Section 3.5: Rational Functions

Objectives: Upon completion of this lesson, you will be able to:

- given a rational function or its graph,
  - determine the domain and range
  - find the $x$– and $y$–intercepts
  - determine the horizontal and vertical asymptotes
  - find holes
  - find where $f(x) > 0$ or $f(x) < 0$ using a sign chart
  - determine end behavior
  - find where the function is increasing or decreasing

- graph of a rational function
- write the equation of rational functions that satisfy specific conditions.

Required Reading, Video, Tutorial

- Read Swokowski/Cole: Section 3.5
  - Disregard Examples 9 and 10
- Watch the Section 3.5 Video
- Complete the Section 3.5 Before Class WA assignment

Discussion

Definition: The line $x = a$ is a **vertical asymptote** for the graph of a function if $f(x) \to \pm\infty$ as $x \to a^{\pm}$.

Notice Figure 4 on page 222 that shows the four cases for vertical asymptotes and the appropriate notation.

Definition: The line $y = c$ is a **horizontal asymptote** for the graph of a function if $f(x) \to c$ as $x \to \pm\infty$.

Notice Figure 5 on page 222 that provides examples of horizontal asymptotes and the appropriate notation.

Generalizations for Horizontal Asymptotes

See the Theorem on Horizontal Asymptotes on page 224 as well.

- When the greatest powered term in the numerator exceeds that of the denominator, there is no horizontal asymptote. The function tends toward either positive or negative infinity as $x \to \pm\infty$.

  Example: $f(x) = \frac{x^3}{x + 5}$

- When the greatest powered term in the denominator exceeds that of the numerator, the horizontal asymptote is $y = 0$. As $x \to \pm\infty$, the denominator grows faster than the numerator, so the fraction tends toward zero.

  Example: $f(x) = \frac{x - 1}{x^3 + x}$

- When the greatest powered term of the numerator is the same as the greatest powered term of the denominator, the horizontal asymptote is the ratio of the coefficients on the greatest powered terms.
Example:  
\[ f(x) = \frac{5x^3 - 1}{x - 2x^3} \]  
The horizontal asymptote in this case is  \[ y = -\frac{5}{2} \].

Example 2 on page 224 illustrates each of the generalizations mentioned above.

**Holes in Rational Functions** - See the discussion on page 221 and Example 4 on page 227 of the text.

For a rational function  \( f(x) \) with a common factor of  \( x - a \) in the numerator and denominator and a reduced version of the function called  \( F(x) \), there is a hole in the function at the point  \((a, F(a))\).

**Example 1**: Find the coordinates of the hole in the function  \( f(x) = \frac{(x + 2)(x + 5)}{(x + 2)(x + 1)} \).

**Solution**:  
\[ f(x) = \frac{(x + 2)(x + 5)}{(x + 2)(x + 1)} = \frac{x + 5}{x + 1}, \]  
where \( x \neq -2 \), so  \( F(x) = \frac{x + 5}{x + 1} \).

Note:  
\[ f(x) = \frac{(x + 2)(x + 5)}{(x + 2)(x + 1)} \neq \frac{x + 5}{x + 1} \]  
since the graph of one has a hole and the other does not. The restriction \( x \neq -2 \) MUST be given for equality.

The function  \( f(x) \) has a hole at  \( x = -2 \) because  \((x + 2)\) is a common factor of both the numerator and the denominator. Using the simplified version of the function  \( F(x) = \frac{x + 5}{x + 1} \), where  \( x \neq -2 \), we see that the hole is at  \((-2, F(-2)) = \left(-2, \frac{-2 + 5}{-2 + 1}\right) \) or  \((-2, -3)\).

This section of the text is very well written with excellent examples. Following is one example of identifying the important attributes of a rational function and its graph and one example of writing a rational function given specific attributes.

**Example 2**: Sketch the graph of  \( f(x) = \frac{x + 3}{x(x - 2)} \).

**Solution**: Notice that this function does not have a hole since there are no common factors in the numerator and denominator.

-  \( x \)-intercept:  \( f(x) = 0 \) when  \( x + 3 = 0 \). Thus,  \( x = -3 \) and  \((-3, 0)\) is the  \( x \)-intercept.
  -  Note: Recall that if  \( \frac{a}{b} = 0 \), then  \( a = 0 \).
-  vertical asymptote:  \( f(x) \) is undefined when  \( x = 0 \) or when  \( x - 2 = 0 \). Thus, vertical asymptotes are located at  \( x = 0 \) or  \( x = 2 \).
-  \( y \)-intercept:  \( f(x) \) does not have a  \( y \)-intercept since the  \( y \)-axis is a vertical asymptote.
-  horizontal asymptote: By the theorem on page 224, since the greatest power on  \( x \) is in the denominator, the horizontal asymptote is  \( y = 0 \).
By constructing a sign diagram with the values from the $x$– intercept and the vertical asymptotes, we can see where the function is above and below the $x$– axis by checking values in the intervals. Select test numbers in each interval of the number line. We will use $-4, -1, 1$, and $3$.

Plug these values into the expression $\frac{x+3}{x(x-2)}$, and indicate a + or - on the sign diagram for the sign of the expression in each interval.

\[
\begin{array}{c|c|c|c|c}
x & -3 & 0 & 2 & +
\end{array}
\]

\[
f(x) = \frac{x+3}{x(x-2)} > 0 \text{ or above the } x-\text{axis on the intervals } (-3, 0) \cup (2, \infty).
\]

\[
f(x) = \frac{x+3}{x(x-2)} < 0 \text{ or above the } x-\text{axis from } (-\infty, -3) \cup (0, 2).
\]

From the sketch of the given function, we can see the following.

- As $x \to 0^-$, $f(x) \to +\infty$.
- As $x \to 0^+$, $f(x) \to -\infty$.
- As $x \to 2^-$, $f(x) \to -\infty$.
- As $x \to 2^+$, $f(x) \to +\infty$.
- As $x \to +\infty$, $f(x) \to 0$.
- As $x \to -\infty$, $f(x) \to 0$.

Remember that the graph of a rational function may never cross a vertical asymptote, but the graph may cross a horizontal asymptote as in the example above. To determine if the graph of a rational function crosses the horizontal asymptote, set the function equal to the value of the horizontal asymptote. In this case set $\frac{x+3}{x(x-2)} = 0$. We see that the function crosses the horizontal axis at $x = -3$, the $x$– intercept and approaches the horizontal asymptote $y = 0$ from the underside.

The author provides **Guidelines for Sketching the Graph of a Rational Function** on page 225. In addition to this set of guidelines, you should also construct a sign diagram as was done in the example above.

**Example 3**: Write an equation of a rational function that satisfies the following conditions.

- vertical asymptotes: $x = -8$ and $x = 5$
- horizontal asymptote: $y = 0$
- $x$– intercept: $(-2, 0)$
- $f(0) = -2$
- hole at $x = 3$
Solution: With vertical asymptotes at $x = -8$ and $x = 5$, we have factors of $(x + 8)$ and $(x - 5)$ in the denominator.

With an $x-$intercept of $(-2, 0)$, we have a factor of $(x + 2)$ in the numerator.

With a hole at $x = 3$, we have a factor of $(x - 3)$ in both the numerator and the denominator.

So far, $f(x) = \frac{k(x + 2)(x - 3)}{(x + 8)(x - 5)(x - 3)}$, where $k$ is a constant.

If the factors above are multiplied out, the denominator has the term with the greatest power on $x$. Thus, the horizontal asymptote is $y = 0$.

Lastly, if $f(0) = -2$, then $-2 = \frac{k(2)}{(8)(-5)}$, so $k = 40$.

Hence, $f(x) = \frac{40(x + 2)(x - 3)}{(x + 8)(x - 5)(x - 3)}$.

Practice Problems
Be sure to work the Not on WA problems in the list below, but do not submit them for grading. They could appear on tests and the final exam. Answers to odd-numbered problems can be found at the end of your text.

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