Section 2.5: *Graphs of Functions*

**Objectives** Upon completion of this lesson, you will be able to:

- sketch the graph of a piecewise function containing any of the Library Functions
  - polynomial functions of degree 3 or less
  - absolute value functions
  - rational functions
  - square root functions
  - exponential functions
  - logarithmic functions
- answer questions about functions of the form
  - as $x \to a$, $f(x) \to \ldots$, as $x \to a^+$, $f(x) \to \ldots$, or as $x \to a^-$, $f(x) \to \ldots$
  - as $x \to -\infty$, $f(x) \to \ldots$, or as $x \to \infty$, $f(x) \to \ldots$
  - $f(a) = \ldots$
- given a function or its graph, sketch the absolute value of the function
- write a mathematical model of a piecewise application.

**Required Reading, Video, Tutorial**

- Read Swokowski/Cole: Section 2.5
  - Focus on Examples 2, 8, 10, and 12
  - Examples 3-7 and Appendix II on pages 792-793 cover transformations; such as shifting vertically and horizontally, stretching and compressing, and reflecting. It is expected that you are proficient in the topic of transformations from previous courses.
- Locate Appendix I and II at the back of your text.
- Watch the Section 2.5 Video
- Complete the Section 2.5 Before Class WA assignment

**Algebra/Geometry Review**

- Increasing/Decreasing: Find the interval on which $f(x) = |x|$ is decreasing.
- What is the difference between the graphs of $f(x) = \ln(x-1)$ and $g(x) = \ln(x-1)$?
- What is the difference between the graphs of $f(x) = e^x$, $g(x) = -e^x$, and $h(x) = e^{-x}$?

**Discussion**

Please review the common graphs and their functions in Appendix I of your text. You are responsible for knowing the graphs of all Library Functions listed in the Objectives above. Our primary focus in Section 2.5 will be on piecewise functions with elements from the Library of Functions including simple transformations, the concept of limit, and absolute value functions.

Sections 2.4 and 2.5 use many basic functions in the examples. Below is a summary of the basic functions you will want to familiarize yourself with.

**Library of Functions**

**Polynomial Functions** are of the form: $f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$. 
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The domain of all polynomial functions is the set of all real numbers $\mathbb{R}$.

We will limit our discussion of polynomials to linear, quadratic, and cubic functions.

- **Linear functions** are of the form: $f(x) = a_{i}x + a_{0}$ or $y = mx + b$.

  Special cases of linear functions include:

  o **Identity function**: $y = x$
    
    This function has a slope of 1 and a $y$-intercept of 0.
    The identity function is an odd function.
    It is symmetric with respect to the origin.

  o **Constant function**: $y = b$
    
    This function has a slope of 0 and a $y$-intercept of $b$.
    It is a horizontal line through $b$.
    Any constant function is an even function.
    It is symmetric with respect to the $y$-axis.

- **Quadratic functions** are of the form: $f(x) = a_{2}x^{2} + a_{1}x + a_{0}$ or $y = ax^{2} + bx + c$.

  The graph of a quadratic function is a parabola with a vertex at
  $$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$.
  
  The axis of symmetry of the parabola is the vertical line $x = -\frac{b}{2a}$.
  The parabola opens up if $a > 0$ and down if $a < 0$.

  The simplest quadratic function is the square function $y = x^{2}$.
  The graph of the square function has its vertex at the origin.
  It is an even function with a domain of $\mathbb{R}$ and a range of $[0, \infty)$.

- **Cubic functions** are of the form: $f(x) = a_{3}x^{3} + a_{2}x^{2} + a_{1}x + a_{0}$ or $y = ax^{3} + bx^{2} + cx + d$.

  The simplest cubic function is the equation $y = x^{3}$.
  The graph of this simple cubic function is odd.
  The domain and range are $\mathbb{R}$.
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**Square root function** - The simplest square root function is $y = \sqrt{x}$.

The domain is $[0, \infty)$, and the range is $[0, \infty)$.

**Rational function** - The simplest rational function is the reciprocal function $y = \frac{1}{x}$.

This is an odd function with a vertical asymptote along the $y$–axis. The domain is $(-\infty, 0) \cup (0, \infty)$ and the range is $(-\infty, 0) \cup (0, \infty)$.

**Absolute value function** - The simplest absolute value function is $y = |x|$.

By definition, $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$

Therefore, the graph is $y = x$ if $x \geq 0$ and $y = -x$ if $x < 0.$

This simple absolute value function is an even function with a domain of $\mathbb{R}$ and a range of $[0, \infty)$.

**Exponential function**: $y = e^x$

- $y$-intercept: $(0, 1)$ since $e^0 = 1$
- $x$-intercept: none since $y = e^x$ never equals 0
- Horizontal asymptote: $y = 0$
- Domain: $\mathbb{R}$
- Range: $(0, \infty)$
Natural logarithmic function: \( y = \ln x \)

- **x-intercept:** \( (1, 0) \) since \( \ln 1 = 0 \)
- **y-intercept:** none since \( \ln 0 \) is undefined
- **vertical asymptote:** \( x = 0 \)
- **domain:** \( (0, \infty) \)
- **range:** \( \mathbb{R} \)

Notice that \( y = e^x \) and \( y = \ln x \) are inverses of each other. The graphs are symmetric with respect to the line \( y = x \).

**Greatest-integer function** - Even though there are many applications of the greatest-integer function, you are not responsible for it.

**Piecewise function** – These functions will consist of pieces of the Library Functions mentioned above, or slight variations to their simplest forms, defined for a limited domain.

Following is an example of a piecewise function. This particular example is composed of three of the basic functions mentioned above.

**Example 1:** Sketch the graph of the given piecewise function.

\[
f(x) = \begin{cases} 
\sqrt{-x} & \text{if } x \leq 0 \\
-5 & \text{if } 0 < x < 2 \\
-3x & \text{if } x \geq 2 
\end{cases}
\]

**Solution:**

- \( f(x) = \sqrt{-x} \) is the square root function defined for the interval \( (-\infty, 0] \).
- \( f(x) = -5 \) is a constant function defined for the interval \( (0, 2) \).
- \( f(x) = -3x \) is a linear function defined for the interval \( [2, \infty) \).

**Note:** When you finish sketching the graph of a piecewise function, check to see that it passes the vertical line test.
From the sketch and the given function, we can see the following.

- \( f(-3) = \sqrt{3} \)
- \( f\left(\frac{3}{2}\right) = -5 \)
- \( f(3) = -9 \)
- \( f(0) = 0 \)
- As \( x \to 0^- \), \( f(x) \to 0 \). This notation means that as \( x \) approaches 0 from the left, \( f(x) \) is approaching 0.
- As \( x \to 0^+ \), \( f(x) \to -5 \). This notation means that as \( x \) approaches 0 from the right, \( f(x) \) is approaching -5.
- \( f(2) = -6 \)
- As \( x \to 2^- \), \( f(x) \to -5 \).
- As \( x \to 2^+ \), \( f(x) \to -6 \).
- As \( x \to -\infty \), \( f(x) \to \infty \).
- As \( x \to \infty \), \( f(x) \to -\infty \).

The concept of limit is extremely important to the study of calculus. Be sure that you understand the “As \( x \to ... \)” items above.

The following is an example of an absolute value function.

**Example 2** Sketch the graph of the given absolute value function.

\[ f(x) = |x + 2| - 3 \]

**Solution:** Think of the function \( f(x) \) as \( f(x) = |g(x)| \) where \( g(x) = |x + 2| - 3 \). The function \( g(x) \) is an absolute value function shifted left 2 units and down 3 units as shown to the right.

The graph of \( f(x) \) is just a modification to the graph of \( g(x) \) in which the negative \( y \) values are reflected to their positive value across the \( x \)-axis. See the sketch to the right.

**Practice Problems**

Be sure to work the Not on WA problem in the list below, but do not submit it for grading. It could appear on tests and the final exam. Answers to odd-numbered problems can be found at the end of your text.

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