Section 2.4: Definition of Function

Objectives
Upon completion of this lesson, you will be able to …

- evaluate a given function at a given value or expression;
- given a function or the graph of a function \( f \), find
  - the domain and range
  - where \( f(x) = 0 \), \( f(x) > 0 \), \( f(x) < 0 \)
  - the intervals where \( f \) is increasing, decreasing, or constant;
- sketch the graph of a function;
- given a linear or quadratic function, find and simplify a difference quotient;
- write a linear function given specific characteristics;
- construct a model in terms of one variable and use the model to answer questions.

Required Reading, Tutorial, and Video
- Read Swokowski: Section 2.4: Definition of Function
  - Disregard Examples 8 and 9
- Watch the Section 2.4 Video
- Complete the Section 2.4 WA Before Class assignment

Discussion
This lesson introduces a special class of equations known as functions. A function is a relation such that each element of a given input set (the domain) is related to exactly one element of an output set (the range).

Example: If \( g(x) = 4 - x^2 \), find (a) \( g(-1) \) and (b) \( g(1-a) \).

Solution: (a) Begin by replacing \( x \) with \(-1\) into the function \( g \).

\[
g(-1) = 4 - (-1)^2 \quad \text{substitute}
\]
\[
= 4 - 1 \quad \text{simplify}
\]
\[
= 3
\]
Since \( g(-1) = 3 \), this means that the point \((-1, 3)\) on the graph of \( g \).

(b) Begin by replacing \( x \) with \(1-a\) into the function \( g \).

\[
g(1-a) = 4 - (1-a)^2 \quad \text{substitute}
\]
\[
= 4 - (1 - 2a + a^2) \quad \text{expand}
\]
\[
= 4 - 1 + 2a - a^2 \quad \text{distribute}
\]
\[
= 3 + 2a - a^2 \quad \text{simplify}
\]

3.1 \( \in \mathbb{R} \) means that 3.1 is an element of the set of real numbers.
The graph of a set of points on a coordinate plane is the graph of a function if every vertical line intersects the graph in at most one point. This is known as the **Vertical Line Test**.

Please familiarize yourself with the notation in the following chart.

<table>
<thead>
<tr>
<th>Terminology</th>
<th>Definition</th>
<th>Graphical Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>f</em> is increasing</td>
<td><em>f</em>(<em>a</em>) &lt; <em>f</em>(<em>b</em>) where <em>a</em> &lt; <em>b</em></td>
<td>![Graph of increasing function]</td>
</tr>
<tr>
<td>on an interval <em>I</em></td>
<td></td>
<td></td>
</tr>
<tr>
<td><em>f</em> is decreasing</td>
<td><em>f</em>(<em>a</em>) &gt; <em>f</em>(<em>b</em>) where <em>a</em> &lt; <em>b</em></td>
<td>![Graph of decreasing function]</td>
</tr>
<tr>
<td>on an interval <em>I</em></td>
<td></td>
<td></td>
</tr>
<tr>
<td><em>f</em> is constant</td>
<td><em>f</em>(<em>a</em>) = <em>f</em>(<em>b</em>) for every <em>a</em> and <em>b</em></td>
<td>![Graph of constant function]</td>
</tr>
<tr>
<td>on an interval <em>I</em></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

When considering whether the graph of a function is increasing, decreasing, or constant, think of moving from left to right along the horizontal axis and notice what is happening to the curve. Is it rising (increasing), falling (decreasing), or remaining constant?

**Example:** Find the domain of the function \( f(x) = \frac{\sqrt{x^2 - 9}}{x^2 - 4x} \).

**Solution:** Recall that the domain of a function is the set of \( x \) – values for which the function is defined. For the given function, the expression under the radical must be non-negative, \( x^2 - 9 \geq 0 \), and the denominator cannot equal 0, \( x^2 - 4x \neq 0 \).

To find where \( x^2 - 9 = (x-3)(x+3) \geq 0 \), set up a sign diagram with \( x = \pm 3 \) as the critical points.

Check values in the intervals and we see that \( x^2 - 9 \geq 0 \) for the intervals \( (-\infty, -3] \cup [3, \infty) \).
Next, consider where the denominator equals 0. \( x^2 - 4x = x(x - 4) = 0 \). We must also restrict \( x = 0 \) and \( x = 4 \) from the domain. Since \( x = 0 \) is already not included in the interval \((-\infty, -3] \cup [3, \infty)\), we just need to restrict \( x = 4 \), so our domain is \((-\infty, -3] \cup [3, 4) \cup (4, \infty)\).

**The Difference Quotient**

A *secant line* is a line connecting two points on a curve. The *difference quotient* is the slope of the secant line.

Refer to Figure 9 on the bottom of page 126 and the following figure.

For the function \( f \) in the figure above, the slope of the secant line is

\[
m_s = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x + h) - f(x)}{(x + h) - x} = \frac{f(x + h) - f(x)}{h}, \quad h \neq 0.
\]

The text uses multiple versions of the difference quotient. They include:

\[
\frac{f(a + h) - f(a)}{h}, \quad h \neq 0; \quad \frac{f(x) - f(a)}{x - a}, \quad x \neq a; \quad \text{and} \quad \frac{f(x + h) - f(x)}{h}, \quad h \neq 0.
\]

Example 5 on page 127 of the text is an excellent example of a difference quotient with a polynomial function.

Below are two more examples of how to find a difference quotient (DQ).

**Example**: Find and simplify the difference quotient \( \frac{f(5 + h) - f(5)}{h} \) for \( f(x) = 3 - 2x \).

**Solution**: Begin by finding \( f(5) = 3 - 2(5) = -7 \) and \( f(5 + h) = 3 - 2(5 + h) = -7 - 2h \).
\[
\frac{f(5-h) - f(5)}{h} = \left[ \frac{-7 - 2h}{h} \right] - \left[ \frac{-7}{h} \right] \\
= \frac{-7 - 2h + 7}{h} \\
= \frac{-2h}{h} \\
= -2
\]

**Example:** Find and simplify the difference quotient \( \frac{f(x+h) - f(x)}{h} \) for \( f(x) = -x^2 + 3x \).

**Solution:** Begin by replacing \( x \) with \((x+h)\) for the given function \( f(x) \).

\[
f(x+h) = -(x+h)^2 + 3(x+h)
\]

\[
\frac{f(x+h) - f(x)}{h} = \left[ \frac{-(x+h)^2 + 3(x+h)}{h} \right] - \left[ \frac{-x^2 + 3x}{h} \right]
\]

substitute expressions into the DQ

\[
= \frac{-x^2 - 2xh - h^2 + 3x + 3h + x^2 - 3x}{h}
\]

expand and distribute

\[
= \frac{-2xh - h^2 + 3h}{h}
\]

combine like terms

\[
= \frac{h(-2x - h + 3)}{h}
\]

factor out common terms

\[
= -2x - h + 3
\]

simplify

**Practice Problems**

The problems listed below are in your textbook and on the WA Section 2.4 HW assignment. Answers to the odd textbook questions are in the back of your textbook.

<table>
<thead>
<tr>
<th>Section</th>
<th>~Time</th>
<th>Practice Problems</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.4</td>
<td>95 min</td>
<td>1, 3, 4, 5, 9, 11, 19, 21, 24, 25, 27, 33, 40, 47, 53, 65, 71</td>
</tr>
</tbody>
</table>