THE ALL-PAY AUCTION WITH INDEPENDENT PRIVATE VALUES

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ABSTRACT. In the independent private values setting, we provide sufficient conditions for the continuity and uniqueness of the all-pay auction equilibrium. We also provide an algorithm to compute the equilibrium.

1. Introduction

To find a Bayesian-Nash equilibrium of all-pay auctions, simplifying assumptions such as – bid strategies are monotone and differentiable – are, explicitly or implicitly, often used. Not surprisingly, applied work employing the all-pay auction framework are called upon such limitations: “The results are interesting but how do we know this is the only equilibrium? Even if the differentiable equilibrium is unique, how do we know there are not other equilibria in discontinuous strategies?” The purpose of this paper is two-fold. First, to provide sufficient conditions for equilibrium continuity, differentiability and uniqueness. Second, in doing so, to validate the first-order approach.

Our contribution is primarily relevant to environments with three or more asymmetric bidders. For two bidders, Amann and Leininger (1996) constructed an algorithm to find the equilibrium and Lizzeri and Persico (1998) proved the equilibrium is unique and differentiable. For symmetric bidders, Krishna and Morgan (1997) characterized the symmetric equilibrium; Parreiras and Rubinchik (2010) proved the symmetric equilibrium its unique.

We restrict attention to risk-neutral bidders with independent, private values for two reasons. First, risk aversion may create discontinuities (Parreiras and Rubinchik, 2010). Secondly, correlation or interdependent values may lead to non existence of monotonic equilibria. (Lu and Parreiras, 2015a).

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We are grateful to Anna Rubinchik and David C. Ulrich. All errors are ours.
It should be self-evident that our model is not nested with models with correlation and/or interdependent values (Krishna and Morgan, 1997; Lizzeri and Persico, 1998; Lu and Parreiras, 2015b).

The more than two asymmetric bidders case, is relevant to many real-life applications. Moreover, the behavior implied by its equilibrium might be qualitatively different from the other cases (Kirkegaard, 2010; Parreiras and Rubinchik, 2010). It appears, the lack results establishing uniqueness and the validity of the first-order approach has been the major impediment to applied work. Notable exceptions are Kirkegaard (2010, 2013) and Parreiras and Rubinchik (2015).

2. Model and Notation

There are \( N \geq 2 \) risk-neutral bidders who compete for a single, indivisible object. Bidder \( i = 1, \ldots, N \) privately learns the realization of her valuation, \( v_i \), which is drawn from an absolutely continuous distribution, \( V_i \sim F_i \), with support on \( [\underline{v}_i, \overline{v}_i] \). Valuations are independently distributed. After learning valuations, bidders simultaneously post their bids thus a bidder’s strategy is a map from the valuations to bids, \( b_i : [\underline{v}_i, \overline{v}_i] \to \mathbb{R}^+ \). The object is allocated to the highest bid. If a tie happens, any random tie-breaking rule can be used.

**Definition 1.** Given a strategy, \( b_i(\cdot) \), the corresponding cumulative probability distribution of bids is \( G_i(b) \triangleq \Pr[b_i(V_i) \leq b] \).

We denote densities (whenever they are well-defined) by the corresponding lower cases; valuations and bids densities are \( f_i(v) \) and \( g_i(b) \).

**Definition 2.** For \( H : \mathbb{R} \to \mathbb{R} \) and all \( a, b \in \mathbb{R} \), we define
\[
\Delta H(a, b) \triangleq H(b) - H(a).
\]

**Definition 3 (Extreme Bids).** Given a strategy, \( b_i(\cdot) \), bidder \( i \)'s top and bottom bids are \( \overline{b}_i \triangleq b_i(\overline{v}_i) \) and \( \underline{b}_i \triangleq b_i(\underline{v}_i) \).

3. Any equilibrium is monotone

Athey (2001) proved the existence of an pure strategy, monotone equilibrium. At the end of this section, we shall see that under independent private values, any equilibrium pure strategy equilibrium is monotone.

**Lemma 1 (No Atoms At Positive Bids).** In equilibrium, the distribution of bids \( G_i \) is continuous.

**Proof.** A discontinuity of \( G_i \) corresponds to bidder \( i \) choosing \( b > 0 \) with positive probability. At this \( b > 0 \), the winning probability of
any other bidder distinct from \( i \) must be discontinuous at \( b \). That is, bidders distinct from \( i \) do not place bids in the some left neighborhood of the bid, \((b - \varepsilon, b)\) where \( \varepsilon > 0 \). It follows that \( i \)'s winning probability must be constant on \((b - \varepsilon, b)\) but then bidding \( b - \varepsilon / 2 \) is strictly better rather than bidding \( b \) for some type \( v_i > 0 \) that is bidding \( b \).

By Lemma 1, in equilibrium ties never happen at positive bids, so we can define:

**Definition 4** (Winning Probability). For \( b > 0 \), define bidder \( i \)'s winning probability, \( W_i(b) \equiv \prod_{j \neq i} G_i(b) \).

**Lemma 2** (Bottom Bids are Zero). In equilibrium, \( b_i = 0 \).

*Proof.* Let \( b = \max_i b_i \). On one hand, if \( b = 0 \) then the proof is done. On the other hand, if \( b > 0 \) then no bidder chooses \( b \) with positive probability (Lemma 1). Since bids at or below \( b \) never win and \( b > 0 \) is costly, \( b \) can not be a best-response. 

**Lemma 3** (No Atoms At Zero For At Least One Bidder). In equilibrium, \( \Pi_{i=1}^{n} G_i(0) = 0 \).

*Proof.* Assume that \( \Pi_{i \neq j} G_i(0) > 0 \). Bidding zero leads to ties with positive probability and so \( W_j \) must be discontinuous at zero. Bidding zero is optimal only for players with zero valuation. But \( v_j = 0 \) implies \( G_j(0) = 0 \) contradicting the assumption. 

The next lemma is well-known; we omit its proof.

**Lemma 4** (Winning Probabilities Are Strictly Increasing.). In equilibrium, \( W_i \) is strictly increasing on \([0, W_i^{-1}(1)]\).

The next lemma is extremely useful.

**Lemma 5** (Ranking Winning Probabilities). For any \( 0 \leq a < b \), \( G_i(a, b) \geq G_j(a, b) \) if and only if \( W_i(a, b) \leq W_j(a, b) \).

*Proof.* It follows immediately from the identity, \( \Delta W_i(a, b) \Delta G_j(a, b) \equiv \Delta W_j(a, b) \Delta G_i(a, b) \).

In words, \( \Delta W_i(a, b) \) is the probability that players other than player \( i \) bid in the interval \((a, b)\) and \( \Delta G_i(a, b) \) is the probability that player \( i \) bids in \((a, b)\). The product \( \Delta W_i(a, b) \Delta G_i(a, b) \) is the probability that all players bid in \((a, b)\) and so it must be invariant with respect to \( i \).

The *interim* payoff of bidder \( i \) with valuation \( v \) who bids \( b \) is, \( U_i(b|v) = W_i(b) \cdot v - b \).
Lemma 6 (Bid ranking). Player $i$ with valuation $v$ weakly prefers to bid $b$ rather than bid $a$, $U_i(b|v) - U_i(a|v) \geq 0$, if and only if:

$$\Delta W_i(a, b) \geq \frac{b - a}{v}.$$ 

Lemma 7 (Incentive Compatibility). For $v_i \leq t < v \leq \bar{v}_i$ and $b_i(\cdot) \in \arg\max_b U_i(b|\cdot)$, we have:

$$\Delta W_i(b_i(t), b_i(v)) \cdot t \leq b_i(v) - b_i(t) \leq \Delta W_i(b_i(t), b_i(v)) \cdot v.$$ 

Lemma 8 (Monotone). Any pure strategy equilibrium, $b = (b_1, \ldots, b_N)$, is non-decreasing, $v \geq t$ implies $b_i(v) \geq b_i(t)$.

Proof. The proof follows immediately from incentive compatibility (lemma 7) and the fact the winning prob. are strictly increasing (lemma 4). □

Corollary 1. In an equilibrium, for all $i$ exists $\hat{v}_i \geq v_i$ such that $b_i(v) = 0$ for all $v \leq \hat{v}_i$ and $b_i$ is strictly increasing in $(\hat{v}_i, \bar{v}_i)$.

Proof. Since valuations are absolutely continuous valuations, if the bid function were constant in a region, it would posses an atom. Lemma 1 rules out atoms at positive bids but not at zero. □

Hereafter, for brevity, we write ‘equilibrium’ instead of ‘pure strategy, monotone equilibrium’.

4. Sufficient Conditions For Continuity

We need some additional definitions. Given a non-decreasing function $H$, its left and right limits at $v$, which always exist, are denoted by $H^-(v) \overset{\text{def}}{=} \sup\{H(x) : x < v\}$ and $H^+(v) \overset{\text{def}}{=} \inf\{H(x) : x > v\}$. Moreover, a non-decreasing $H$ is said to be discontinuous if and only if $H^-(v) < H^+(v)$ for some $v$. In this instance, we say $v$ is a discontinuity point of $H$ and $(H^-(v), H^+(v))$ is a gap (in the image) of $H$.

As any equilibrium is monotone, any discontinuity induces a gap and vice-versa. When $(c, d)$ is a gap of $b_i$, by taking limits at bid ranking inequality (lemma 6), we see that player $i$ must be indifferent between $c$ and $d$.

Next, we extend, to the all-pay auction, Bernard LeBrun’s\(^1\) results for the first-price auction.

\(^1\)See Lebrun (1997). Beware that many results are not included in the published version, Lebrun (1999).
Lemma 9 (Bidding in i’s Gap, Reveals A Weakly Higher Valuation). Given $c < d$, if $b_i(v_i) = c$, $G_i(c, d) = 0$ and $b_j(v_j) \in (c, d]$. then:

$$v_j \geq v_i.$$

Proof. Type $v_i$ of player $i$ weakly prefers $c$ to any $b$ in $(c, d)$ so:

$$\frac{b - c}{v_i} \geq \Delta W_i(c, b) \quad \forall b \in (c, d). \quad (4.1)$$

Type $v_j$ of player $j$ prefers $b = b_j(v_j) \in (c, d]$ to $c$, so by lemma 6:

$$\Delta W_j(c, b) \geq \frac{b - c}{v_j}. \quad (4.2)$$

Moreover, as $\Delta G_j(c, b) \geq \Delta G_i(c, b) = 0$ by lemma 5:

$$\Delta W_i(c, b) \geq W_j(c, b). \quad (4.3)$$

Combining inequalities 4.1, 4.2 and 4.3 we obtain: $u \geq v$. 

We shall use the following extensions of lemma 9:

Corollary 2. If $\Delta G_j(c, \hat{b}) > 0$, 4.3’s inequality is strict and so $v_j > v_i$.

Corollary 3 (No Gaps at Zero). If $v_i = v$ for all bidders then in any gap $(c, d), c > 0$.

Proof. Bidders with gaps $(0, d_i)$ and discontinuity points $v_i > v$ must bid zero with positive probability. Any bidder $j$ who bid continuously on $(0, \varepsilon)$, where $\varepsilon > \min d_i > 0$, has $v_j > v_i$. That is, any such bidder $j$ also bids zero with positive probability. So, all must bid zero with positive probability contradicting Lemma 3. 

Lemma 10 (Gaps Do Not Overlap). The intersection of the gap $(c, d)$ of bidder $i$ with the gap $(e, f)$ of bidder $j$ is empty, $(c, d) \cap (e, f) = \emptyset$.

Proof. Assume to the contrary that there is an overlap of $j$’s gap $(e, f)$ and $i$’s gap $(c, d)$, that is either $c < e < d < f$ or $e < c < f < d$. Let $v_i$ and $v_j$ be the corresponding discontinuity points of $b_i$ and $b_j$ that are associated to the gaps. Bidding in the gap of another player reveals a weakly higher valuation (lemma 9), using this observation twice yields, $v_i = v_j$.

Without loss of generality, let’s assume $c < e < d < f$ and so:

$$\frac{e - c}{v_i} \geq \Delta W_i(c, e) \geq \Delta W_j(c, e) \geq \frac{e - c}{v_j}. \quad (4.4)$$

The first and the last inequality follows from incentive compatibility (lemma 7), the second by the ranking winning probabilities result (lemma 5).
Equation 4.4 together with \( v_i = v_j \) implies \( \Delta W_i(c, e) = \Delta W_j(c, e) \), which in turn, means that bidder \( j \) like bidder \( i \) does not bid in \((c, e)\). In sum, \( \Delta G_j(c, e) = \Delta G_i(c, e) = 0 \).

However, by definition, \( e = b_j(v_j) \), so there is a sequence \( v^n_j \uparrow v_j \) such that \( b_j(v^n_j) \uparrow e \). Pick \( n_0 \) be such that for \( n < n_0 \), \( b_j(v^n_j) \in (c, e) \). We obtain a contradiction since \( 0 < \Delta F_j(v^n_j, v_j) \geq \Delta G_j(c, e) = 0 \).

\[ \blacksquare \]

**Lemma 11 (Minimal Gap).** In any discontinuous equilibrium there is a bidder \( i \) and an interval \((c, d)\) such that:

1. The interval \((c, d)\) is a gap for bidder \( i \).
2. Any bidder \( j \) either does not bid in \( i \)'s gap, \( \Delta G_j(c, d) = 0 \), or bids continuously on it, \( \Delta G_j(x, y) > 0 \), \( 0 < x < y \leq d \).

**Proof.** Since gaps do not overlap (lemma 10), gaps are either disjoint or ordered by inclusion. Since there is a finite number of bidders, any chain of gaps has a minimal gap which clearly satisfies (1) and (2). \[ \blacksquare \]

**Remark 1.** Although, gaps do not overlap, gaps may coincide, be nested or disjoint.

Our next step is to characterize bidders’ behavior at the gap. To analyze bidders’ marginal incentives, we need the additional notation.

**Definition 5.** The set of active bidders at \( b \) is:
\[
J(b) \overset{\text{def}}{=} \{ j : \forall \varepsilon > 0, \Delta G_j(b - \varepsilon, b + \varepsilon) > 0 \}.
\]

**Definition 6.** The set of bids where bid densities are well defined is:
\[
B \overset{\text{def}}{=} \{ b : \forall i, G^+_i(b) = G^-_i(b) \}.
\]

Note that since \( G_i \) is non-decreasing, the set \( B \) has full-measure.

For a probability cumulative distribution \( H \) with density \( h \), the reversed hazard rate or growth rate of \( H \) is \( h/H \).

**Lemma 12 (Marginal Winning Probabilities).** For \( b \in B \), the reverse hazard rate of the bid distribution of an active bidders is:
\[
g_i(b) = \frac{\left( \sum_{j \in J(b)} \frac{G_i(b)}{F_j^{-1}(G_j(b))} \right) - (\#J(b)-1)G_i(b)}{\left( \#J(b)-1 \right) \Pi_{j=1}^n G_j(b)}, \quad \forall i \in J(b), \quad \text{(rhr)}
\]

and marginal winning probabilities are:
\[
W'_i(b) = \begin{cases} 
\frac{1}{F_j^{-1}(G_j(b))} & \text{if } i \in J(b), \\
\frac{\sum_{j \in J(b)} \frac{G_j(b)}{F_j^{-1}(G_j(b))}}{\left( \#J(b)-1 \right) G_i(b)} & \text{otherwise}.
\end{cases} \quad \text{(MW)}
\]
Proof of \textit{rhr}. Please, see Parreiras and Rubinchik (2010, lemma 7, p. 711).

\textbf{Proof of MW.} Note the set $B$ has full measure since the $G_i$ are monotone. Pick $b \in B$. We have $W'_i(b) = \sum_{j \neq i} g_j(b) \Pi_k G_k(b)$ so
\begin{equation*}
\frac{W'_i(b)}{G_j(b)} = \sum_{j \notin J(b)} \frac{\sum_{k \in J(b)} \frac{G_k(b)}{F_j^{-1}(G_k(b))} - (#J-1) \frac{G_i(b)}{F_j^{-1}(G_i(b))}}{(#J-1) \Pi_k G_k(b)} = \begin{cases}
(#J-1) \sum_{j \in J(b)} \frac{G_j(b)}{F_k^{-1}(G_j(b))} & \text{if } i \in J(b) \\
(#J-1) \sum_{j \notin J(b)} \frac{G_j(b)}{F_k^{-1}(G_j(b))} - \sum_{j \in J(b)} \frac{G_j(b)}{F_j^{-1}(G_j(b))} & \text{if } i \notin J(b)
\end{cases}
\end{equation*}

Finally, simplifying the above expression one gets MW.

\textbf{Remark 2.} If $J(b)$ is constant in a neighborhood of $b$ and $i \in J(b)$ then by \textit{rhr}, $g_i$ is continuous in this neighborhood.

\textbf{Lemma 13} (Local Incentives at The Gap). Let $(c,d)$ be a minimal gap, as defined in Lemma 11, corresponding to a discontinuity of $b_i(\cdot)$ at $v_i$ and let $J$ be the set of bidders that bid continuously in $(c,d)$. We have:
\begin{equation}
\sum_{j \in J} \frac{G_j(c)}{F_j^{-1}(G_j(c))} \leq (#J-1) \frac{F_i(v_i)}{v_i} \leq \sum_{j \in J} \frac{G_j(d)}{F_j^{-1}(G_j(d))}.
\end{equation}

\textbf{Proof.} First, the sets $J(b)$ and $J$ are identical for any $b \in B \cap (c,d)$. Secondly, the type $v_i$ is indifferent between $c$ and $d$ and weakly prefers $c$ or $d$ to any other bid $b$ in $(c,d)$. Third, for any sequences $c_n \searrow c$ and $d_n \nearrow d$ in $B$ there exists $n_0$ such that for $n \geq n_0$:
\begin{equation}
W'_i(c_n) \leq \frac{1}{v_i} \leq W'_i(d_n)
\end{equation}

By \textit{MW} since $i$ does not bid in $(c_n,d_n)$ and $J(c_n) = J(d_n) = J$:
\begin{equation}
\sum_{j \in J} \frac{G_j(c_n)}{F_j^{-1}(G_j(c_n))} \leq (#J-1) \frac{G_i(c_n)}{v_i} \leq \sum_{j \in J} \frac{G_j(d_n)}{F_j^{-1}(G_j(d_n))}
\end{equation}

Since $G_i(c_n) = G_i(d_n) = F_i(v_i)$ as $G_i$ is constant on $(c,d)$ and moreover as $F_j^{-1}$ and $G_j$ are continuous for all $j$, taking the limit $n \nearrow +\infty$ in
Lemma 14. If $c > 0$, the first inequality in Lemma 13 binds,
\[
\sum_{j \in J} \frac{G_j(c)}{F_j^{-1}(G_j(c))} \leq \frac{F_i(v_i)}{v_i} \leq \sum_{j \in J} \frac{G_j(d)}{F_j^{-1}(G_j(d))}.
\]

Proof. As before $J$ is the set of bidders the bid continuously on $(c, d)$. Let $K$ be the set of bidders with $(c, d)$ as a gap. For notational simplicity, write
\[
\alpha_j \overset{\text{def}}{=} \frac{G_j(c)}{F_j^{-1}(G_j(c))} \forall j \in J \quad \text{and} \quad \beta_k \overset{\text{def}}{=} \frac{G_k(c)}{F_k^{-1}(G_k(c))} \forall k \in K.
\]

Pick $\varepsilon > 0$ so that the set of bidders who bid continuously on $(c - \varepsilon, c)$ is the union $J \cup K$. For any sequence $c_n \nearrow c$, $g_k(c_n) \geq 0$ and so by taking limits on rhr we get that for any $k$ in $K$,
\[
\sum_{j=1}^{\#J} \alpha_j + \sum_{k=1}^{\#K} \beta_k \geq (\#J + \#K - 1) \cdot \beta_k,
\]
and by lemma 13,
\[
\sum_{j=1}^{\#J} \alpha_j \leq (\#J - 1) \cdot \beta_k.
\]

Summing 4.10 and 4.11 over $K$ yields
\[
\#K \sum_{i=1}^{\#J} \alpha_j \geq (\#J - 1) \sum_{k=1}^{\#K} \beta_k
\]
and
\[
\#K \sum_{i=1}^{\#J} \alpha_j \leq (\#J - 1) \sum_{k=1}^{\#K} \beta_k.
\]
Both inequalities in both 4.10 and 4.11 are binding, which implies the $\beta_k$ are identical. ■

Corollary 4. If $i$ and $j$ have a common minimal gap $(c, d)$ with $c > 0$, then $v_i = v_j$ and $F_i(v_i) = F_j(v_j)$, where $v_i$ and $v_j$ are their discontinuity points.

Proof. First, by lemma 9, the discontinuity points $v_k = F_k^{-1}(G_k(c))$ are identical. Secondly, by the lemma’s proof, the $\beta_k$ are identical. Finally, remember that $\beta_k = F_k(v_k)/v_k$. ■

For each valuation (random variable) $V_i$ consider an auxiliary random variable, $H_i \sim U[0, v]$. Shaked and Shanthikumar (2007) show that $V_i$ is larger (smaller) than $H_i$ in the reverse hazard rate order, if and only if, the ratio $F_i(v)/F_{H_i}(v) = F_i(v) \cdot v_i/v$ is increasing (decreasing) in $v$. 

Proposition 1 (Continuity Part I). Any equilibrium is continuous on condition that:

1. \( F_i(v)/v \) is non-decreasing \( \forall i \);
2. \( \exists j \) such that \( F_j(v)/v \) is increasing and
3. \( v_i = v \ \forall i \).

Proof. As the sum of the ratios is increasing and \( G_i \) is constant on \((c, d)\), the marginal winning probability \( W_i' \) is increasing on \((c, d)\) by MW. Noted that corollary 3 and the assumption all \( v_i \) are identical implies \( c > 0 \) by . Therefore,

\[
\frac{dW}{v_i} = \Delta W_i(c, d) \geq \int_c^d W_i'(b)db > (d - c) \cdot W_i^+(c).
\]

In 4.12, the equality holds true because \( i \) is indifferent between \( c \) and \( d \), the weak inequality holds because \( W_i \) might not be absolutely continuous, the strict inequality because \( W_i' \) is increasing. From 4.12, we deduced that \( \frac{1}{v_i} > W_i^+(c) \), which contradicts the equality in 4.8. ■

When bidders have distinct valuation lower bounds, \( v_i \neq v_j \):

Corollary 5. Suppose that:

1. \( F_i(v)/v \) is non-increasing \( \forall i \) and
2. \( \exists j \) such that \( F_j(v)/v \) is decreasing, then

the gap of any discontinuity has zero as lower bound.

Proof. Let \((c, d)\) be a minimal gap associate to some discontinuity. By the proof of Proposition 1, \( c = 0 \). Now, consider some gap which is not minimal, \((e, f)\). As gaps are ordered (lemma 10), there is some minimal gap \((0, d) \subseteq (e, f)\), which implies \( e = 0 \). ■

Next we extend Proposition 1 to cover other cases:

Proposition 2 (Continuity Part II). Any equilibrium is continuous provided that:

1. \( F_i(v)/v \) is non-increasing \( \forall i \);
2. \( \exists j \) such that \( F_j(v)/v \) is decreasing.

Proof. In this case, \( \sum_{j \in J} \frac{G_j(c)}{F_j'(G_j(c))} \) is decreasing in \( b \) so 4.8 is violated. ■

Note that \( F_i(v)/v \) non-increasing implies \( v_i = 0 \). An example for Proposition 2 is \( F_i(v) = (v/\bar{\pi}_i)_{\alpha_i} \), where \( 0 < \alpha_i < 1 \).

Proposition 3 (Continuity Part III). Any equilibrium is continuous provided that \( F_i(v)/v \) is constant \( \forall i \).
**Proof.** In this case, valuations are uniformly distributed\(^2\) with \(V_i \sim U[0, \pi_i]\) and so \(F_i(v)/v = 1/\pi_i\).

Consider a minimal gap \((c, d)\) and the respective sets \(J\) and \(K\) of bidders who bid continuously on \((c, d)\) and of bidders who have \((c, d)\) as a gap. As \(F_i(v)/v = 1/\pi_i\), Lemma 13 implies,

\[
\overline{v}_i^{-1} = (\#J - 1) \sum_{j \in J} \overline{v}_j^{-1}, \quad \forall k \in K.
\] (4.13)

Pick \(\delta > 0\) such that for \(0 < \varepsilon < \delta\), the set of active bidders at \(b \in (d - \varepsilon, d)\) is \(J(b) = J \cup K\). But then, the equality 4.13 and rhr imply \(g_k = 0\) on \((d - \varepsilon, d)\) which contradicts \(k \in K\) being active. ■

5. ANY CONTINUOUS EQUILIBRIUM IS DIFFERENTIABLE

**Proposition 4.** If the equilibrium is continuous then it is differentiable everywhere with possibly the exception of \(n + 1\) points: \(0, \overline{b}_1, \ldots, \overline{b}_N\).

Before proceeding with the proof, we need the auxiliary result:

**Lemma 15.** If \(b_i(\cdot)\) is continuous then \(W_i\) differentiable on \((0, \overline{b}_i)\).

**Proof.** Pick \(v_i\) such that \(\overline{v}_i > v_i\), set \(b = b_i(v) > 0\) and consider some sequence \(b^n \uparrow b\). As \(b_i(\cdot)\) is strictly increasing and continuous, there are \(v^n\) with \(b_i(v^n) = b^n\). Incentive compatibility (lemma 7) implies,

\[
\frac{1}{v^n} \leq \frac{\Delta W_i(b^n, b)}{b - b^n} \leq \frac{1}{v}.
\] (5.1)

Taking the limit, we obtain \(W_i^-(b) = \frac{1}{v}\). By considering a sequence \(b^n \uparrow b_i\), we similarly obtain \(W_i^+(b) = \frac{1}{v}\). ■

**Remark 3.** If \(b \not\in [0, \overline{b}_i]\) then the bid distribution \(G_i\) is differentiable at \(b\) since \(G_i\) is constant outside this interval as \(G_i(b) = 0\) for \(b < \overline{b}_i\) and \(G_i(b) = 1\) for \(b > \overline{b}_i\).

Next we prove Proposition 4:

**Proof of Proposition 4.** First, we safely ignore the bids \(b > \max \overline{b}_i\), since no one bids in this range. Second, we linearize the identity \(W_i = \prod_{j \neq i} G_j\) by taking logs, \(\hat{G}_i = \ln(G_i)\) and \(\hat{W}_i = \ln(W_i)\), so that:

\[
M \cdot \left( \begin{array}{c} \hat{G}_1 \\ \vdots \\ \hat{G}_N \end{array} \right) = \left( \begin{array}{c} \hat{W}_1 \\ \vdots \\ \hat{W}_N \end{array} \right)
\]

\(^2\)Parreiras and Rubinchik (2015) obtain closed-form expressions for the equilibrium of the uniform case with \(\overline{v}_i = 0\).
where $M$ is an invertible $N \times N$-matrix with
\[
M_{ij} = \begin{cases} 
0 & \text{if } i = j, \\
1 & \text{otherwise}.
\end{cases}
\]

Pick $b \notin \{0, b_1, \ldots, b_N\}$, re-ordering the bidders if necessary, we can assume there exists $m \geq 0$ such that $b < b_i$ for $i \leq m$ and $b > b_i$ for $i > m$. Decomposing $M$ in blocks, with $M_1$ being $n \times n$,
\[
M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix},
\]
we obtain:
\[
\begin{pmatrix} \hat{G}_1 \\ \vdots \\ \hat{G}_m \end{pmatrix} = M_1^{-1} \cdot \begin{pmatrix} \hat{W}_1 \\ \vdots \\ \hat{W}_m \end{pmatrix} - M_2 \cdot \begin{pmatrix} \hat{G}_{m+1} \\ \vdots \\ \hat{G}_N \end{pmatrix}
\]
(5.2)
\[
\begin{pmatrix} \hat{W}_{m+1} \\ \vdots \\ \hat{W}_N \end{pmatrix} = M_3 \cdot M_1^{-1} \cdot \begin{pmatrix} \hat{W}_1 \\ \vdots \\ \hat{W}_m \end{pmatrix} + (M_4 - M_3 \cdot M_1^{-1} M_2) \cdot \begin{pmatrix} \hat{G}_{m+1} \\ \vdots \\ \hat{G}_N \end{pmatrix}
\]
By lemma 15 $W_i$ is differentiable at $b$ for $i \leq m$. By remark 3, $G_i$ is differentiable at $b$ for $i > m$. In sum, $(W_1, \ldots, W_m, G_{m+1}, \ldots, G_N)$ is differentiable. As a result, 5.2 implies $(G_1, \ldots, G_m, W_{m+1}, \ldots, W_N)$ is also differentiable. \hfill \blacksquare

Next we prove $W_i$ and $G_i$ are differentiable everywhere with the possible exception of $\overline{b}_1$ and zero.

**Lemma 16.** Suppose: 1) $f : [0, 1] \to [0, 1]$ is continuous, differentiable in $[0, 1] \setminus \{a\}$ and non-decreasing; 2) There is a full Lebesgue-measure set $X$ such that for any sequence $x_n \in X$, $x_n \xrightarrow{n \to \infty} a \Rightarrow f'(x_n) \xrightarrow{n \to \infty} c$. Then: $f$ is absolutely continuous on $[0, 1]$, differentiable at $a$ with $f'(a) = c$.

We thank David C. Ullrich for the lemma’s proof.\footnote{See \url{http://math.stackexchange.com/questions/1377034}.}

**Proof.** There exist some measure $\mu$ such that $\mu([0, x]) = f(x)$, which we decompose into absolutely continuous part and singular parts, $\mu = \mu_{ac} + \mu_s$. Since $f$ is continuous at $b$, $0 = \mu(\{a\}) \geq \mu_s(\{a\}) \geq 0$.

We also have that $\mu_s([0, 1] \setminus \{a\}) = 0$. Otherwise $f$ would not be differentiable at some point in $[0, 1] \setminus \{a\}$ because $d\mu_s/dx = +\infty$ almost everywhere wrt. the measure $\mu_s$ (Rudin, 1987, Theorem 7.15, p. 143).
As \( \mu_k \equiv 0 \), \( f \) is absolutely continuous, so:
\[
\frac{f(b) - f(x)}{b - x} = \int_x^b f'(z)dz.
\]
Since for all \( \varepsilon > 0 \), exists \( \delta > 0 \) such that \( |f'(z) - c| < \varepsilon \) for almost all \( x \) with \( |b - z| < \delta \) we have that \( f'(b) = c \). \hfill \blacksquare

**Corollary 6** (Bid Densities Are Continuous At The Top). If i’s top bid is not the highest top bid, \( \bar{b}_i < \max \bar{b}_j \), then \( G_i \) is differentiable and \( g_i(\bar{b}_i) = 0 \).

**Proof.** Proposition 4 implies there is \( \varepsilon > 0 \) such that \( G_i \) is differentiable everywhere in \((\bar{b}_i - \varepsilon, \bar{b}_i)\). Remark 3 implies \( G_i \) is differentiable in \((\bar{b}_i, +\infty)\).

Recall that \( B \) is the (full-measure) set of bids where all bid densities are well defined. We want to show that \( g_i(b_n) \xrightarrow{n\to\infty} 0 \) as \( b_n \xrightarrow{n\to\infty} \bar{b}_i \) for \( b_n \in B \). Once we do that, Lemma 16 applies and the proof is done.

Consider the sets \( K \) and \( J \) of bidders for which \( \bar{b}_i \) is the top bid and bidders who bid in a neighborhood of \( \bar{b}_i \): \( K = \{ k : b_k(\bar{v}_k) = \bar{b}_i \} \) and \( J = \{ j : b_j(\bar{v}_j) < \bar{b}_i \} \). We have that:
\[
\frac{(\#J - 1)}{\bar{v}_k} \geq \sum_{j \in J} \frac{G_j(\bar{b}_i)}{F_j^{-1}(G_j(\bar{b}_i))} \quad \text{and} \quad (5.3)
\]
\[
\sum_{\ell \in J \cup K} \frac{G_\ell(\bar{b}_i)}{F_\ell^{-1}(G_\ell(\bar{b}_i))} \geq \frac{\#J - 1}{\bar{v}_k} \quad \forall k \in K \quad (5.4)
\]
Eq. 5.3 follows from \( \lim_{b_n \to \bar{b}_i} \bar{v}_k \cdot W'_k(b_n) - 1 \leq 0 \), MW and \( G_k(\bar{b}_i) = 1 \). Eq. 5.4 follows from \( \text{rhr} \) and \( \lim_{b_n \to \bar{b}_i} g_k(b_n) \geq 0 \). Adding these two equations and simplifying, we obtain: \( 0 \geq 0 \). Thus, the inequality in 5.4 must bind and so \( \lim_{b_n \to \bar{b}_i} g_k(b_n) = 0 \). Clearly, we also have \( \lim_{b_n \to \bar{b}_i} g_k(b_n) = 0 \) as \( g_k(b_n) = 0 \) for \( b_n > \bar{v}_i \). \hfill \blacksquare

To recover the equilibrium from the first-order conditions, we need it to be absolutely continuous, this is done next.

**Corollary 7.** The winning probability \( W_i \) is differentiable at \( \bar{b}_i \).

**Proof.** The idea and notation is the same as in the proof of the previous corollary. Proposition 4 implies \( W_i'(\bar{b}_i) = \frac{1}{\bar{v}_i} \) and MW implies \( \lim_{b_n \to \bar{b}_i} W'_i(b_n) = \sum_{\ell \in J \cup K} \frac{G_\ell(\bar{b}_i)}{F_\ell^{-1}(G_\ell(\bar{b}_i))} / (\#K + \#J - 1) \). Since 5.4 binds and \( i \in K \), \( \lim_{b_n \to \bar{b}_i} W'_i(b_n) = W_i'(\bar{b}_i) \) and Lemma 16 applies. \hfill \blacksquare

To recover the equilibrium by integrating the first-order conditions, we need it to be absolutely continuous.
Proposition 5. The winning probabilities $W_i$ and the bid distributions $G_i$ are absolutely continuous on $[0, \bar{b}_1]$.

Proof. It follows directly from Lemma 16. ■

6. Determining Active Bidders

The first-order conditions only apply to active bidders. Usually in other auction formats, all bidders are active. In contrast, in the all-pay auction, we do not have prior knowledge of set of active bidders, which is endogenously determined.\footnote{Of course, unless bidders are symmetric or if there are only two bidders.} At first-glance it seems hopeless to apply the first-order approach. As it turns out, however, it suffices to rank the top bids.

Lemma 17 (Top Bids Are Ordered By Top Valuations). A strictly higher top bid imply a strictly higher valuation, $\bar{b}_j < \bar{b}_i \implies v_j < v_i$.

Proof. If $\bar{b}_j < \bar{b}_i$ then $\Delta G_j(\bar{b}_j, \bar{b}_i) = 0 < \Delta G_i(\bar{b}_i, \bar{b}_i)$, which then implies $\Delta W_j(\bar{b}_j, \bar{b}_i) > \Delta W_i(\bar{b}_j, \bar{b}_i)$. Given this, the result follows by incentive compatibility (lemma 7):

$$\Delta W_j(\bar{b}_j, \bar{b}_i) \leq v_{\bar{b}_j} - v_{\bar{b}_i} \leq \Delta W_i(\bar{b}_j, \bar{b}_i) v_i.$$ ■

Hereafter, without any loss of generality, we assume:

Notation. Bidders are ordered by top valuations: $\bar{v}_1 \geq \bar{v}_2 \geq \ldots \geq \bar{v}_n$.

Lemma 17 does not imply that if $\bar{b}_i = \bar{b}_j$ then $\bar{v}_i = \bar{v}_j$, which clearly is false as $\bar{b}_1 = \bar{b}_2$ in any equilibrium even if $\bar{v}_1 > \bar{v}_2$. However we can say that:

Corollary 8 (Ordering Bid’s Supports). If $\bar{b}_i < \bar{b}_1$ and $\bar{b}_j = \bar{b}_i$ then $\bar{v}_i = \bar{v}_j$.

Proof. It follows from $g_j(\bar{b}_j) = g_i(\bar{b}_i) = 0$ (corollary 6) and rhr. ■

Corollary 8 is silent regarding who places the highest top bid $\bar{b}_1$. The next two corollaries remedy this omission.

Corollary 9. If the ratio $F_i(v)/v$ is decreasing for all $i$, the set of bidders that choose $\bar{b}_1$ is

$$J(\bar{b}_1) = \{ j : \sum_{i=1}^{j} \frac{1}{\bar{v}_i} \geq (j-1) \frac{1}{\bar{v}_j} \}.$$
Proof. Assume $\overline{b}_1 = \ldots = \overline{b}_i > \overline{b}_{i+1}$ in equilibrium.

Part I: Since $g_i(\overline{b}_1) \geq 0$,

$$\sum_{j=1}^{i} 1/\overline{v}_j \geq (i-1)/\overline{v}_i$$

and since $\overline{v}_j \geq \overline{v}_i$ for any $1 \leq j \leq i$, it follows that $\sum_{k=1}^{j} 1/\overline{v}_k \geq (j-1)/\overline{v}_j$ for all $1 \leq j \leq i$, which proves that $J(\overline{b}_1) \supset \{j : \sum_{i=1}^{j} 1/\overline{v}_i \geq (j-1)\frac{1}{\overline{v}_j}\}$.

Part II: In equilibrium $g_{i+1}(\overline{b}_{i+1}) = 0$ (corollary 6) and so by rhr,

$$\sum_{j=1}^{i+1} F_j(v_j)/v_j = i/\overline{v}_{i+1}$$

where $v_j = F_j^{-1}(G_j(\overline{b}_{i+1}))$, that is type $v_j$ bids as type $\overline{v}_{i+1}$, $b_j(v_j) = \overline{b}_{i+1}$. As the ratios $F_j(v)/v$ are decreasing,

$$\sum_{j=1}^{i+1} 1/\overline{v}_j < \sum_{j=1}^{i+1} F_j(v_j)/v_j = i/\overline{v}_{i+1}.$$ 

We repeat the argument with players $k \geq i + 1$ and obtain that $\sum_{j=1}^{k} 1/\overline{v}_j < \sum_{j=1}^{k} F_j(v_j)/v_j = (k-1)/\overline{v}_k$, which proves that $J(\overline{b}_1) \subset \{j : \sum_{i=1}^{j} 1/\overline{v}_i \geq (j-1)\frac{1}{\overline{v}_j}\}$.

\[\square\]

**Corollary 10.** If the ratio $F_i(v)/v$ is increasing for all $i$, the set of bidders that bid at the top bid is $J(\overline{b}_1) = \{j : \sum_{i=1}^{j} 1/\overline{v}_i \geq (j-1)\frac{1}{\overline{v}_j}\}$.

Proof. Assume $\overline{b}_1 = \ldots = \overline{b}_i > \overline{b}_{i+1}$ in equilibrium. The argument used in first part of the proof of corollary 9, give us that

$$\sum_{k=1}^{j} 1/\overline{v}_k \geq (j-1)/\overline{v}_j, \quad \forall 1 \leq j \leq i.$$ 

Moreover, by Lemma 17, $g_{i+1}(\overline{b}_{i+1}) = 0$ and so, by MW, $W'_{i+1}(\overline{b}_{i+1}) = \sum_{j=1}^{i} \frac{G_j(\overline{b}_{i+1})}{F_j^{-1}(G_j(\overline{b}_{i+1}))} \frac{1}{\overline{v}_i} = 1/\overline{v}_{i+1}$. As the ratios $F_j(v)/v$ are increasing, so are the ratios $G_j(b)/F_j^{-1}(G_j(b))$ and so is the marginal winning probability, $W'_{i+1}(b)$. We have $\Delta W_{i+1}(\overline{b}_1, \overline{b}_1) = \int_{\overline{b}_1}^{\overline{b}_{i+1}} W'_{i+1}(b) \, db > (\overline{b}_1 - \overline{b}_{i+1}) \cdot W'_{i+1}(\overline{b}_1) = \frac{\overline{b}_1 - \overline{b}_{i+1}}{\overline{v}_{i+1}}$, so $\Delta W_{i+1}(\overline{b}_1, \overline{b}_1) > \frac{\overline{b}_1 - \overline{b}_{i+1}}{\overline{v}_{i+1}} \Leftrightarrow \Delta W_{i+1}(\overline{b}_1, \overline{b}_1) < \frac{\overline{b}_{i+1} - \overline{b}_1}{\overline{v}_{i+1}}$, which contradicts $\overline{b}_{i+1}$ is weakly preferred to $\overline{b}_1$ by type $\overline{b}_{i+1}$ (lemma 6).

\[\square\]

7. **Uniqueness**

First consider the following algorithm.
(1) Order bidders by top valuations, \( \bar{v}_1 \geq \bar{v}_2 \geq \ldots \geq \bar{v}_N \).
(2) Set \( J = \{ j : \sum_{i=1}^{j} \frac{1}{\bar{v}_i} \geq (j - 1) \frac{1}{\bar{v}_j} \} \) and \( k = \# J \).
(3) Set \( b = \bar{v}_1 \) and \( G_i(\bar{v}_1) = 1 \) all \( i \).
(4) Set \( J(b) = J \) as the set of active bidders.
(5) Solve the system of differential equations given by \( \text{rhr} \).
   (a) Use (3) as the set of initial conditions.
   (b) Solve \( \text{rhr} \) locally.
   (c) Start at the current value of \( b \).
   (d) Keep decreasing \( b \) until the (i) or (ii) are met.
      (i) While \( \sum_{i=1}^{k} \frac{G_i(b)}{F_i^{-1}(G_i(b))} = \frac{k-1}{\bar{v}_{k+1}} \) do
          \( J = J \cup \{ k + 1 \} \) and \( k = k + 1 \). Go to (4).
      (ii) If \( G_j(b) = 0 \) then go to (6).
(6) Pick the current value of \( b \) and re-scale bids, \( \bar{b} = \bar{v}_1 - b \).
(7) End.

**Proposition 6.** Assume \( F_i(v)/v \) is increasing for \( i = 1, \ldots, N \) and \( \bar{v}_i \) are the same. There is a unique equilibrium. Moreover, the equilibrium can be obtained by the algorithm.

**Proof.** Step (2) pins-down the set of bidders that bid at \( \bar{b}_1 \), see cor. 10. Step (5di) uniquely identifies bidders that become active: Since \( \sum_{i=1}^{j} G_i(b)/F_i^{-1}(G_i(b)) - (k - 1)/\bar{v}_{k+1} \) is strictly increasing in \( b \) it must cross zero only once, exactly at where bidder \( k + 1 \) becomes active (note that \( \sum_{i=1}^{k} G_i(b)/F_i^{-1}(G_i(b)) = (k - 1)/\bar{v}_{k+1} \) is equivalent to \( g_{k+1}(b) = 0 \) by corollary 6).

Since the set of active bidders is uniquely determined, any equilibrium must satisfy the first-order conditions or equivalently, \( \text{rhr} \). For any \( \varepsilon > 0 \), the right hand side of \( \text{rhr} \) is Lipschitz continuous in the \( G_i \)'s provided \( G_i > \varepsilon \). The solution of \( \text{rhr} \) can be extended uniquely in the region where \( G_i > \varepsilon \). Now we take the limit \( \varepsilon \downarrow 0 \) and extend \( G_i \) by the continuity. The solution is uniquely determined. \( \blacksquare \)

**Proposition 7.** Assume \( F_i(v)/v \) is constant for \( i = 1, \ldots, N \) There is a unique equilibrium. Moreover, the equilibrium can be obtained by the algorithm.

**Proof.** The proof is similar to the above and so omitted. The main difference is the set of active bidders in this case is constant for all \( b \). A bidder either is always active or always inactive: step (5di) of the algorithm is never reached. See Parreiras and Rubinchik (2015) for an explicit derivation of equilibrium strategies. \( \blacksquare \)
8. CONCLUSIONS

We provided sufficient conditions for the equilibrium of the all-pay auction be continuous and differentiable, which allow us to use the first-order approach.

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