Section 2.4: Definition of Function – Part I

Objectives: Upon completion of this lesson, you will be able to:
- given a function, find and simplify a difference quotient for
  - polynomial functions of degree 3 or less
  - rational functions
  - square root functions
- sketch the graph of a polynomial, square root, or absolute value function and find
  - the domain and range
  - intervals on which the function is increasing, decreasing, or constant.

Required Reading, Video, Tutorial
- Read Swokowski/Cole: Section 2.4
  - Read and study all examples through Example 5 as a review, but focus on Examples 3, 4, and 5
- Please review the details of Interval Notation from the handout on WebAssign
- Locate the ALGEBRA information page at the beginning of your textbook.
- Watch the Section 2.4 Part I Video
- Complete the Section 2.4 Part I Before Class WA assignment

Algebra/Geometry Review
- Evaluate a function
  - Given \( g(x) = -x^2 + \sqrt{x+2} \), find the exact value of \( g(5) \).
  - Given \( g(x) = -x^2 + \sqrt{x+2} \), find \( g(-h) \).
- Special factoring formulas
  - True/False. \( (x+5)^2 = x^2 + 25 \) If false, give correct expansion for \( (x+5)^2 \).
  - Factor \( x^3 - 8 \).
- Least Common Denominator (LCD)
  - Find the LCD for \( \frac{1}{a} + \frac{1}{a+h} + \frac{1}{2a^2} \).
- The Conjugate
  - Find the conjugate for \( 3 - \sqrt{x+h} \).

Discussion
Section 2.4 is a review of functions. In Part I of this section, we will concentrate on the Difference Quotient and identifying the domain and range of a function and where the graph of a function is increasing or decreasing.

Definition: A function is a relation such that each element of a given input set (the domain) is related to exactly one element of an output set (the range).

Example 1: If \( g(x) = 4 - x^2 \), find \( g(1-a) \).

Solution: Begin by replacing \( x \) with \( 1-a \) into the function \( g \).
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\[ g(1 - a) = 4 - (1 - a)^2 \]

substitute

\[ = 4 - (1 - 2a + a^2) \]

expand

\[ = 4 - 1 + 2a - a^2 \]

distribute

\[ = 3 + 2a - a^2 \]

simplify

\[ 3.1 \in \mathbb{R} \] means that 3.1 is an element of the set of real numbers.

The graph of a set of points on a coordinate plane is the graph of a function if every vertical line intersects the graph in at most one point. This is known as the Vertical Line Test.

A **one-to-one function** is a function such that each element of the range is related to exactly one element of the domain. Graphs of one-to-one functions pass both the vertical line test and the horizontal line test. A function must be 1-1 on a given interval in order for its inverse to exist.

Please familiarize yourself with the notation in the following chart. It will be useful again in Section 5.3 when the graphs of the trigonometric functions are introduced.

<table>
<thead>
<tr>
<th>Terminology</th>
<th>Definition</th>
<th>Graphical Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Increasing</strong></td>
<td>( f(a) &lt; f(b) ) where ( a &lt; b )</td>
<td><img src="image1.png" alt="Graph of Increasing Function" /></td>
</tr>
<tr>
<td>on an interval ( I )</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Decreasing</strong></td>
<td>( f(a) &gt; f(b) ) where ( a &lt; b )</td>
<td><img src="image2.png" alt="Graph of Decreasing Function" /></td>
</tr>
<tr>
<td>on an interval ( I )</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Constant</strong></td>
<td>( f(a) = f(b) ) for every ( a ) and ( b )</td>
<td><img src="image3.png" alt="Graph of Constant Function" /></td>
</tr>
<tr>
<td>on an interval ( I )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
When considering whether the graph of a function is increasing, decreasing, or constant, think of moving from left to right along the horizontal axis and notice what is happening to the curve. Is it rising (increasing), falling (decreasing), or remaining constant?

For the rest of this section, we will focus on the difference quotient and writing functional models in terms of a single variable.

**The Difference Quotient**

A **secant line** is a line connecting two points on a curve. The **difference quotient** is the slope of the secant line.

Refer to Figure 9 on the bottom of page 126 and the following figure.

For the function \( f \) in the figure above, the slope of the secant line is

\[
m_s = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}, \text{ where } h \neq 0.
\]

The text uses multiple versions of the difference quotient. They include:

\[
\frac{f(a+h) - f(a)}{h}, \ h \neq 0; \quad \frac{f(x) - f(a)}{x-a}, \ x \neq a; \text{ and } \frac{f(x+h) - f(x)}{h}, \ h \neq 0.
\]

Example 5 in the text is an excellent example of a difference quotient with a polynomial function.

The next three examples will demonstrate techniques to use when simplifying a difference quotient (DQ) for polynomial, rational, and square root functions.

**Example 2**: Find and simplify the difference quotient \( \frac{f(x+h) - f(x)}{h} \) for \( f(x) = -x^2 + 3x \).

**Solution**: Begin by replacing \( x \) with \( (x+h) \) for the given function \( f(x) \).

\[
f(x+h) = -(x+h)^2 + 3(x+h)
\]
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\[
\frac{f(x+h) - f(x)}{h} = \frac{\left[-(x+h)^2 + 3(x+h)\right] - \left[-x^2 + 3x\right]}{h}
\]

substitute expressions into the DQ

\[
= \frac{-x^2 - 2xh - h^2 + 3x + 3h + x^2 - 3x}{h}
\]

expand and distribute

\[
= \frac{-2xh - h^2 + 3h}{h}
\]

combine like terms

\[
= \frac{h(-2x - h + 3)}{h}
\]

factor out common terms

\[
= -2x - h + 3
\]

simplify


Example 3: Find and simplify the difference quotient \( \frac{f(x+h) - f(x)}{h} \) for \( f(x) = \frac{1}{x+3} \).

Solution: Begin by replacing \( x \) with \( (x+h) \) for the given function \( f(x) \).

\[
f(x+h) = \frac{1}{(x+h)+3}
\]

\[
\frac{f(x+h) - f(x)}{h} = \frac{1}{x+h+3} - \frac{1}{x+3}
\]

substitute expressions into the DQ

\[
= \left(\frac{1}{x+h+3} - \frac{1}{x+3}\right) \frac{(x+h+3)(x+3)}{h}
\]

multiply by the common denominator

\[
= \frac{(x+3) - (x+h+3)}{h(x+h+3)(x+3)}
\]

distribute and combine like terms

\[
= \frac{-h}{h(x+h+3)(x+3)}
\]

simplify

\[
= \frac{-1}{(x+h+3)(x+3)}
\]

Leave answers in factored form.

Example 4: Find and simplify the difference quotient \( \frac{f(x+h) - f(x)}{h} \) for \( f(x) = \sqrt{2x} \).

Solution: Begin by replacing \( x \) with \( (x+h) \) for the given function \( f(x) \).

\[
f(x+h) = \sqrt{2(x+h)}
\]
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\[
\frac{f(x + h) - f(x)}{h} = \sqrt{\frac{2(x + h) - 2x}{h}}
\]

substitute expressions into the DQ

\[
= \left( \frac{\sqrt{2(x + h) - 2x}}{h} \right) \left( \frac{\sqrt{2(x + h) + 2x}}{\sqrt{2(x + h) + 2x}} \right)
\]
multiply by the conjugate of the numerator

\[
= \frac{2(x + h) - 2x}{h(\sqrt{2(x + h) + 2x})}
\]
distribute and combine like terms

\[
= \frac{2h}{h(\sqrt{2(x + h) + 2x})}
\]
simplify

\[
= \frac{2}{\sqrt{2(x + h) + 2x}}
\]

Practice Problems
Answers to odd-numbered problems can be found at the end of your text.

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