases}
\]
So far we have that:

\[
BR_1(e_2) = \begin{cases} 
\frac{e}{(1-e_2)L} & \text{if } -e^2 \geq (1-e_2)L \left(\frac{(1-e_2)L}{4} - 1\right) \text{ or } e \leq \frac{(1-e_2)}{2} \\
\frac{(1-e_2)L}{2} & \text{otherwise}
\end{cases}
\]

and

\[
BR_2(e_1) = \begin{cases} 
0 & \text{if } e_1 < e \\
\frac{(1-e_1)L}{2} & \text{if } e_1 \geq e \text{ and } (1-e_1)L \leq 2, \\
1 & \text{if } e_1 \geq e \text{ and } (1-e_1)L > 2.
\end{cases}
\]
Let’s see if there is any Nash equilibrium where the injurer is not negligent, $e_1 = e$. In this case,

$$BR_2(e) = \begin{cases} \frac{(1-e)L}{2} & \text{if } (1-e)L \leq 2, \\ 1 & \text{otherwise.} \end{cases}$$

Clearly if $e_2 = 1$ then $BR_1(1) = 0 \neq e$ so let’s assume $(1-e)L \leq 2$ which implies: $e_2 = BR_2(e) = \frac{(1-e)L}{2}$.

We have $BR_1\left(\frac{(1-e)L}{2}\right) = e$ if either

$$\frac{(1 - \frac{(1-e)L}{2}) L}{2} \geq e$$

or

$$u_1\left(e, \frac{(1-e)L}{2}\right) \geq \frac{L}{16} (2 + (-1 + e)L)(-8 + 2L + (-1 + e)L^2)$$

$$u_1\left(\frac{(1-(1-e)L)}{2}, \frac{(1-e)L}{2}\right)$$
Let’s see if there is any Nash equilibrium where the injurer is negligent, in this case $e_2 = 0$ and $e_1 = \frac{L}{2}$.

So it must be that $e > \frac{L}{2}$.

Clearly the victim has no incentive to deviate if $e > e_1 = \frac{L}{2}$. As for the injurer, we must have that

$$-e^2 \leq -(1 - \frac{L}{2})L - \frac{L^2}{4} \quad \text{or equivalently}$$

$$e \geq \sqrt{(1 - \frac{L}{2})L + \frac{L^2}{4}}.$$
**Definition:**

In a strategic game, the strategy $\hat{s}_i$ is **strictly dominated** if there is some other strategy $\tilde{s}_i$ such that $u_i(\hat{s}_i, s_{-i}) < u_i(\tilde{s}_i, s_{-i})$ for all $s_{-i}$.

**Definition:**

In a strategic game, the strategy $\hat{s}_i$ is **weakly dominated** if there is some other strategy $\tilde{s}_i$ such that $u_i(\hat{s}_i, s_{-i}) \leq u_i(\tilde{s}_i, s_{-i})$ for all $s_{-i}$ AND exists some $s_{-i}$ such that $u_i(\hat{s}_i, s_{-i}) < u_i(\tilde{s}_i, s_{-i})$.
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Dominated Strategies

**Definition:**
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Iterative Elimination of Strictly Dominated Actions

1 Start with a game $\Gamma^0 = (I, \Pi_{i \in I} S_i^0, \{u_i : S^0 \to \mathbb{R}\}_{i \in I})$.

Set $k = 1$.

For each player $i$

$S_i^k \leftarrow$ strategies of player $i$ that are not strictly dominated in the game $\Gamma^{k-1}$ (remove the strictly dominated strategies)

Define the game $\Gamma^k$ as having the same players as $\Gamma^{k-1}$ but the players’ strategy sets are now $S_i^k$ and payoffs are restricted to the set of strategy profiles, $S^k = \Pi_{i \in I} S_i^k$.

If $\Gamma^k = \Gamma^{k-1}$ stop, otherwise go to step 3.

The set of strategies that survive the process of iterative deletion of strictly dominated strategies is a solution concept for a strategic games.

It is weaker than Nash... Why?
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It is weaker than Nash... Why?
Iterative Elimination of Strictly Dominated Actions

1. Start with a game $\Gamma^0 = (I, \Pi_{i \in I} S_i^0, \{ u_i : S^0 \rightarrow \mathbb{R} \}_{i \in I})$.
2. Set $k = 1$.
3. For each player $i$:
   \[ S^k_i \leftarrow \text{strategies of player } i \text{ that are not strictly dominated in the game } \Gamma^{k-1} \] (remove the strictly dominated strategies)
4. Define the game $\Gamma^k$ as having the same players as $\Gamma^{k-1}$ but the players’ strategy sets are now $S^k_i$ and payoffs are restricted to the set of strategy profiles, $S^k = \Pi_{i \in I} S^k_i$.
5. If $\Gamma^k = \Gamma^{k-1}$ stop, otherwise go to step 3.

The set of strategies that survive the process of iterative deletion of strictly dominated strategies is a solution concept for a strategic games.

It is weaker than Nash... Why?
Iterative Deletion of Strictly Dominated Strategies
Example, Cournot Duopoly

\[ S_1^0 = S_2^0 = [0, +\infty) \] and \[ u_i(s_1, s_2) = (1 - s_1 - s_2) \cdot s_i \] for \( i = 1, 2 \).

We have that:

\[ \Gamma^1 : S_1^1 = S_2^1 = [0, \frac{1}{2}] \]
\[ \Gamma^2 : S_1^2 = S_2^2 = [\frac{1}{4}, \frac{1}{2}] \]
\[ \Gamma^3 : S_1^3 = S_2^3 = [\frac{1}{4}, \frac{3}{8}] \]

Notice...

\( \square \) The strategy \( s_1 = \frac{1}{8} \) is not strictly dominated in \( \Gamma^0 \) but it is strictly dominated in \( \Gamma^1 \). Why?

\( \square \) The strategy \( s_1 = \frac{1}{2} \) is not strictly dominated in \( \Gamma^1 \) but it is strictly dominated in \( \Gamma^2 \). Why?

\( \square \) Which strategies are strictly dominated in \( \Gamma^3 \)?
Iterative Deletion of Strictly Dominated Strategies
Example, Cournot Duopoly

\[ S_1^0 = S_2^0 = [0, +\infty) \text{ and } u_i(s_1, s_2) = (1 - s_1 - s_2) \cdot s_i \text{ for } i = 1, 2. \]

We have that:

\[ \Gamma^1 \quad S_1^1 = S_2^1 = [0, \frac{1}{2}]. \]
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\[ \Gamma^3 \quad S_1^3 = S_2^3 = [\frac{1}{4}, \frac{3}{8}]. \]

Notice...

1. The strategy \( s_1 = \frac{1}{8} \) is not strictly dominated in \( \Gamma^0 \) but it is strictly dominated in \( \Gamma^1 \). Why?

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3. Which strategies are strictly dominated in \( \Gamma^3 \)?
Iterative Deletion of Strictly Dominated Strategies
Example, Cournot Duopoly

\[ S_1^0 = S_2^0 = [0, +\infty) \] and \( u_i(s_1, s_2) = (1 - s_1 - s_2) \cdot s_i \) for \( i = 1, 2 \).

We have that:

\[
\begin{align*}
\Gamma^1 & \quad S_1^1 = S_2^1 = [0, \frac{1}{2}]. \\
\Gamma^2 & \quad S_1^2 = S_2^2 = [\frac{1}{4}, \frac{1}{2}]. \\
\Gamma^3 & \quad S_1^3 = S_2^3 = [\frac{1}{4}, \frac{3}{8}].
\end{align*}
\]

Notice...

\[ \square \] The strategy \( s_1 = \frac{1}{3} \) is not strictly dominated in \( \Gamma^0 \) but it is strictly dominated in \( \Gamma^1 \). Why?

\[ \square \] The strategy \( s_1 = \frac{1}{2} \) is not strictly dominated in \( \Gamma^1 \) but it is strictly dominated in \( \Gamma^2 \). Why?

\[ \square \] Which strategies are strictly dominated in \( \Gamma^3 \)?
Iterative Deletion of Strictly Dominated Strategies
Example, Cournot Duopoly

\[ S_1^0 = S_2^0 = [0, +\infty) \] and \[ u_i(s_1, s_2) = (1 - s_1 - s_2) \cdot s_i \] for \( i = 1, 2 \).

We have that:

- \( \Gamma^1 \quad S_1^1 = S_2^1 = [0, \frac{1}{2}] \).
- \( \Gamma^2 \quad S_1^2 = S_2^2 = [\frac{1}{4}, \frac{1}{2}] \).
- \( \Gamma^3 \quad S_1^3 = S_2^3 = [\frac{1}{4}, \frac{3}{8}] \).

Notice...

- The strategy \( s_1 = \frac{1}{8} \) is not strictly dominated in \( \Gamma^0 \) but it is strictly dominated in \( \Gamma^1 \). Why?
- The strategy \( s_1 = \frac{1}{2} \) is not strictly dominated in \( \Gamma^1 \) but it is strictly dominated in \( \Gamma^2 \). Why?
- Which strategies are strictly dominated in \( \Gamma^3 \)?
Iterative Deletion of Strictly Dominated Strategies
Example, Cournot Duopoly

\[ S_1^0 = S_2^0 = [0, +\infty) \] and \( u_i(s_1, s_2) = (1 - s_1 - s_2) \cdot s_i \) for \( i = 1, 2 \).

We have that:

\[ \Gamma^1 \quad S_1^1 = S_2^1 = [0, \frac{1}{2}] \]
\[ \Gamma^2 \quad S_1^2 = S_2^2 = [\frac{1}{4}, \frac{1}{2}] \]
\[ \Gamma^3 \quad S_1^3 = S_2^3 = [\frac{1}{4}, \frac{3}{8}] \]

Notice...

1. The strategy \( s_1 = \frac{1}{8} \) is not strictly dominated in \( \Gamma^0 \) but it is strictly dominated in \( \Gamma^1 \). Why?
2. The strategy \( s_1 = \frac{1}{2} \) is not strictly dominated in \( \Gamma^1 \) but it is strictly dominated in \( \Gamma^2 \). Why?
3. Which strategies are strictly dominated in \( \Gamma^3 \)?
Mixed Strategy Nash Equilibrium

<table>
<thead>
<tr>
<th>Pepé Le Pew</th>
<th>V</th>
<th>M</th>
</tr>
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<tbody>
<tr>
<td>V</td>
<td>1,-1</td>
<td>-1,1</td>
</tr>
<tr>
<td>M</td>
<td>-1,1</td>
<td>1,-1</td>
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</tbody>
</table>

Pepé Le Pew and Penelope Pussycat
# Mixed Strategy Nash Equilibrium

<table>
<thead>
<tr>
<th>Nicky</th>
<th>Terry</th>
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<tbody>
<tr>
<td>Empire State</td>
<td>Empire State</td>
</tr>
<tr>
<td></td>
<td>1,1</td>
</tr>
<tr>
<td>Grand Central</td>
<td>0,0</td>
</tr>
</tbody>
</table>
Consider a finite game in the strategic form with \( I \) players where each player has \( n_i \) strategies, \( S_i = \{ s_i^1, \ldots, s_i^k, \ldots, s_i^{n_i} \} \).

**Definition:** A probability distribution over pure strategies in \( S_i \), \( \sigma_i = (\sigma_i^1, \ldots, \sigma_i^k, \ldots, \sigma_i^{n_i}) \in \Sigma_i \) is called a **mixed strategy** for player \( i \).

Remarks: \( \sigma_i^k \) represents the probability that player \( i \) chooses the pure strategy \( s_i^k \) and \( \Sigma_i \) is the set of all mixed strategies available to player \( i \).

**Definition:** Expected Payoffs (of mixed strategies)

\[
U_i(\sigma) = \sum_{s_1^{k_1} \in S_1} \cdots \sum_{s_I^{k_I} \in S_I} \left( \prod_{j=1}^{I} \sigma_j^{k_j} \right) u_i(s_1^{k_1}, \ldots, s_I^{k_I}).
\]
Given \(\sigma_{-i}\), consider the maximization problem whose set of solutions is the best response:

\[
\max_{\sigma_i^1, \ldots, \sigma_i^k, \ldots, \sigma_i^{n_i}} \sum_{k=1}^{n_i} \sigma_i^k \left[ \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i}) u_i(s_i^k, s_{-i}) \right] \\
\text{subject to} \\
\forall k, -\sigma_i^k \leq 0 \\
\sum_{k=1}^{n_i-1} \sigma_i^k \leq 1
\]
Mixed Strategy Nash Equilibrium

\[
\begin{align*}
\max_{\sigma_1^i, \ldots, \sigma_k^i, \ldots, \sigma_n^i} & \quad \sum_{k=1}^{n_i-1} \sigma_i^k \left[ \sum_{s_i \in S_i} \sigma_i(s_i) u_i(s_i, s) \right] + \left( 1 - \sum_{k=1}^{n_i-1} \sigma_i^k \right) u_i(s_i, s) \\
\text{subject to} & \quad \forall k, -\sigma_i^k \leq 0 \\
& \quad \sum_{k=1}^{n_i-1} \sigma_i^k \leq 1
\end{align*}
\]

\[
L(\sigma_1^i, \ldots, \sigma_n^i, \lambda_1, \ldots, \lambda_n, \mu) = \sum_{k=1}^{n_i} \sigma_i^k U_i(s_i^k, \sigma_i) + \\
+ \left( 1 - \sum_{k=1}^{n_i-1} \sigma_i^k \right) u_i(s_i^{n_i}, s) + \sum_{k=1}^{n_i-1} \lambda_k \sigma_i^k - \mu \left( \sum_{k=1}^{n_i-1} \sigma_i^k \right)
\]
Mixed Strategy Nash Equilibrium

\[ \mathcal{L} \left( \sigma^1_i, \ldots, \sigma^{n_i}_i, \lambda_1, \ldots, \lambda_{n_i}, \mu \right) = \sum_{k=1}^{n_i} \sigma^k_i U_i(s^k_i, \sigma_{-i}) + \]  
\[ + \left( 1 - \sum_{k=1}^{n_i-1} \sigma^k_i \right) u_i(s_i, s_{-i}) + \sum_{k=1}^{n_i-1} \lambda_k \sigma^k_i - \mu \left( \sum_{k=1}^{n_i-1} \sigma^k_i \right) \]

\[ \frac{\partial}{\partial \sigma^k_i} \mathcal{L} = U_i(s^k_i, \sigma_{-i}) + \lambda_k - \mu = 0, \]

\[ \lambda_k \sigma^k_i = 0, \quad \lambda_k \geq 0, \]

\[ \mu \left( 1 - \sum_{k=1}^{n_i-1} \sigma^k_i \right) = 0, \quad \mu \geq 0. \]
Mixed Strategy Nash Equilibrium

\[ U_i(s_i^k, \sigma_{-i}) + \lambda_k + \mu = 0, \]
\[ \lambda_k \sigma_i^k = 0, \quad \lambda_k \geq 0. \]

Now for any \( \sigma_i \), define the set \( \text{supp}(\sigma_i) \equiv \{ s_i^k : \sigma_i^k > 0 \} \) the support of \( \sigma_i \) — that is, the strategies that \( \sigma_i \) select with positive probability.

When \( \sigma_i \in BR_i(\sigma_{-i}) \), we have two important conditions:

\[ \forall s_i^l, s_i^m \in \text{supp}(\sigma_i) \quad \implies \quad U_i(s_i^l, \sigma_{-i}) = U_i(s_i^m, \sigma_{-i}) \quad \text{(Indiff)} \]
\[ \forall s_i^l \notin \text{supp}(\sigma_i), s_i^m \in \text{supp}(\sigma_i) \quad \implies \quad U_i(s_i^l, \sigma_{-i}) \leq U_i(s_i^m, \sigma_{-i}) \quad \text{(Opt)} \]
Mixed Strategy Nash Equilibrium

\[ U_i(s_i^k, \sigma_{-i}) + \lambda_k + \mu = 0, \]
\[ \lambda_k \sigma_i^k = 0, \quad \lambda_k \geq 0. \]

**Indifference** says that: all strategies that are played with positive prob. in eq. must give the same payoff.

**Proof of the indifference condition:**
\[ \forall s_i^l, s_i^m \in \text{supp}(\sigma_i) \Rightarrow \lambda_l = \lambda_k = 0 \Rightarrow U_i(s_i^l, \sigma_{-i}) = U_i(s_i^m, \sigma_{-i}) = -\mu \]

**Optimality** says that: strategies that are not played with positive prob. in eq. may not give a higher payoff than strategies that are played with positive prob. in eq.

**Proof of the optimality condition:**
\[ \forall s_i^l \notin \text{supp}(\sigma_i), s_i^m \in \text{supp}(\sigma_i) \Rightarrow \lambda_l \geq \lambda_k = 0 \Rightarrow \]
\[ \Rightarrow U_i(s_i^l, \sigma_{-i}) + \lambda_l = U_i(s_i^m, \sigma_{-i}) \Rightarrow U_i(s_i^l, \sigma_{-i}) \leq U_i(s_i^m, \sigma_{-i}) \]
Finding Nash Eq. under Mixed Strategies

We use the indifference and the optimality conditions to compute Nash equilibria in mixed strategies. Thus the *hard* item in computing a Nash eq. is to identify the strategies that are played with positive probability and those that are played with zero prob.

If a player has $n$ strategies then we there are $2^n - 1$ different ways that strategies can have positive or zero prob. !

Example: say a player has three strategies: $a$, $b$ and $c$. Then we have the cases where: 1) only one of them has positive prob. (by the way this is the case of pure strategies – 3 cases); 2) only two of them has positive prob. (3 cases); 3) all of the have positive probability (1 case). So in the example we have $2^3 - 1 = 8 - 1 = 7$ cases.
Finding Nash Eq. under Mixed Strategies
An Example

Consider the following game:

<table>
<thead>
<tr>
<th></th>
<th>$b_1$</th>
<th>$b_2$</th>
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<tbody>
<tr>
<td>$a_1$</td>
<td>3,3</td>
<td>3,2</td>
</tr>
<tr>
<td>$a_2$</td>
<td>2,2</td>
<td>5,6</td>
</tr>
<tr>
<td>$a_3$</td>
<td>0,3</td>
<td>6,1</td>
</tr>
</tbody>
</table>

Let $q = \sigma_2^1$ and $1 - q = \sigma_2^2$ the probabilities that player 2 chooses $b_1$ and $b_2$ respectively. Also let $p_1 = \sigma_1^1$, $p_2 = \sigma_1^2$ and $1 - p_1 - p_2$ the probabilities that player 1 chooses $a_1$, $a_2$ and $a_3$ respectively.

\[
U_1(a_1, \sigma_2) = 3(1 - q) + 3q = 3 \\
U_1(a_2, \sigma_2) = 5(1 - q) + 2q = 5 - 3q \\
U_1(a_3, \sigma_2) = 5(1 - q) + 2q = 6(1 - q) = 6 - 6q
\]
Finding Nash Eq. under Mixed Strategies
An Example

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\[
U_1(a_1, \sigma_2) = 3(1 - q) + 3q = 3 \\
U_1(a_2, \sigma_2) = 5(1 - q) + 2q = 5 - 3q \\
U_1(a_3, \sigma_2) = 6(1 - q) = 6 - 6q
\]

Thus,

\[
BR_1(\sigma_2) = \begin{cases} 
  a_3 & \text{if } q \leq \frac{1}{3}, \\
  a_2 & \text{if } \frac{1}{3} \leq q \leq \frac{2}{3}, \\
  a_1 & \text{if } \frac{2}{3} \leq q.
\end{cases}
\]
Finding Nash Eq. under Mixed Strategies
An Example

Since

\[ BR_1(\sigma_2) = \begin{cases} 
    a_3 & \text{if } q \leq \frac{1}{3}, \\
    a_2 & \text{if } \frac{1}{3} \leq q \leq \frac{2}{3}, \\
    a_1 & \text{if } \frac{2}{3} \leq q.
\end{cases} \]

it follows that by the indifference and opt. conditions, if 1 is mixing in the equilibrium then either 1 mixes between \( a_3 \) and \( a_2 \) (when \( q = 1/3 \)) or 1 mixes between \( a_2 \) and \( a_1 \) (when \( q = 2/3 \)). But player 1 will never mix between all the strategies nor between \( a_3 \) and \( a_1 \).
Finding Nash Eq. under Mixed Strategies
An Example

<table>
<thead>
<tr>
<th></th>
<th>$b_1$</th>
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<tbody>
<tr>
<td>$a_1$</td>
<td>3,3</td>
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<td>$a_3$</td>
<td>0,3</td>
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</tbody>
</table>

$$U_2(b_1, \sigma_2) = 3p_1 + 2p_2 + 3(1 - p_1 - p_2) = 3 - p_2$$
$$U_2(b_2, \sigma_2) = 2p_1 + 6p_2 + (1 - p_1 - p_2) = 1 + p_1 + 5p_2$$

Thus,

$$BR_2(\sigma_1) = \begin{cases} b_1 & \text{if } 3 - p_2 \geq 1 + p_1 + 5p_2, \\ b_2 & \text{if } 1 + p_1 + 5p_2 \geq 3 - p_2. \end{cases}$$
Bertrand Equilibrium
A Mixed Strategy Example

Two firms 1 and 2 choose prices $p_1, p_2 \in [0, +\infty)$. The demand is inelastic, more exactly it satisfies $Q(p) \geq 0$, $Q' < 0$, $\lim_{p \to \infty} p \, Q(p) = \infty$ and $(p - c) \, Q'(p) + Q(p) > 0$ for all $p$.

$$u_i(p_i, p_j) = \begin{cases} 0 & \text{if } p_i > p_j, \\ (p_i - c) \frac{Q(p_i)}{2} & \text{if } p_i = p_j, \\ (p_i - c) \, Q(p_i) & \text{if } p_i > p_j. \end{cases}$$

Exercise: Show that the best response correspondence is:

$$BR_i(p_j) = \begin{cases} [c, +\infty) & \text{if } p_j \leq c, \\ \emptyset & \text{if } p_j > c. \end{cases}$$
Bertrand Equilibrium
A Mixed Strategy Example

Since

\[ BR_i(p_j) = \begin{cases} [c, +\infty) & \text{if } p_j \leq c, \\ \emptyset & \text{if } p_j > c. \end{cases} \]

the only Nash equilibrium in pure-strategies is \((c, c)\). But this game has other equilibrium in mixed strategies.

**Exercise:** Pick any \(p_0 > c\) and assume the prob. \(j\)'s price is below or at \(p\) is

\[ G_j(p) = \begin{cases} 0 & \text{if } p \leq c, \\ 1 - \frac{(p_0-c)Q(p_0)}{(p-c)Q(p)} & \text{if } p > p_0. \end{cases} \]

1. Compute the expected payoff of player \(i\) for \(p_i \in (c, p_0)\) and \(j\) mixes accordingly to \(G_j\).
2. Compute \(i\)'s expected payoff if \(i\) chooses \(p_i > p_0\).
3. Over what range of prices is \(i\) willing to mix? Explain.
Mixed Strategies
Finding all equilibria

The hard item in finding a mixed strategy Nash equilibrium is to find out which strategies are played with positive probabilities. But this leaves us with many cases to consider. If each player $i$ has $n_i$ strategies, then there are $2^{n_i} - 1$ cases to analyze.

In the game below, where player 1 has 3 pure strategies, he can place positive or zero prob. in 7 different ways: $(+, 0, 0), (0, +, 0), (0, 0, +), (+, +, 0), (+, 0, +), (0, +, +), (+, +, +)$. Where $(+, 0, +)$ means that $U$ and $B$ have positive probability but $M$ has zero prob.; while $(+, +, +)$ means that all strategies have positive prob., etc...

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<tr>
<th></th>
<th>$L$</th>
<th>$M$</th>
<th>$R$</th>
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</thead>
<tbody>
<tr>
<td>$U$</td>
<td>1, -2</td>
<td>0, 0</td>
<td>0, 0</td>
</tr>
<tr>
<td>$M$</td>
<td>0, 0</td>
<td>-2, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td>$B$</td>
<td>0, 0</td>
<td>0, 0</td>
<td>1, 1</td>
</tr>
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</table>
The game below, that we saw previously, has only one Nash equilibrium \((B, R)\) where at least one of the players uses a pure strategy, but how do you know if there are no mixed strategy equilibria? Well, we have to analyze the cases: \((+, +, +)\), \((+, +, 0)\), \((+, 0, +)\) and \((0, +, +)\) for player 1 and also the similar cases for player 2, so the total number of cases is actually \(4 \times 4 = 16\)! But we shall see that the task is not hard as it seems at first glance...

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<tr>
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<th>L</th>
<th>M</th>
<th>R</th>
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<tbody>
<tr>
<td><strong>U</strong></td>
<td>1,-2</td>
<td>0,0</td>
<td>0,0</td>
</tr>
<tr>
<td><strong>M</strong></td>
<td>0,0</td>
<td>-2,1</td>
<td>0,0</td>
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<tr>
<td><strong>B</strong></td>
<td>0,0</td>
<td>0,0</td>
<td>1,1</td>
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</tbody>
</table>
Mixed Strategies
Finding all equilibria

\[ U_1(U, \sigma_2) = p_1 + (1 - p_1 - p_3)(-2) + p_3 0 = 3p_1 + 2p_2 - 2 \]
\[ U_1(M, \sigma_2) = p_1(-2) + (1 - p_1 - p_3)(1) + p_3 0 = 1 - 3p_1 - p_3 \]
\[ U_1(D, \sigma_2) = p_1 0 + (1 - p_1 - p_3) 0 + p_3 = p_3 \]

\[ BR_1(\sigma_2) = \begin{cases} 
U & \text{if } 3p_1 + 2p_3 - 2 \geq \max(1 - 3p_1 - p_3, p_3) \\
M & \text{if } 1 - 3p_1 - p_2 \geq \max(p_1 + 2p_2 - 2, p_3) \\
D & \text{if } p_3 \geq \max(3p_1 + 2p_3 - 2, 1 - 3p_1 - p_2) 
\end{cases} \]

We can simplify the \( BR_1 \) – I just wrote them this way so one may better understand the process we use to obtain it.
Mixed Strategies
cont.

$$BR_1(\sigma_2) = \begin{cases} 
U & \text{if } 3p_1 + 2p_3 - 2 \geq \max(1 - 3p_1 - p_3, p_3) \\
M & \text{if } 1 - 3p_1 - p_2 \geq \max(3p_1 + 2p_3 - 2, p_3) \\
D & \text{if } p_3 \geq \max(3p_1 + 2p_3 - 2, 1 - 3p_1 - p_3)
\end{cases}$$

Let’s consider the cases (+, +, +) and (+, +, 0). If player 1 puts positive prob. on both $U$ and $M$ then both must be best responses to $\sigma_2$ so we must have:

$$3p_1 + 2p_3 - 2 = 1 - 3p_1 - p_3 \geq p_3.$$  

Using the first equality we get $p_1 = \frac{3-3p_3}{6} = \frac{1-p_3}{2}$ and so we have that the payoff of playing $U$ or $M$ is 

$$1 - 3p_1 - p_3 = 1 - \frac{3}{2} + \frac{3}{2}p_3 - p_3 = -\frac{1}{2} + \frac{p_3}{2}$$

but this is always less than the payoff of playing $D$ which is $p_3$. We reached a contradiction so it is not possible that player 1 will place positive prob. to both $U$ and $M$. 
Mixed Strategies
cont.

From the previous analysis, we know that the only remaining cases for player 1 mixing are (0, +, +) and (+, 0, +).

Equally importantly, notice we can use a similar argument to establish that player 2 will not place positive prob. to both $L$ and $M$. 
Mixed Strategies
cont.

\[ BR_1(\sigma_2) = \begin{cases} 
U \text{ if } 3p_1 + 2p_3 - 2 \geq \max(1 - 3p_1 - p_3, p_3) \\
M \text{ if } 1 - 3p_1 - p_2 \geq \max(3p_1 + 2p_3 - 2, p_3) \\
D \text{ if } p_3 \geq \max(3p_1 + 2p_3 - 2, 1 - 3p_1 - p_3) 
\end{cases} \]

Case (+, 0, +): If 1 places positive prob. on both \( U \) and \( D \) and zero prob. on \( M \) then we must have:

\[ p_3 = 3p_1 + 2p_3 - 2 \geq 1 - 3p_1 - p_3. \]

Using the first inequality we obtain that \( p_3 = 2 - 3p_1 \) and this implies that \( p_1 > 0 \) otherwise \( p_3 \) would be greater than one if \( p_1 = 0 \).
Claim: But if player 2 is playing $L$ with positive prob. ($p_1 > 0$) then it must be that 1 plays $M$ with positive probability, which contradicts the assumption that $M$ receives zero prob. in the case $(+, 0, +)$.

Exercise: Derive the expected payoffs of player 2 for each of her pure strategies: $U_2(\sigma_1, L)$, $U_2(\sigma_1, M)$ and $U_2(\sigma_1, R)$; compute her best response to a mixed strategy of player 1 and prove the above claim.
Consider the following matrix depicting the payoffs of player 1 given player 1 and player 2 actions. Let $q$ be the probability player 2 chooses $C$, $p_1$ the prob. that 1 chooses $A$ and $p_2$ the prob. that 1 chooses $B$.

<table>
<thead>
<tr>
<th></th>
<th>$q$</th>
<th>$1 - q$</th>
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<tbody>
<tr>
<td>$C$</td>
<td>2</td>
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</tr>
<tr>
<td>$D$</td>
<td>0</td>
<td>1</td>
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</tbody>
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<tr>
<th></th>
<th>$p_1$</th>
<th>$p_2$</th>
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</thead>
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<tr>
<td>$A$</td>
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<td>$(1 - p_1 - p_2)$</td>
</tr>
<tr>
<td>$B$</td>
<td>0</td>
<td>1</td>
<td>$(1 - p_1 - p_2)$</td>
</tr>
<tr>
<td>$C$</td>
<td>-2</td>
<td>3</td>
<td>$(1 - p_1 - p_2)$</td>
</tr>
<tr>
<td>$D$</td>
<td>0</td>
<td>1</td>
<td>$(1 - p_1 - p_2)$</td>
</tr>
</tbody>
</table>

**a** Write down the expected payoff of player 1 as a function of $q$, $p_1$ and $p_2$:

$$U_1(p_1, p_2, q) = p_1 \cdot (q \cdot 2 + 0 \cdot (1 - q)) + p_2 \cdot (0 \cdot q + (1 - q) \cdot 1) + (1 - p_1 - p_2)(q \cdot (-2) + (1 - q) \cdot 3) = p_1(2q) + p_2(1 - q) + (1 - p_1 - p_2)(3 - 5q) = 3 + (7q - 3)p_1 + (4q - 2)p_2$$
Consider the following three constraints: 1) $-p_1 \leq 0$, 2) $-p_2 \leq 0$, and 3) $p_1 + p_2 \leq 1$; and write down the Lagrangian corresponding to the maximization of the payoff you found in item (a) subject to the constraints (1), (2) and (3).

$$\mathcal{L}(p_1, p_2, \lambda_1, \lambda_2, \lambda_3) = (7q - 3)p_1 + (4q - 2)p_2 - \lambda_1 (-p_1) - \lambda_2 (-p_2) - \lambda_3 (p_1 + p_2) = (7q - 3)p_1 + (4q - 2)p_2 + \lambda_1 p_1 + \lambda_2 p_2 - \lambda_3 (p_1 + p_2)$$
c Assume $q$ is given and the only choice variables are $p_1$ and $p_2$. Write down the first-order conditions, FOCs, for maximization of the expected payoff subject to the constraints.

\[
\mathcal{L}(p, \lambda) = 3 + (7q - 3)p_1 + (4q - 2)p_2 + \lambda_1 p_1 + \lambda_2 p_2 - \lambda_3 (p_1 + p_2)
\]

\[
\frac{\partial}{\partial p_1} \mathcal{L}(p, \lambda) = 7q - 3 + \lambda_1 - \lambda_3 = 0 \quad FOC_{p_1}
\]

\[
\frac{\partial}{\partial p_2} \mathcal{L}(p, \lambda) = 4q - 2 + \lambda_2 - \lambda_3 = 0 \quad FOC_{p_2}
\]
d Let the Lagrange multiplier associated to the $k_{th}$ constraint be denoted by $\lambda_k$. Write down the corresponding complementary slackness conditions.

\[ \lambda_1 \cdot p_1 = 0 \quad \text{CS}_1 \]
\[ \lambda_2 \cdot p_2 = 0 \quad \text{CS}_2 \]
\[ \lambda_3 \cdot (1 - p_1 - p_2) = 0 \quad \text{CS}_3 \]
e Assume that $\lambda_1 > 0$, $\lambda_2 = 0$ and $\lambda_3 = 0$. Using the
first-order conditions and the complementary
slackness conditions compute the value of $p_1$ and
$p_2$ (possibly as a function of $q$). For which values
of $q$, these values of $p_1$ and $p_2$ satisfy the
constraints 1,2, and 3?

By $CS_1$, $\lambda_1 > 0 \Rightarrow p_1 = 0$. By the FOCS,

$$\frac{\partial}{\partial p_1} \mathcal{L}(p, \lambda) = 7q - 3 + \lambda_1 = 0 \Rightarrow \lambda_1 = 3 - 7q$$

$$\frac{\partial}{\partial p_2} \mathcal{L}(p, \lambda) = 4q - 2 = 0 \Rightarrow q = \frac{1}{2} \Rightarrow \lambda_1 = 3 - \frac{7}{2} = \frac{-1}{2} < 0$$

So this case is NOT consistent.
Repeat question e but instead, now assume that 
\( \lambda_1 = 0 \) and \( \lambda_2 > 0 \) and \( \lambda_3 = 0 \). 

By \( CS_2 \), \( \lambda_2 > 0 \) \( \Rightarrow \) \( p_2 = 0 \). By the FOCS,

\[
\frac{\partial}{\partial p_1} \mathcal{L}(p, \lambda) = 7q - 3 = 0 \Rightarrow q = \frac{3}{7}
\]

\[
\frac{\partial}{\partial p_2} \mathcal{L}(p, \lambda) = 4q - 2 + \lambda_2 = 0 \Rightarrow \lambda_2 = 2 - \frac{12}{7} = \frac{2}{7} > 0
\]

So this case is consistent, moreover \( p_2 = 0 \) and any value of \( p_1 \in [0, 1] \) solve the constraints.
Consider the following matrix depicting the payoffs of player 1 and 2 given their 2 actions. Let $q$ be the probability player 2 chooses $E$, $p_1$ the prob. that 1 chooses $A$ and $p_2$ the prob. that 1 chooses $B$.

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<th>$q$</th>
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<tr>
<td>$F$</td>
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<td></td>
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<tr>
<td>$p_1$</td>
<td>$A$</td>
<td>2,-2</td>
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<tr>
<td>$p_2$</td>
<td>$B$</td>
<td>0,0</td>
</tr>
<tr>
<td>$1 - p_1 - p_2$</td>
<td>$C$</td>
<td>-2,2</td>
</tr>
</tbody>
</table>

a) $U_1(A, q) = 2q$.

b) $U_1(B, q) = 1 - q$.

c) $U_1(C, q) = -2q + 3(1 - q) = 3 - 5q$. 
\[ d \quad U_2(p_1, p_2, E) = p_1(-2) + p_2(0) + (1 - p_1 - p_2)(2) = 2 - 4p_1 - 2p_2. \]

\[ e \quad U_2(p_1, p_2, F) = p_1(0) + p_2(-1) + (1 - p_1 - p_2)(-3) = -3 + 3p_1 + 2p_2. \]

\[ f \quad \text{If } q > 0 \text{ and } 1 - q > 0 \text{ and 2 is best responding,} \]
\[ U_2(p_1, p_2, E) = U_2(p_1, p_2, F) \iff 2 - 4p_1 - 2p_2 = -3 + 3p_1 + 2p_2. \]
\( g \) \( U_1(A, q) = 2q = U_1(C, q) = \frac{3}{1} - 5q \geq U_1(B, q) = 1 - q \). So \( q = \frac{3}{7} \).

\( h \) \( p_2 = 0 \Rightarrow U_2(p_1, p_2, E) = U_2(p_1, p_2, F) U_2(p_1, p_2, E) = U_2(p_1, p_2, F) \Leftrightarrow p_1 = \frac{5}{7} \) and of course \( q = \frac{3}{7} \) as before.
Consider the following matrix depicting the payoffs of player 1 and 2 given their 2 actions. Let $q$ be the probability player 2 chooses $C$, $p_1$ the prob. that 1 chooses $A$. Assume that $R > 1$.

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<th>$q$</th>
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<tr>
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<td>$E$</td>
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<td>$p$</td>
<td>$A$</td>
<td>R,R</td>
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<tr>
<td>$1 - p$</td>
<td>$B$</td>
<td>1,0</td>
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</table>

**a** Which game is this? **Stag-hunt**. Give an example of economic situation(s) or a story that this game is intended to model. **Bank runs** R is the risk-action (all get high payoff if all choose it), S is the safe action (it gets a lower but sure payoff regardless of what others do).

**b** Compute all the Nash equilibria in pure strategies. $(R, R)$ and $(S, S)$
c Compute the mixed strategy Nash equilibrium.

\[ U_1(R, q) = R \cdot q + (1 - q) \cdot 1 = U_1(S, q) = 1 \Leftrightarrow q = \frac{1}{R}. \]

By symmetry, \( p = \frac{1}{R} \).

d What happens, in the equilibrium in mixed strategies, with the prob. that 1 plays A as the value of R increases? Explain. It decreases, if R becomes more valuable then the only way of player 1 to remain indifferent between R and S is for player 2 to choose R less often.
Dynamic Games
Entry Deterrence

Player 1

- stay out → 0, 100

Player 2

- enter
  - accommodate → 50, 50
  - fight → -5, -15
Dynamic Games
Entry Deterrence

Player 1

- **stay out** → 0, 100

Player 2

- **enter**
  - **accomodate** → 50, 50
  - **fight** → -5, -15
Dynamic Games
Entry Deterrence

Player 1
- **stay out** → 0, 100
- **enter**

Player 2
- **accomodate** → 50, 50
- **fight** → −5, −15
Dynamic Games
Strategy Profiles: Actions and strategies are now different concepts!

Player 1

Player 2

\[ s_2 = (a, b) \] means \( a \) if 1 chose \( u \) and \( b \) if 1 chose \( d \).
Dynamic Games
Strategy Profiles: Actions and strategies are now different concepts!

Player 1

Player 2

$S_1 = \{u, d\}$

$s_2 = (a, b)$ means $a$ if 1 chose $u$ and $b$ if 1 chose $d$. 

Player 2

Player 2

$S_2 = \{(u, u), (u, d), (d, u), (d, d)\}$. 

0,0
Dynamic Games

Strategy Profiles: Actions and strategies are now different concepts!

Player 1

Player 2

$S_1 = \{u, d\}$

$S_2 = \{(u, u), (u, d), (d, u), (d, d)\}$

$s_2 = (a, b)$ means $a$ if 1 chose $u$ and $b$ if 1 chose $d$. 
Dynamic Games
Strategy Profiles: Actions and strategies are now different concepts!

Player 1

Player 2

\[ S_1 = \{ u, d \} \]

\[ S_2 = \{ (u, u), (u, d), (d, u), (d, d) \} \]

\[ s_2 = (a, b) \text{ means } a \text{ if 1 chose } u \text{ and } b \text{ if 1 chose } d. \]
The Centipede Game
with 5 moves
A strategy for player 1 is triple \((a_1, a_2, a_2)\) where \(a_i \in \{S, C\}\) is the action that player 1 executes at her \(i_{th}\) node.

A strategy for player 2 is duple \((b_1, b_2)\) where \(b_i \in \{S, C\}\) is the action that player 2 executes at his \(i_{th}\) node.
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</table>
A strategy for player 1 is triple \((a_1, a_2, a_2)\) where \(a_i \in \{S, C\}\) is the action that player 1 executes at her \(i_{th}\) node.

A strategy for player 2 is duple \((b_1, b_2)\) where \(b_i \in \{S, C\}\) is the action that player 2 executes at his \(i_{th}\) node.

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</table>

Payoff of \(BR_1\)
The Centipede Game
with 5 moves

A strategy for player 1 is triple \((a_1, a_2, a_2)\) where \(a_i \in \{S, C\}\) is the action that player 1 executes at her \(i_{\text{th}}\) node.

A strategy for player 2 is duple \((b_1, b_2)\) where \(b_i \in \{S, C\}\) is the action that player 2 executes at his \(i_{\text{th}}\) node.

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</tbody>
</table>

Payoff of \(BR_2\)
The Centipede Game
with 5 moves

A strategy for player 1 is triple \((a_1, a_2, a_2)\) where \(a_i \in \{S, C\}\) is the action that player 1 executes at her \(i_{th}\) node.

A strategy for player 2 is duple \((b_1, b_2)\) where \(b_i \in \{S, C\}\) is the action that player 2 executes at his \(i_{th}\) node.

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Nash eq. Payoffs
The Centipede Game
with 5 moves

A strategy for player 1 is triple \((a_1, a_2, a_2)\) where \(a_i \in \{S, C\}\) is the action that player 1 executes at her \(i_{th}\) node.

A strategy for player 2 is duple \((b_1, b_2)\) where \(b_i \in \{S, C\}\) is the action that player 2 executes at his \(i_{th}\) node.

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Subgame Perfect Nash eq. Payoffs
Bargaining
Dynamic games

The ultimatum game

Player 1 makes demand $x$ and 2 after observing it, says either yes or no. Payoffs are: $u_1(x, y) = x$, $u_2(x, y) = 1 - x$, and $u_1(x, n) = u_2(x, n) = 0$.

- Player 2’s best response is to say yes to any $x < 1$.
- Player 2 is indifferent to say yes or no to $x = 1$.
- Thus player 2 has two strategies that can be item of a subgame perfect equilibrium:

$$s^*(x) = y \text{ for all } x \quad \text{and}$$

$$s^{**}(x) = \begin{cases} y & \text{if } x < 1 \text{ and} \\ n & \text{if } x = 1. \end{cases}$$
We saw player 2 has two strategies that can be item of a subgame perfect equilibrium:

\[ s^*(x) = y \text{ for all } x \text{ and } \]
\[ s^{**}(x) = \begin{cases} 
  y & \text{if } x < 1 \text{ and } \\
  n & \text{if } x = 1.
\end{cases} \]

**Exercise:**

1. Show that \((1, s^*)\) is a SPNE.
2. Show that if 2 uses \(s^{**}\) then \(BR_1(s^{**}) = \emptyset\).
3. Conclude that there does not exist any SPNE where player 2 uses the strategy \(s^{**}\).
Bargaining
A dynamic game

Let’s modify the ultimatum game to allow for many rounds of bargaining. Each round one player makes an offer $x \in [0, 1]$ (a fraction of the pie). After the offer is made, the other player either says "yes" or "no". If the offer is accepted, then the game ends and the player who made the offer gets $x \pi_t$ while the other player gets $(1 - x) \pi_t$ where $\pi_t$ is the size of the pie after $t$ offers were made. If the offer is rejected then the size of the pie shrinks ($\pi_{t+1} = \delta \pi_t$ and $\pi_1 = 1$, where $\delta \in (0, 1)$) and now it is turn of the other player to make an offer. After $T$ offers were made and rejected then he game ends with both players getting zero.
Bargaining
Many rounds of bargaining

If $T = 1$ then the game is the ultimatum game. Let’s then first analyze the game with $T = 2$

- Consider all subgames where the initial offer of player 1 was rejected. These subgames are identical to an ultimatum game where player 2 makes an offer first and pie has size $\delta$. So by the previous analysis, player 1 accepts any offer of player 2 and player 2 demands the whole pie for herself. The payoff equilibrium at these subgames is $(0, \delta)$.

- Consider all subgames where player 1 made a demand of $x$ and player 2 has to say "yes" or "no". If 2 rejects player 1, she will get $\delta$ in the continuation game. Thus 2 is willing to accept any offer such that $1 - x \geq \delta$.

- Now consider the beginning of the game when player 1 makes an offer. Any demand $x$ with $x < 1 - \delta$ is rejected and player 1’s payoff is zero. Any demand with $x \leq 1 - \delta$ is accepted and 1’s payoff is $1 - \delta$. So 1 demands $1 - \delta$. 
In sum, the unique subgame perfect Nash equilibrium when $T = 2$ is $(s_1, s_2)$ given by

$$s_1(\emptyset) = 1 - \delta; \quad s_1(x, "no", y) = "yes";$$

$$s_2(x) = \begin{cases} 
    yes & \text{if } x \leq 1 - \delta \\
    no & \text{otherwise}
\end{cases}; \quad s_2(x, "no") = 1.$$
Now let’s study the game with $T = 3$ (potential) proposals. Consider all subgames where player 1 made an offer that was refused, that is all histories $h = (x, ”no”).$ Anyone of these subgames is equivalent to a game with $T = 2$ where player 2 moves first and the size of the pie is $\delta$. So by the previous analysis in any of these subgames, player 2’s payoff is $\delta(1 - \delta)$ and player 1’s payoff is $\delta \delta$ (can you see why this is the case?). Now consider the subgames where 1 made his first offer, $h = (x)$. Since 2 can guarantee a payoff of $\delta(1 - \delta)$ then player 2 will say "no" to any demand such that $1 - x < \delta(1 - \delta)$ and accept any $x$ such that $x \leq 1 - \delta(1 - \delta)$. Thus at the beginning of the game, $h = \emptyset$, player 1 offers $1 - \delta(1 - \delta)$.
Bargaining
Many rounds of bargaining

In sum, the unique subgame perfect Nash equilibrium when $T = 3$ is $(s_1, s_2)$ given by

$$s_1(\emptyset) = 1 - \delta(1 - \delta); \quad s_1(x, "no", y) = \begin{cases} yes & \text{if } y \leq 1 - \delta \\ no & \text{otherwise} \end{cases};$$

$$s_1(x, "no", y, "no") = 1; \quad s_2(x) = \begin{cases} yes & \text{if } x \leq 1 - \delta(1 - \delta) \\ no & \text{otherwise} \end{cases};$$

$$s_2(x, "no") = 1 - \delta; \quad s_2(x, "no", y, "no", z) = "yes".$$
The basic model
Two players 1 and 2 have to divide a pie of size 1.

A static game
Player 1 makes demand $x$ and 2 makes demand $y$. If demands are feasible $x + y \leq 1$ then player 1 payoff is $x$ and 2’s payoff is $y$. But if demands are infeasible, both players get zero payoff.

Exercise: Prove that any pair $(x, y)$ that is feasible $x + y \leq 1$ is a Nash equilibrium of the static bargaining game.
In an extensive tree if player \( i \) does not observe all (but perhaps observes some) of the past moves and it is now \( i \)’s turn to play, we say that the \( i \)’s current \textit{information set} is the collection of all nodes where player \( i \) moves that are consistent with all the actions player \( i \) observes.

Important: since a player can not distinguish between different nodes in the same information set, the player has to take the same action at every node in the same information set. Hence, strategies are functions from information sets into actions. The less informed is a player, the less strategies the player has. The more informed, the more strategies.
The information below is for all questions:

- **Players:** *(dramatis personae)* are Nature, Hamlet, and Claudius.
- Nature chooses whether to kill Hamlet’s father with a viper bit *B* or not *N*.
- If Nature chooses *N* then Claudius moves and chooses whether to poison Hamlet’s father *P* or not *N*.
- After Nature chooses *B* or Claudius chooses *P*, Hamlet moves and chooses vengeance *(i.e., kill Caludius)* *V* or not *N*. 
Write down the game tree (without the payoffs) of the extensive games where (you have to drawn one tree for each item):

1. All players observe all past moves.
2. Claudius observes Nature’s moves and Hamlet does not observe any moves and they know that.
3. Claudius observes if Nature chose $B$ and Hamlet may or may not observe the past moves (here you will need to add an additional node where Nature moves before choosing $B$ or $N$) and they know that.
Bayesian Games

Is more information always better?

If we have only one player, more information is always better. However, take a look at this example:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
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<tbody>
<tr>
<td>**G1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>U</td>
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<td>0,0</td>
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<tr>
<td>**G2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>U</td>
<td>5,5</td>
<td>-100,5.1</td>
</tr>
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</tbody>
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Scenario: both players are uncertain about the game they are playing, each player believes the game is G1 with probability 1/3 and G2 with prob. 2/3. We have two Bayesian-Nash eq.: \((U, L)\) and \((D, R)\). Both players prefer \((U, L)\).

Alternative scenario: player 2 has two types, \(t_1\) when the game is G1 and \(t_2\) when the game is G2. Player 1 is uninformed. Now the only Bayesian-Nash equilibrium is \((D, (R, R))\).
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Bayesian Nash Equilibrium
A Cournot Game

- Two firms 1 and 2.
- Market demand \( P = \alpha - Q \) where \( Q = q_1 + q_2 \).
- Cost functions: \( C_1(q_1) = q_1^2 \) and \( C_2(q_2) = \beta q_2^2 \).
- Uncertainty about \( \alpha \sim U[a, \bar{a}] \) and \( \beta \sim U[b, \bar{b}] \).
- Player 1 knows \( \alpha \). Player 2 knows \( \beta \).
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**A Question From a Past Exam**

Player 1 moves and chooses between $L$ and $R$. After 1’s move and without knowing it, player 2 decides between spying or not spying ($S$ or $N$). If player 2 decides to spy, her final payoff will be reduced by 0.3 units regardless of the other actions. If player 2 decides to spy, she learns what was player 1’s choice of action and knowing this, she must then choose between $l$ and $r$. But if player 2 decides to not spy, player 2 does not observe player 1 action and has also to chose between $l$ and $r$ without knowledge what was 1’s move.

**Payoffs**

$$u_1(R, S, r) = u_1(R, N, r) = u_1(L, S, l) = u_1(L, N, l) = 1$$

$$u_1(R, S, l) = u_1(R, N, l) = u_1(L, S, r) = u_1(L, N, r) = 0;$$

for player 2, without spying: $u_2(R, N, r) = u_2(L, N, l) = 0$, and $u_2(R, N, l) = u_2(L, N, r) = 1$, and with spying:

$$u_2(R, S, r) = u_2(L, S, l) = -0.3,$$

and

$$u_2(R, S, l) = u_2(L, S, r) = 0.7.$$  

1. Draw the game tree of the game, including payoffs and information sets.
2. List all strategies of player 1. Explain your labeling clearly.
3. List all strategies of player 2. Explain your labeling clearly.
4. Write the game in the normal (strategic) form. You may write the reduced normal form (eliminate payoff equivalent strategies) instead if you prefer.
5. Find the set of sequential Nash equilibrium for the extensive game.
6. Is there any Nash equilibrium (possibly in mixed strategies) that fails to be part of a sequential Nash equilibrium? Explain.
The Tree of The Espionage Game

The diagram represents a strategic interaction between two players, where the payoffs are depicted in terms of payoffs for each player at each node. The nodes are labeled with payoffs and probabilities, indicating the outcomes of the game at different stages. The edges are labeled with strategic moves and probabilities, reflecting the decision-making process of the players.

The game tree includes several decision points labeled with strategic choices, such as 'S' for 'Strategy', 'L' for 'Left', and 'R' for 'Right', along with associated payoffs and probabilities. The payoffs are typically represented as tuples where the first element is the payoff for Player 1 and the second element is the payoff for Player 2. Probabilities are indicated next to the edges, showing the likelihood of a particular move being chosen.

The diagram captures the essence of a sequential decision-making process in a strategic game, where players make choices based on the outcomes of previous decisions, leading to different payoffs as the game progresses.
A Card Game
A Card Game
Defining the available strategies

Pure strategies of player 1, $S_1 = \{(R, R), (R, N), (N, R), (R, R)\}$ where $(x, y)$ means play $x$ after player zero chooses $H$ and play $y$ after $L$.

Pure strategies of player 2, $S_1 = \{(M, M), (M, P), (P, M), (P, P)\}$ where $(x, y)$ means play $x$ after player 1 chooses $R$ and play $y$ after $N$.

Mixed strategy for player 1:

$$s_1 = \begin{pmatrix} p \\ q \end{pmatrix} \quad \text{prob. of R after } H \quad \text{prob. of R after } L$$

Mixed strategy player 2:

$$s_2 = \begin{pmatrix} r \\ s \end{pmatrix} \quad \text{prob. of M after } R \quad \text{prob. of M after } N$$
We have to compute expected payoffs in order to obtain the above payoffs. See the two examples below:

\[
\begin{align*}
    u_1(NR, PM) &= Pr[H] \cdot u_1(H, N, M) + Pr[L] \cdot u_1(L, R, P) = \\
    &= \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 \text{ and} \\
    u_1(RN, PM) &= Pr[H] \cdot u_1(H, R, P) + Pr[L] \cdot u_1(L, N, M) = \\
    &= \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (-1).
\end{align*}
\]
A card game

Using the strategic form to compute best response in pure strategies, we can see that there are no Nash equilibria in pure strategies.

In the previous page, for each strategy of player 1, the payoff in blue corresponds to the payoff of the best-response of player 2.

for each strategy of player 2, the payoff in red corresponds to the payoff of the best-response of player 1.
To obtain 2’s best-response after player 1 chooses R, $a_1 = R$, we need to compare the respective payoffs:

$$
U_2(p, q, M, |a_1 = R) = \frac{1}{2} p \cdot u_2(H, R, M) + \frac{1}{2} q \cdot u_2(L, R, M)
$$

$$
= \frac{p}{p + q} \cdot (-2) + \frac{q}{p + q} \cdot (2) = 2 \cdot \frac{q - p}{p + q}
$$

and

$$
U_2(p, q, P, |a_1 = R) = \frac{1}{2} p \cdot u_2(H, R, P) + \frac{1}{2} q \cdot u_2(L, R, P) = -1
$$

So $BR_2(p, q| R) = \begin{cases} M & \text{if } 2 \cdot \frac{q - p}{p + q} > -1 \\ M \text{ or } P & \text{if } 2 \cdot \frac{q - p}{p + q} = -1 \\ P & \text{if } 2 \cdot \frac{q - p}{p + q} < -1 \end{cases}$, but notice that

$$
2 \cdot \frac{q - p}{p + q} = -1 \Leftrightarrow p = 3q
$$

So we can simplify the above:

$$BR_2(p, q| R) = \begin{cases} M & \text{if } p < 3q \\ M \text{ or } P & \text{if } p = 3q \end{cases}$$
Best-Response of Player 2

To obtain 2’s best-response after player 1 chooses N, \( a_1 = N \), we need to compare the respective payoffs:

\[
U_2(p, q, M, \mid a_1 = N) = \frac{1}{2}(1 - p) \cdot u_2(H, N, M) + \frac{1}{2}(1 - q) \cdot u_2(L, N, M) = \frac{1 - p}{2 - p - q} \cdot (-1) + \frac{1 - q}{2 - p - q} \cdot (1)
\]

\[
= \frac{p - q}{2 - p - q} \quad \text{and} \quad U_2(p, q, P, \mid a_1 = N) = -1
\]

So \( BR_2(p, q \mid N) = \begin{cases} M & \text{if } \frac{p - q}{2 - p - q} > -1 \\ M \text{ or } P & \text{if } 2 \cdot \frac{p - q}{2 - p - q} = -1 \quad \text{, but notice} \\ P & \text{if } \frac{p - q}{2 - p - q} < -1 \end{cases} \)

that \( \frac{p - q}{2 - p - q} = -1 \iff q = 1 \) so we can simplify the above:

\[
BR_2(p, q \mid N) = \begin{cases} M & \text{if } q \neq 1 \text{ and} \\ M \text{ or } P & \text{if } q = 1 \end{cases}
\]
A Card Game
How to proceed, left as an exercise...

1. Derive the best response of player 1 to \((r, s)\) (mixed strategy of 2) when player 1 has a high card.

2. Same as in 1 but now for when player 1 has a low card.

3. Check if there is a mixed strategy Nash equilibrium where \(1 > p > 0\) and \(1 > q > 0\). Player 1 must be indifferent between \(M\) and \(N\) (both must be best-responses!) when he has a high card (so \(1 > p > 0\)) and when he has a low card (so \(1 > q > 0\)).

4. Check if there is a mixed strategy Nash equilibrium where \(1 > p > 0\) and but \(q = 0\) or \(q = 1\). Player 1 must be indifferent between \(M\) and \(N\) when he has a high card (so \(1 > p > 0\)) but when he has a low card, he does not mixes.

5. Check if there is a mixed strategy Nash equilibrium where \(1 > q > 0\) and but \(p = 0\) or \(p = 1\). Now 1 is indifferent between \(M\) and \(N\) when he has a low card (so \(1 > q > 0\)) but not when the card is high.
The model:

- In the population, agents have wealth \( \omega \) and an agent with "type" \( \lambda \) suffers a loss of size \( \ell \in (0, \omega) \) with probability \( \lambda \).
- An insurance contract is a pair \((P, C)\) where \( P \) is the premium and \( C \) is the coverage.
- All agents have utility function for money \( u \) that satisfies \( u(0) = 0 \), \( u' > 0 \) and \( u'' < 0 \).
- A fraction \( \pi \) of the population has a higher risk, \( \Pr[\lambda = \lambda_H] = \pi \) and \( \Pr[\lambda = \lambda_L] = 1 - \pi \).
The assumption $u'' < 0$ implies agents are risk-averse. Let $V(P, C, \lambda)$ be the expected utility for an agent of buying the insurance contract $(P, C)$ when the agent probability of a loss is $\lambda$. Thus,

$$V(P, C, \lambda) = \lambda u(\omega - \ell + C - P) + (1 - \lambda) u(\omega - P).$$

Notice $V(0, 0, \lambda)$ is the utility of not buying insurance (or buying a free insurance contract that pays nothing in the case of a loss).
Lemma (1)

An agent with type $\lambda$ prefers to buy $(P, C)$ than to not buy insurance if and only if $V(P, C, \lambda) - V(0, 0, \lambda) \geq 0$.

Proof.

Just use the definition of $V$.

Lemma (2)

If an agent with type $\lambda_L < \lambda_H$ prefers to buy $(P, C)$ than to not buy insurance then an agent with type $\lambda_H$ also prefers to buy.

Proof.

Show that if $V(P, C, \lambda) - V(0, 0, \lambda) \geq 0$ then

$$\frac{\partial}{\partial \lambda} [V(P, C, \lambda) - V(0, 0, \lambda)] \geq 0.$$
Lemma 2 is very important because it says that the insurer can exclude the low risk-agent (by offering a contract he would not buy) but if the insurer wants to sell to the low risk-agent then the agent will also have to sell to the high-risk ones.
Repeated Games

Consider the (static) game in the strategic (normal) form: \( G = (I, A, u) \).

The finitely repeated game \((\Gamma, T)\) is the game where agents play \( G \) for \( T \) periods; at each period they observe all the past choices; and their final payoff is the sum of period payoffs.

In the infinitely repeated game \((\Gamma, \delta)\) they play \( G \) in the first period, at any period if they always played \( G \) in the past, they play it again and with probability \( \delta \) they move to the next period and with prob. \( (1 - \delta) \) the game ends.
Repeated Games

In the infinitely repeated game \((\Gamma, \delta)\) the expected payoff is

\[
U_i(a) = (1 - \delta) \cdot \sum_{t=0}^{+\infty} \delta^t \cdot u_i(a^t)
\]

where \(a^t\) is the action profile played at period \(t\) if players follow the strategy profile \(s\) and \(\delta \in (0, 1)\) is the discount factor (inverse of the interest rate) or, alternatively, \(\delta\) is the probability the game will continue in the next period.
What is a strategy for a player? A strategy is a mapping from her information sets into her actions available at the info. set. In a repeated game, they are always playing the same (stage) game, so the exact same set of actions, $A_i$, is available at any information set.

What are the information sets of player $i$? He or she observes the past play, so each possible history of the game corresponds to an information set for the player. So the set of all information sets (histories) is $\mathcal{H} = \bigcup_{k=1}^{\infty} A^k \cup \{\emptyset\}$. 
A strategy for player $i$ is a mapping from her information sets into her actions available at the info:

$$s_i : \mathcal{H} \rightarrow A_i$$

Main results:

1. Any strategy that calls for players playing some Nash eq. of $G$ (possibly distinct) at every period is a SPNE of $\Gamma$.

2. However $\Gamma$ has other SPNE that are very different from just playing a Nash eq. of $G$.

3. For example, for high values of $\delta$ it is possible to have cooperation in the repeated prisoners dilemma as a SPNE.
Repeated Games
Representing Strategies as Automata

For example consider the strategy profile in the Prisoner’s Dilemma described by:

Start cooperating, and keep cooperating if all cooperate (cooperation phase). If someone did not cooperate, then do not cooperate for three periods. After that resume the cooperation phase. (punishment phase)
Who watches the watchmen?
Quis custodiet ipsos custodes?
Consider the entry-deterrence game we study before but where the payoffs are

Player 1

- **stay out**
  - 0, a
- **enter**

Player 2

- **accommodate**
  - b, 0
- **fight**
  - b - 1, -1 when

the incumbent type is "normal" ...
and if the incumbent type is "crazy", payoffs are:

Player 1

- **stay out** → 0, a

Player 2

- **enter**
  - **accommodate** → b, −1
  - **fight** → b − 1, 0

where \( a > 1 > b > 0 \).
In the beginning of the game (and once for all) Nature chooses the type of the incumbent (crazy or normal). The incumbent who knows his type will face a finite sequence $t = 1, 2, \ldots, T$ of entrants in different markets. The entrants move sequentially, they do not know the type of the incumbent but they observe the actions played by the incumbent and previous entrants.

Let $p_1$ be the probability that Nature chooses the incumbent is "crazy" and let $p_t(h)$ be the probability the entrant is "crazy" given the history $h$ leading to round $t$. 
Reputation

Prove that in the last round, in any subgame perfect Nash equilibrium, the normal incumbent always chooses to accommodate and the crazy incumbent always chooses to fight.

Given the behavior of the incumbent, what is the expected utility of the last entrant $t = T$ if she enters? if she stays out? What is her best response (as a function of $p_T$)?
Reputation

Once you solved for the behavior of the last entrant (as a function of $p_T$), derive the value of $p_T$ if the incumbent fought at $T - 1$ (using Bayes rule) as a function of $p_{T-1}$ and the probability the incumbent fights at $T - 1$, which we denote as $\sigma_{T-1}$.

Also derive the value of $p_T$ if the incumbent did not fight at $T - 1$ (using Bayes rule) as a function of $p_{T-1}$ and the probability the incumbent fights at $T - 1$, which we denote as $\sigma_{T-1}$. 
Definition of a coalitional game

1. A set of players $I = \{1, 2, \ldots, n\}$.
2. For each subset of players $P \subset I$, a set of feasible actions $A(P)$ for the coalition $P$.
3. For each player, a numerical ranking (payoff) over the actions of any coalition the player is a member, $u_i : \bigcup_{P:i \in P} A(P) \to \mathbb{R}$. 

Coalition Games
Definition of a coalitional game with transferrable utility

For any subset of players $P \subset I$ and any action $a$ in $A(P)$, the sum of players’ payoffs $\sum_{i \in P} u_i(a) = c(P)$ depends only on $P$ – with respect to the actions in $A(P)$ the sum of payoffs is a constant.
**Definition of the Core**

The core is the subset of actions of the grand coalition $C \subset A(I)$ such that for any $c$ in $C$ and any other coalition $P$, there is no $a \in A(P)$ such that $u_i(a) > u_i(c)$ for all $i$ in $P$. 