Section 6.3: *The Addition and Subtraction Formulas*

**Objectives:** Upon completion of this lesson, you will be able to:
- rewrite trigonometric expressions as a cofunction of a complementary angle
- use an addition or subtraction formula to evaluate or simplify a trigonometric expression
- use an addition or subtraction formula to verify an identity or rewrite an expression.

**Required Reading, Video, Tutorial**
- Read Swokowski/Cole: Section 6.3
  - Disregard Examples 7 and 8
- Watch the Section 6.3 Video
- Complete the Section 6.3 Before Class assignment

**Discussion**
Recall from geometry that supplementary angles sum to \(180^\circ\) and complementary angles sum to \(90^\circ\). Note that \(\alpha\) and \(\beta\), the two acute angles in the right triangle to the right, sum to \(90^\circ\) and are therefore complementary angles. So \(\alpha + \beta = 90^\circ\) and then \(\beta = 90^\circ - \alpha\).

Notice that \(\sin \alpha = \frac{a}{c} = \cos \beta\).

We conclude that \(\sin \alpha = \cos \beta\). Do we agree that \(30^\circ\) and \(60^\circ\) are complementary angles and that \(\sin 30^\circ = \cos 60^\circ\)?

Since \(\beta = 90^\circ - \alpha\), then \(\sin \alpha = \cos(90^\circ - \alpha)\). From this result, we have that the function value of an angle equals the cofunction of the complementary angle.

Sine and cosine are cofunctions.
Secant and cosecant are cofunctions.
Tangent and cotangent are cofunctions.

**Cofunction Formulas** where \(\theta\) is measured in radians

\[
\begin{align*}
\sin \left( \frac{\pi}{2} - \theta \right) &= \cos \theta \\
\tan \left( \frac{\pi}{2} - \theta \right) &= \cot \theta \\
\sec \left( \frac{\pi}{2} - \theta \right) &= \csc \theta \\
\cos \left( \frac{\pi}{2} - \theta \right) &= \sin \theta \\
\cot \left( \frac{\pi}{2} - \theta \right) &= \tan \theta \\
\csc \left( \frac{\pi}{2} - \theta \right) &= \sec \theta
\end{align*}
\]

The cofunction formulas can be used to convert from one function to another as demonstrated in the next three examples and later in Example 8.

**Example 1:** Express \(\cos 42^\circ\) as a cofunction of a complementary angle.

**Solution** \(\cos 42^\circ = \sin(90^\circ - 42^\circ) = \sin 48^\circ\)

**Example 2:** Express \(\tan \left( \frac{\pi}{5} \right)\) as a cofunction of a complementary angle.
Solution \( \tan \left( \frac{\pi}{5} \right) = \cot \left( \frac{\pi}{2} - \frac{\pi}{5} \right) = \cot \left( \frac{3\pi}{10} \right) \)

**Example 3:** Express \( \sec \left( \frac{1}{6} \right) \) as a cofunction of a complementary angle.

**Solution** \( \sec \left( \frac{1}{6} \right) = \csc \left( \frac{\pi}{2} - \frac{1}{6} \right) = \csc \left( \frac{3\pi - 1}{6} \right) \)

Let’s turn our attention to the addition and subtraction formulas.

Note that \( \sin \left( \frac{\pi}{2} + \frac{\pi}{2} \right) = \sin \pi = 0 \); however, \( \sin \left( \frac{\pi}{2} + \frac{\pi}{2} \right) \neq \sin \left( \frac{\pi}{2} \right) + \sin \left( \frac{\pi}{2} \right) = 1 + 1 = 2 \).

NEVER use the distributive property with trigonometric functions as shown above! Instead, we use the Addition and Subtraction Formulas to evaluate problems of this nature.

\[
\begin{align*}
\sin(\alpha \pm \beta) &= \sin \alpha \cos \beta \pm \cos \alpha \sin \beta \\
\cos(\alpha \pm \beta) &= \cos \alpha \cos \beta \mp \sin \alpha \sin \beta \\
\tan(\alpha \pm \beta) &= \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta}
\end{align*}
\]

The text does a great job of deriving the formulas given above. You are not responsible for knowing how to derive them, nor do you need to memorize them, but be proficient at using them. The formulas listed above will be provided on the cover sheet for Test 2, Test 3, and the Final Exam. The six equations as presented in the text are combined into three equations using the \( \pm / \mp \) notation. Familiarize yourself with the format in which these formulas will be given. Please print the Math 130 Formula Sheet on WA to familiarize yourself with its format and keep it handy as you work the practice problems.

**Example 4:** Verify that \( \cos \left( \frac{3\pi}{4} + \frac{7\pi}{6} \right) \neq \cos \left( \frac{3\pi}{4} \right) + \cos \left( \frac{7\pi}{6} \right) \).

**Solution** Begin by selecting the correct expansion formula.

\[
\cos \left( \frac{3\pi}{4} + \frac{7\pi}{6} \right) = \cos \left( \frac{3\pi}{4} \right) \cos \left( \frac{7\pi}{6} \right) - \sin \left( \frac{3\pi}{4} \right) \sin \left( \frac{7\pi}{6} \right) \\
= \left( -\frac{\sqrt{2}}{2} \right) \left( -\frac{\sqrt{3}}{2} \right) - \left( \frac{\sqrt{2}}{2} \right) \left( \frac{1}{2} \right) \\
= \frac{\sqrt{6} + \sqrt{2}}{4}
\]

On the other hand, \( \cos \left( \frac{3\pi}{4} \right) + \cos \left( \frac{7\pi}{6} \right) = \left( -\frac{\sqrt{2}}{2} \right) + \left( -\frac{\sqrt{3}}{2} \right) = -\frac{\sqrt{2} + \sqrt{3}}{2} \).

As \( \frac{\sqrt{6} + \sqrt{2}}{4} \neq -\frac{\sqrt{2} + \sqrt{3}}{2} \), then \( \cos \left( \frac{3\pi}{4} + \frac{7\pi}{6} \right) \neq \cos \left( \frac{3\pi}{4} \right) + \cos \left( \frac{7\pi}{6} \right) \).
Example 5: If \( \sin \alpha = -\frac{2}{5} \) and \( \cos \alpha < 0 \), find the exact value of \( \sin \left( \alpha - \frac{\pi}{3} \right) \).

Solution  Since \( \sin \alpha < 0 \) and \( \cos \alpha < 0 \), \( \alpha \) is in quadrant III.

Sketch a right triangle. Select the appropriate expansion formula.

\[
\sin \left( \alpha - \frac{\pi}{3} \right) = \sin \alpha \cos \left( \frac{\pi}{3} \right) - \cos \alpha \sin \left( \frac{\pi}{3} \right) \quad \text{expand}
\]

\[
= \left( -\frac{2}{5} \right) \left( \frac{1}{2} \right) \quad \left( -\frac{\sqrt{21}}{5} \right) \left( \frac{\sqrt{3}}{2} \right) \quad \text{substitute values}
\]

\[
= \frac{-2 + \sqrt{63}}{10} = \frac{\sqrt{63} - 2}{10} \quad \text{simplify}
\]

Example 6: Expand and simplify the following.

a) \( \sin \left( x - \frac{\pi}{2} \right) \)

b) \( \cos (x + 3\pi) \)

Solution Begin by expanding each given expression using the appropriate addition/subtraction formula.

a) \( \sin \left( x - \frac{\pi}{2} \right) = \sin x \cdot \cos \left( \frac{\pi}{2} \right) - \cos x \cdot \sin \left( \frac{\pi}{2} \right) = \sin x \cdot (0) - \cos x \cdot (1) = 0 - \cos x = -\cos x \)

b) \( \cos (x + 3\pi) = \cos x \cdot \cos(3\pi) - \sin x \cdot \sin(3\pi) = \cos x \cdot (-1) - \sin x \cdot (0) = -\cos x \)

Notice that when adding or subtracting an odd multiple of \( \frac{\pi}{2} \), the answer will be in terms of a cofunction. When adding or subtracting a multiple of \( \pi \), the answer will remain in terms of the same function.

Example 7: Expand and simplify \( \sin \left( \theta - \frac{\pi}{6} \right) \).

Solution Begin by selecting the appropriate addition/subtraction formula.

\[
\sin \left( \theta - \frac{\pi}{6} \right) = \sin \theta \cdot \cos \left( \frac{\pi}{6} \right) - \cos \theta \cdot \sin \left( \frac{\pi}{6} \right)
\]

\[
= \sin \theta \cdot \left( \frac{\sqrt{3}}{2} \right) - \cos \theta \cdot \left( \frac{1}{2} \right)
\]

\[
= \frac{1}{2} \left( \sqrt{3} \sin \theta - \cos \theta \right)
\]
Example 8: If \( \sin \theta = \frac{1}{4} \) and \( \cos \theta < 0 \), find the exact value of \( \tan \left( \frac{\pi}{2} - \theta \right) \).

Solution Notice that \( \sin \theta > 0 \) and \( \cos \theta < 0 \), hence \( \theta \) is in quadrant II. At first glance, we might think to try using the difference of two angles formula for tangent; however, the tangent of \( \frac{\pi}{2} \) is undefined so the formula cannot be used. We will use a cofunction instead.

\[
\tan \left( \frac{\pi}{2} - \theta \right) = \cot \theta \quad \text{convert to a cofunction}
\]

\[
\cot \theta = \frac{-\sqrt{15}}{1} = -\sqrt{15} \quad \text{ratio of adj/opp}
\]

Remember when finding an exact value, this implies do not use a calculator!

Exact values will often contain a radical or \( \pi \).

Rationalizing denominators is not necessary, but do simplify as much as possible.

Practice Problems
Answers to odd-numbered problems can be found at the end of your text.

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