Constant Rank Constraint Qualification: Geometric Properties and Perturbation Analysis

Shu Lu

Department of Statistics and Operations Research
University of North Carolina at Chapel Hill

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Consider a parametric set

\[ S(u) = \{ x \in \bar{X} \mid g_i(x, u) \leq 0, \quad i \in I, \quad g_i(x, u) = 0, \quad i \in J \} \]

where \( \bar{X} \) is an open set in \( \mathbb{R}^n \), \( u \) is a parameter taking values in an open set \( \bar{U} \subset \mathbb{R}^m \), \( I \) and \( J \) are disjoint finite index sets, and \( g_i(x, u) \) for each \( i \in I \cup J \) is a \( C^1 \) function from \( \bar{X} \times \bar{U} \) to \( \mathbb{R} \).

Denote the index set of active constraints for each \( (x, u) \in \bar{X} \times \bar{U} \) by

\[ I(x, u) = \{ i \in I \cup J : g_i(x, u) = 0 \}. \]
Constraint qualifications

At a pair \((x, u)\) with \(x \in S(u)\),

- LICQ holds if \(\{\nabla_x g_i(x, u), \ i \in I(x, u)\}\) is linearly independent

- MFCQ \([\text{Mangasarian and Fromovitz 1967}], [\text{Gauvin 1977}]\) holds if
  
  (a) there exists a vector \(w \in \mathbb{R}^n\) such that
  
  \[
  \langle \nabla_x g_i(x, u), w \rangle < 0, \quad i \in l(x, u) \cap I, \\
  = 0, \quad i \in J;
  \]

  (b) the family \(\{\nabla_x g_i(x, u), \ i \in J\}\) is linearly independent

- CRCQ \([\text{Spingarn 1983}], [\text{Janin 1984}]\) holds if for each \(K \subset I(x, u)\) the family \(\{\nabla_x g_i(x', u'), i \in K\}\) is of constant rank for all \((x', u')\) near \((x, u)\)
Each of the above CQs leads to the following expressions for the tangent cone and the normal cone.

\[
T_{S(u)}(x) = \{ v \in \mathbb{R}^n \mid \langle \nabla_x g_i(x, u), v \rangle \leq 0, i \in I(x, u) \cap I, \\
\quad \langle \nabla_x g_i(x, u), v \rangle = 0, i \in J \},
\]

\[
N_{S(u)}(x) = \text{pos}\{\nabla_x g_i(x, u), i \in I(x, u) \cap I\} + \text{span}\{\nabla_x g_i(x, u), i \in J\}.
\]
### Outline of topics

1. Functional dependence under CRCQ
2. Without perturbation
   - Active index sets v.s. faces of the normal cone
   - Relation between CRCQ and MFCQ
3. With perturbation
   - Parametric prox-regularity
   - Continuity
4. Euclidean projectors
5. Variational conditions (variational inequalities defined on nonconvex sets)
**Functional dependence under CRCQ**

**Lemma.** Assume that CRCQ holds at \((x, u)\). Let \(K\) be a subset of \(I(x, u)\) such that \(\{\nabla_x g_i(x, u), i \in K\}\) is linearly independent. Suppose that \(j \in I(x, u) \setminus K\) satisfies

\[
\nabla_x g_j(x, u) = \sum_{i \in K} \lambda_i \nabla_x g_i(x, u)
\]

for some \(\lambda \in \mathbb{R}^{\left|K\right|}\). Then there exist neighborhoods \(X\) of \(x\) in \(\tilde{X}\), \(U\) of \(u\) in \(\tilde{U}\) and \(Y\) of \(g_K(x, u)\) in \(\mathbb{R}^{\left|K\right|}\), with \(Y\) being convex, and a \(C^1\) function \(\phi : Y \times U \to \mathbb{R}\), such that \(g_K(x', u') \in Y\) and \(g_j(x', u') = \phi(g_K(x', u'), u')\) for each \((x', u') \in X \times U\). Moreover, for each \(i \in K\) and each \((y, u') \in Y \times U\),

\[
\text{sgn} \frac{\partial}{\partial y_i} \phi(y, u') = \text{sgn} \lambda_i.
\]

\(^1\)This is a version of the classical constant rank theorem.
Geometric properties under CRCQ \((u \text{ fixed})\)

- If LICQ holds at \(x\) in \(S(u)\), then \(S(u)\) is locally diffeomorphic to \(T_{S(u)}(x)\) \([\text{Robinson 1984}, \text{Guddat, Vasquez and Jongen 1990}]\)
- If CRCQ holds at \(x\) in \(S(u)\), then there is a one-to-one correspondence between the local “face structure” of \(S(u)\) around \(x\) and the face structure of \(N_{S(u)}(x)\)
- Under CRCQ, we can define “curved faces” of \(S(u)\) to be sets of points in \(S(u)\) with the same active constraints, plus boundary (may differ from faces defined in convex analysis for nonpolyhedral convex sets)
Theorem. If CRCQ holds at \( x \in S(u) \), then there exists a neighborhood \( X \) of \( x \) in \( \overline{X} \) such that there is a bijective map between

\[
\{ l(x', u), x' \in S(u) \cap X \} \quad \text{and} \quad \{ \text{nonempty faces of } N_{S(u)}(x) \}.
\]

That map is given by

\[
l(x', u) \mapsto \text{pos}\{ \nabla_x g_i(x, u), i \in l(x', u) \cap I \} + \text{span}\{ \nabla_x g_i(x, u), i \in J \}.
\]

Note: points in \( S(u) \) with active index set \( \{1\} \) all have normal cones similar to a face of \( N_{S(u)}(x) \). It therefore makes sense for them to belong to the same “face.”
An example in which CRCQ fails

Let \( S(u) = \{ x \in \mathbb{R}^2 \mid x_1^2 \leq x_2, \ 0 \leq x_2 \} \), and let \( x = 0 \). MFCQ holds at \( x \) but CRCQ fails. The normal cone \( N_{S(u)}(x) \) is the vertical half-line pointing down through the origin. Let \( X \) be a neighborhood of \( x \).

Note: For this example, we should not define “curved faces” of \( S(u) \) using active index sets, because the geometry at \( x \) is not essentially different from nearby points.
An example in which both MFCQ and CRCQ fail

Let \( S(u) = \{ x \in \mathbb{R}^2 \mid x_2 \leq x_1^2, \; x_2 \geq -x_1^2, \; x_1 \geq 0 \} \). Let \( x = 0 \). Both MFCQ and CRCQ fail at \( x \). The normal cone \( N_{S(u)}(x) \) is \( \mathbb{R}^- \times \mathbb{R} \). Let \( X \) be a neighborhood of \( x \).

![Diagram of normal cone and faces](image)

**two nonempty faces of** \( N_{S(u)}(x) \)

\[
\{ I(x', u), x' \in S(u) \cap X \} = \{ \{1, 2, 3\}, \{1\}, \{2\}, \emptyset \}
\]

“curved faces” of \( S(u) \) around \( x \)
A short summary

If CRCQ holds at $x$ in $S(u)$, then the geometry of $N_{S(u)}(x)$, which is a polyhedral convex cone, contains some essential information about the local geometry of $S(u)$ around $x$. 
Relation between CRCQ and MFCQ

- In the parametric setting, CRCQ is neither weaker nor stronger than MFCQ [Janin 1984]
- For a fixed $u$, CRCQ implies MFCQ to hold in an alternative expression of $S(u)$ obtained by eliminating some indices and moving some indices from $I$ to $J$, see below

**Theorem.** Assume that CRCQ holds at $x \in S(u)$. There exist disjoint subsets $I'$ and $I^*$ of $I$, and a subset $J^*$ of $J$, such that the set

$$S^*(u) = \{ x' \in \mathbb{R}^n \mid g_i(x', u) \leq 0, i \in I', \quad g_i(x', u) = 0, i \in J^* \cup I^* \}$$

locally coincides with $S(u)$ around $x$, and MFCQ holds at $x \in S^*(u)$. 

Some related questions

- In the preceding theorem, does there exist a neighborhood of $u$ such that $S^*(u')$ locally coincide with $S(u')$ for each $u'$ in that neighborhood?
  - No. If it were true, then it would imply $S^*(u')$ to be nonempty for $u'$ near $u$

- For a fixed $u$, does MFCQ imply CRCQ to hold in an alternative expression?
  - Example: $S = \{(x, y) \mid y - x^3 \leq 0, y - x^5 \leq 0\}$
  - MFCQ holds, but one cannot find an alternative expression in which CRCQ holds by changing inequalities to equalities or constraint elimination
Parametric prox-regularity

- If MFCQ holds at \((x, u)\), then the indicator function \(\delta_{S(u)}(x)\) is \textit{prox-regular in} \(x\) \textit{at} \(x\) \textit{for each} \(v \in N_{S(u)}(x)\) \textit{with compatible parametrization by} \(u\) \textit{at} \(u\). [Levy, Poliquin and Rockafellar, 2000]

- The above property also holds under CRCQ, see below. We will use this property to study the Euclidean projector.

**Theorem.** Assume that CRCQ holds at \((x, u)\), and that \(g_i\) for each \(i \in I \cup J\) is a \(C^2\) function on \(\bar{X} \times \bar{U}\). Let \(v \in N_{S(u)}(x)\). Then there exist neighborhoods \(X\) of \(x\) in \(\bar{X}\), \(U\) of \(u\) in \(\bar{U}\) and \(V\) of \(v\) in \(\mathbb{R}^n\), and a real number \(\rho > 0\), such that

\[
\langle v', x'' - x' \rangle \leq (\rho/2)\|x'' - x'\|^2
\]

whenever \(u' \in U\), \(x' \in X \cap S(u')\), \(x'' \in X \cap S(u')\), and \(v' \in V \cap N_{S(u')}(x')\).
A continuity property analogue to Aubin property

It is well known that if MFCQ holds at \((x, u)\) then \(S\) satisfies the Aubin property: i.e., there exist neighborhoods \(X\) of \(x\) in \(\bar{X}\) and \(U\) of \(u\) in \(\bar{U}\) and a real number \(\kappa > 0\) such that

\[
S(u'') \cap X \subset S(u') + \kappa \| u' - u'' \| \mathbb{B}
\]

for each \(u', u'' \in U\). Here \(\mathbb{B}\) is the unit open ball.

An analogue property holds under CRCQ.

**Theorem.** If CRCQ holds at \((x, u)\) then there exist neighborhoods \(X\) of \(\bar{x}\) in \(\bar{X}\) and \(U\) of \(\bar{u}\) in \(\bar{U}\) and a real number \(\kappa > 0\) such that

\[
S(u'') \cap X \subset S(u') + \kappa \| u' - u'' \| \mathbb{B}
\]

for each \(u', u'' \in U\) satisfying \(S(u') \cap X \neq \emptyset\) and \(S(u'') \cap X \neq \emptyset\).
### Euclidean projector under CRCQ

Let $\Pi_{S(u)}(z)$ denote the set of points in $S(u)$ closest to $z$. Consider a base point $(\bar{x}, \bar{u})$, with $\bar{x} \in S(\bar{u})$. Let $\bar{z} := \bar{x}$. Assume in the remaining part of this presentation that $g_i$ for each $i \in I \cup J$ is $C^2$.

**Theorem.** If CRCQ holds at $(\bar{x}, \bar{u})$, then there exist neighborhoods $X_0$ of $\bar{x}$ in $\bar{X}$, $U_0$ of $\bar{u}$ in $\bar{U}$ and $Z_0$ of $\bar{z}$ in $\mathbb{R}^n$ such that $\Pi_{S(u)}(z) \cap X_0$ contains a single point, denoted by $\pi(u, z)$, for each $z \in Z_0$ and each $u \in U'_0 := \{ u \in U_0 \mid S(u) \cap X_0 \neq \emptyset \}$. Moreover, $\pi(u, z)$ is a continuous function on $U'_0 \times Z_0$, is selected from finitely many $C^1$ functions, and is also the unique solution in $X_0$ of the generalized equation $z - x \in N_{S(u)}(x)$.

![Diagram](image-url)
Continuity of the set of multipliers

For each \((u, z) \in U_0' \times Z_0\), define the set of multipliers as

\[
M(u, z) := \{ \lambda \in \mathbb{R}^{|I|+|J|} \mid z - \pi(u, z) = \sum_{i \in I \cup J} \lambda_i \nabla_x g_i(\pi(u, z), u), \lambda_i \geq 0 \text{ for each } i \in I(\pi(u, z), u) \cap I, \lambda_i = 0 \text{ for each } i \in I \setminus I(\pi(u, z), u) \}. 
\]

**Theorem.** For each \(u \in U_0'\), \(M(u, \cdot)\) is a continuous multifunction on \(Z_0\).
An example in which CRCQ fails

Consider $S = \{x \in \mathbb{R}^2 \mid x_1^2 \leq x_2, \ 0 \leq x_2\}$ again. Let $t < 0$ be a scalar.

- The Euclidean projection $\Pi_S(0, t)$ is the origin of $\mathbb{R}^2$, with
  \[
  M(0, t) = \{\lambda \in \mathbb{R}^2_+ : \lambda_1 + \lambda_2 = -t\}.
  \]

- The Euclidean projection $\Pi_S(\epsilon, t)$ with $\epsilon \neq 0$ is not the origin of $\mathbb{R}^2$, and $M(\epsilon, t)$ is a singleton.

- One can verify that $M$ is not continuous at $(0, t)$.

- Consequently there is not a neighborhood $Z_0$ of the origin in $\mathbb{R}^2$ on which $M$ is continuous.
B-derivative of the Euclidean projector

- For each fixed $u \in U'_0$ the function $\pi(u, \cdot)$ is $PC^1$ on $Z_0$, and is therefore B-differentiable there [Scholtes 1994]

- Denote the B-derivative of $\pi(u, \cdot)$ at $z$ for a direction $h$ by $d_z\pi(u, z)(h)$

- Define the critical cone to $S(u)$ associated with $z$ by $K(u, z) = T_{S(u)}(\pi(u, z)) \cap \{z - \pi(u, z)\}^\perp$

**Proposition.** For each $(u, z) \in U'_0 \times Z_0$, let $x = \pi(u, z)$, $\lambda \in M(u, z)$, $h \in \mathbb{R}^n$, and write $A := I_n + \sum_{i \in I \cup J} \lambda_i \nabla_x^2 g_i(x, u)$ and $w := d_z\pi(u, z)(h)$. Then $w \in K(u, z)$ and solves the variational inequality

$$\langle h - Aw, v - w \rangle \leq 0 \text{ for each } v \in K(u, z).$$

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²Originally proved in [Pang and Ralph 1996] for the case $S(u)$ is convex.
Variational condition

Now consider the parametric variational condition

\[-f(x, u) \in N_S(u)(x)\]

where \(f\) is a \(C^1\) function from \(\bar{X} \times \bar{U}\) to \(\mathbb{R}^n\). Assume that
\[-f(\bar{x}, \bar{u}) \in N_S(\bar{u})(\bar{x}),\]
and let \(\hat{z} := \bar{x} - f(\bar{x}, \bar{u})\). Assume WLLG that \(\hat{z} \in Z_0\).

Define \(B_1\) to be a certain family of index sets \(K \subset I(\bar{x}, \bar{u})\). Each \(K \in B_1\)
is associated with two \(C^1\) functions, namely \(x^K : U_0 \times Z_0 \to X_0\) and \(\lambda^K : U_0 \times Z_0 \to \mathbb{R}^{|I|+|J|}\). The way we define \(B_1\) guarantees that for each \((u, z)\) close to \((\bar{u}, \hat{z})\) there exists \(K \in B_1\) such that

\[x^K(u, z) = \pi(u, z)\] and \(\lambda^K(u, z) \in M(u, z)\).
Strong coherent orientation condition

For each $K \in B_1$ define a matrix

$$\Lambda^K(\bar{u}, \hat{z}) = \begin{bmatrix} \nabla_x f(\bar{x}, \bar{u}) + \sum_{i \in K} \nabla_x^2 g_i(\bar{x}, \bar{u}) \lambda^K_i(\bar{u}, \hat{z}) & \nabla_x g_K(\bar{x}, \bar{u})^T \\ -\nabla_x g_K(\bar{x}, \bar{u}) & 0 \end{bmatrix}.$$ 

The SCOC [Luo, Pang and Ralph 1996], [Facchinei and Pang 2003] requires the matrices $\Lambda^K(\bar{u}, \hat{z})$ for all $K \in B_1$ be of the same nonzero determinantal sign.

Define a localized normal map [Robinson 2004] $F : U'_0 \times Z_0 \rightarrow \mathbb{R}^n$ by

$$F(u, z) = f(\pi(u, z), u) + z - \pi(u, z).$$

The SCOC implies the B-derivatives of $F$ w.r.t. $z$ at each $(u, z)$ close to $(\bar{u}, \hat{z})$ to be coherently oriented piecewise linear functions.
Solution existence, uniqueness and continuity

**Theorem.** Assume that CRCQ and SCOC hold at $(\bar{x}, \bar{u})$. There exist neighborhoods $U_2$ of $\bar{u}$ in $U_0$, $Z_2$ of $\hat{z}$ in $Z_0$, $X_2$ of $\bar{x}$ in $X_0$, and continuous functions $z$ and $x$ from $U_2' := \{ u \in U_2 \mid S(u) \cap X_0 \neq \emptyset \}$ to $Z_2$ and $X_2$ respectively, such that the following hold for each $u \in U_2'$.

(a) $z(u)$ is the unique solution to the equation $F(u, \cdot) = 0$ in $Z_2$, and $x(u)$ is the unique solution to $-f(x, u) \in N_{S(u)}(x)$ in $X_2$.

(b) The points $x(u)$ and $z(u)$ satisfy

$$z(u) = x(u) - f(x(u), u) \quad \text{and} \quad x(u) = \pi(u, z(u)).$$

(c) The functions $z$ and $x$ are selected from finitely many $C^1$ functions.
Lipschitz continuity in special cases

In the following two cases, the set $\mathcal{U}'_2$ can be chosen to be convex, and then functions $z(\cdot)$ and $x(\cdot)$ will be Lipschitz continuous.

- When MFCQ holds at $(\bar{x}, \bar{u})$ as well. On convex $S(u)$: [Luo, Pang and Ralph, 1996], [Facchinei and Pang, 2003]
  - In this case, the set $\mathcal{U}'_2$ contains a neighborhood of $\bar{u}$ in $\mathbb{R}^m$.

- When the functions $g_i$ are affine. [Lu and Robinson, 2008]
  - In this case, there is a polyhedral convex neighborhood of $\bar{u}$ in $\mathcal{U}'_2$.
Sensitivity analysis of nonlinear programs

Consider a parametric nonlinear program

\[
\min_{x \in S(u)} \varphi(x, u)
\]  \hspace{1cm} (1)

where \( \varphi \) is a \( C^2 \) function from \( \bar{X} \times \bar{U} \) to \( \mathbb{R} \). Define \( f(x, u) := \nabla_x \varphi(x, u) \). Assume that \(-f(\bar{x}, \bar{u}) \in N_{S(\bar{u})}(\bar{x})\). Let \( \hat{z} = \bar{x} - f(\bar{x}, \bar{u}) \) and assume WLLG that \( \hat{z} \in Z_0 \).

Theorem. Assume that \((\bar{x}, \bar{u})\) satisfies CRCQ, and that for each \( K \in \mathcal{B}_1 \) and each \( v \in \mathbb{R}^n \setminus \{0\} \) with

\[
\nabla_x g_K(\bar{x}, \bar{u})v = 0,
\]

one has \( \langle v, \nabla_x f(\bar{x}, \bar{u})v + \sum_{i \in K} \nabla_x^2 g_i(\bar{x}, \bar{u}) \lambda^K_i(\bar{u}, \hat{z})v \rangle > 0 \). Then the conclusions of the preceding theorem hold. Furthermore, \( x(u) \) is the unique local minimum of (1) in \( X_2 \).
Conclusions

- **Without perturbation:**
  - The face structure of $N_{S(u)}(x)$ corresponds to the “face structure” of $S(u)$
  - CRCQ implies MFCQ to hold in an alternative expression

- **With perturbation:**
  - Parametric prox-regularity
  - Continuity

- Sensitivity analysis of Euclidean projections
- Sensitivity analysis to variational conditions and nonlinear programs
This presentation is based on the following papers.

- Shu Lu. Implications of the Constant Rank Constraint Qualification. Mathematical Programming. Published online at DOI 10.1007/s10107-009-0288-3


Thank you!